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# CRITICAL BRANCHING PROCESSES WITH NONHOMOGENEOUS MIGRATION

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This paper deals with a modification of Galton-Watson processes allowing random migration in the following way: with a probability $p_n$(in the nth generation) one particle is eliminated and does not take part in further evolution, or with a probability $r_n$ takes place immigration of new particles according to a p.g.f. $G(s)$, and, finally, with a probability $q_n$ there is not any migration, $p_n + q_n + r_n = 1$, $n = 0, 1, 2, \cdots$. We investigate a critical case when the offspring mean is equal to one and $r_n G'(1) \equiv p_n \to 0$. Depending on the rate of this convergence we obtain different types of limit theorems.

**1. Introduction.** Let us have on the probability space $(\Omega, \mathscr{F}, P)$ two independent sets of integer-valued random variables (r.v.) $X = \{X_{jn}(k)\}$ and $\varphi = \{\varphi_{jn}(m)\}$ where $X_{jn}(k)$ are independent r.v. with p.g.f. $F_{jn}(s) = Es^{X_{jn}(k)}$ and $\varphi$ is the set of control functions. Sevastyanov and Zubkov [11] have defined controlled branching processes $\{Z_n\}$ in the following way:

$$Z_{n+1} = \textstyle\sum_{j\in J} \sum_{k=1}^{\varphi_{jn}(Z_n)} X_{jn}(k), \quad n = 0, 1, 2, \cdots, \tag{1}$$

where $J$ is an index set (which may be infinite).

Definition (1) describes a very large class of random processes. For example, if $J = \{1\}$, $F_{1n}(s) \equiv F(s)$ and $\varphi_{1n}(m) \equiv m$ a.s., then $\{Z_n\}$ is a classical Galton-Watson process. If $J = \{1, 2\}$ and a.s. $\varphi_{1n}(m) \equiv m$, $\varphi_{2n}(m) \equiv 1$, then $\{Z_n\}$ is a branching process with immigration. If $J = \{1, 2\}$, $F_{1n}(s) \equiv f(s)$, $F_{2n}(s) \equiv g(s)$ and a.s. $\varphi_{1n}(m) \equiv m$, $m \geq 0$, $\varphi_{2n}(m) \equiv 0$, $m \geq 1$, $\varphi_{2n}(0) \equiv 1$, then we obtain the model of Foster [3] and Pakes [9]. The Foster-Pakes processes with $F_{2n}(s) \equiv g_n(s)$ are investigated by Mitov and Yanev [6], [7]. Sevastyanov and Zubkov [11] studied the probabilities of extinction or nonextinction in the case $J = \{1\}$ and $\varphi_{1n}(m) \equiv \varphi(m)$ a.s., where the control function $\varphi(m)$ is nonrandom and integer-valued. Zubkov [22] considered processes with $\varphi_{jn}(m) \equiv \varphi_j(m)$ a.s., where $\varphi_j(m)$ are nonrandom and integer-valued functions. Yanev [13] obtained conditions for extinction or nonextinction when $J = \{1\}$ and $\varphi_n = \{\varphi_{1n}(0), \varphi_{1n}(1), \varphi_{1n}(2), \cdots\}$, $n = 0, 1, 2, \cdots$, are independent identically distributed random processes. These results are generalized for controlled processes in random environments (see Yanev [14]). Vatutin [12] considered a case $J = \{1\}$ and $\varphi_{1n}(m) \equiv \max(m-1, 0)$ a.s., i.e. a branching process with constant emigration of one particle. Note that Yanev and Mitov [15–18], and Nagaev and Han [8], [4] proved asymptotic results for some particular cases of definition (1). Finally, it is not difficult to see that definition (1) describes all Markov chains. However, the most interesting case

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for the theory of branching processes is where a.s. $\sum_{j\in J}\varphi_{jn}(m)\to\infty$, $m\to\infty$. Note that this condition is fulfilled in the papers cited above.

In the present paper we supose that $J=\{1, 2\}$ and

$$\varphi_{1n}(m)=\max\{\min(m, m+Y_n), 0\},\quad \varphi_{2n}(m)=\max(0, Y_n), \tag{2}$$

where the independent random variables $\{Y_n\}_{n=0}^{\infty}$ have distributions:

$$P\{Y_n=-1\}=p_n,\quad P\{Y_n=0\}=q_n,\quad P\{Y_n=1\}=r_n, \quad p_n+q_n+r_n=1. \tag{3}$$

It follows from (1)–(3) that $Z_n$ is a nonhomogeneous Markov process and can be described in the following way:

$$Z_{n+1}=\begin{cases}\sum_{k=1}^{\max(Z_n-1,0)}X_{1n}(k) & \text{with a probab. } p_n,\\ \sum_{k=1}^{Z_n}X_{1n}(k) & \text{with a probab. } q_n,\\ \sum_{k=1}^{Z_n}X_{1n}(k)+X_{2n}(1) & \text{with a probab. } r_n.\end{cases} \tag{4}$$

Without any restriction we can suppose that $Z_0=0$ a.s. It will be assumed that

$$\begin{cases}F(s)=Es^{X_{1n}(k)}=\sum_{i=0}^{\infty}f_is^i, & G(s)=Es^{X_{2n}(k)}=\sum_{i=0}^{\infty}g_is^i,\\ H_n(s)=Es^{Z_n}=\sum_{i=0}^{\infty}P\{Z_n=i\}s^i, & |s|\le 1.\end{cases} \tag{5}$$

It follows from definition (4) that if $q_n\equiv 1$ then $\{Z_n\}$ will be a classical Galton-Watson process characterized by the independence of particle evolutions. In general, definition (4) describes models of branching processes without this restriction, i.e. processes with particle interactions.

Note that if $r_n\equiv 1$ we obtain the well-known Galton-Watson process with immigration (see [1]). The critical case with $p_n\equiv 1$ is investigated by Vatutin [12].

Subcritical and critical processes with $p_n\equiv p$, $q_n\equiv q$, $r_n\equiv r$, $(p+q+r=1)$ are investigated in papers [15–18]. A similar model in the critical and supercritical cases is studied by Nagaev and Han [8], [4].

In paper [19] we considered a model (4) with $F'(1)\le 1$ and $p_n\to 0$, $q_n\to q$, $r_n\to r$, $q+r=1$ and the obtained results are similar to ones for the classical Galton-Watson processes with immigration.

On the other hand, if $F'(1)=1$, $0<F''(1)=2b<\infty$, $r_n\sim C/\log n$ and $p_n=o(r_n)$, then in [20] we found that

$$\lim P\{Z_n>0\}=1-e^{-\theta},\quad \theta=C/b>0,$$

$$\lim P\{(\log Z_n)/\log n\le x\}=e^{-\theta(1-x)},\quad 0\le x\le 1,$$

and

$$\lim P\left\{1-\frac{\log Z_n}{\log n}\le x\,|\,Z_n>0\right\}=\frac{1-e^{-\theta x}}{1-e^{-\theta}},\quad 0\le x\le 1.$$

Let $F'(1)=1$, $0<F''(1)=2b<\infty$, $r_n\sim L(n)/\log n$ and $p_n\sim C/\log n$, where $C>0$ and $L(n)$ is a s.v.f., $L(n)\to\infty$, $r_n\to 0$. Then for the process (4) we prove in [21] that $\lim P\{Z_n > 0\} = 1$ and for $x \geq 0$

$$\lim P\left\{L(n)\left(1 - \frac{\log Z_n}{\log n}\right) \leq x\right\} = 1 - e^{-x/b}.$$

Now we will investigate the process (4) in the critical case $F'(1) = 1$ when $r_n G'(1) \equiv p_n \to 0$. Depending on the rate of this convergence we obtain different types of limit theorems.

**2. Statement of results.** It follows from (4) and (5) that

$$(6)\quad \begin{cases} H_{n+1}(s) = E\{E(s^{Z_{n+1}} \mid Z_n)\} = a_n(s)H_n(F(s)) + p_n H_n(0)\{1 - 1/F(s)\}, \\ H_0(s) = 1, \end{cases}$$

where

$$(7)\quad a_n(s) = p_n/F(s) + q_n + r_n G(s).$$

Repeated application of relation (6) gives

$$(8)\quad H_{n+1}(s) = U_n(n, s) + \textstyle\sum_{k=0}^{n} p_{n-k} H_{n-k}(0)(1 - 1/F_{k+1}(s))U_{k-1}(n, s),$$

where

$$(9)\quad U_k(n, s) = \textstyle\prod_{i=0}^{k} a_{n-i}(F_i(s)), \quad U_{-1}(n, s) \equiv 1,$$

and $F_i(s)$ denote the $i$th functional iterate of $F(s)$, i.e. $F_0(s) = s$ and $F_{i+1}(s) = F(F_i(s))$.

From now on it will be assumed that

$$(10)\quad \begin{cases} F'(1) = 1, \quad 0 < F''(1) = 2b < \infty, \\ 0 < m = G'(1), \quad d = G''(1) < \infty, \\ mr_n \equiv p_n \to 0, \quad n \to \infty. \end{cases}$$

Set $A_n = H_n'(1) = EZ_n$, $B_n = H_n''(1) = EZ_n(Z_n - 1)$, $H_n = H_n(0)$, $R_n = 1 - H_n = P\{Z_n > 0\}$ and $R_n(s) = 1 - H_n(s)$.

The moments of $Z_n$ can be obtained by differentiating (6) or (8) and using (10):

$$(11)\quad A_{n+1} = \textstyle\sum_{k=0}^{n} p_k H_k, \quad A_0 = 0,$$

$$(12)\quad B_{n+1} = 2b \textstyle\sum_{i=0}^{n} p_i H_i(n - i) + (d + 2m(1 - b)) \sum_{k=0}^{n} r_k - 2A_n.$$

Detailed asymptotic investigation of relations (6)–(12) gives the following results.

THEOREM 1. *Suppose* (10) *and* $p_n \sim L(n)n^{-v}$, *where* $0 \leq v < 1$ *and* $L(n)$ *is a slowly varying function (s.v.f.).*

(i) *If* $v = 0$ *and* $L(n) \sim K/\log n$, $0 < \alpha = K/b < 1$, *then* $\lim R_n = \alpha/(1 + \alpha) = \beta$, $A_n \sim b\beta n/\log n$ *and* $B_n \sim \beta b^2 n^2/\log n$;

(ii) *If* $0 < v < 1$ *or* $v = 0$ *and* $L(n) = o(1/\log n)$, *then* $R_n \sim b^{-1}p_n \log n$, $A_n \sim np_n/(1 - v)$, and $B_n \sim 2bn^2 p_n/(1 - v)(2 - v)$.

(iii) *In both cases* (i) *and* (ii) *if* $x \in (0, 1)$ *then*

$$\lim P\{(\log Z_n)/(\log n) \le x \mid Z_n > 0\} = x. \tag{13}$$

THEOREM 2. *Under the condition* (10) *if* $p_n \sim L(n)n^{-1}$ *and*

$$M(n) = \textstyle\sum_{k=0}^{n} p_k \to \infty, \quad n \to \infty,$$

*where* $L(n)$ *is a s.v.f., then* $R_n \sim (bn)^{-1}(L(n)\log n + M(n))$, $A_n \sim M(n)$, *and* $B_n \sim 2bM(n)n$.

*In addition, if there exists*

$$\lim[L(n)\log n/M(n)] = K, \quad 0 \le K \le \infty, \tag{14}$$

*then*

(i) *for* $0 < x < 1$

$$\lim P\{(\log Z_n)/\log n \le x \mid Z_n > 0\} = Kx/(1 + K) \equiv P_1(x); \tag{15}$$

(ii) *for* $x > 0$

$$\begin{aligned} &\lim P\{Z_n/bn \le x \mid Z_n > 0\} \\ &\qquad = K/(1 + K) + (1 - \exp(-x))/(1 + K) \equiv P_2(x). \end{aligned} \tag{16}$$

THEOREM 3. *Assume* (10) *and* $\sum_{k=0}^{\infty} p_k < \infty$. *Then* $R_n \sim \Delta/bn$, $\lim A_n = \Delta$ *and* $B_n \sim 2b\Delta n$, *where* $0 < \Delta = \sum_{k=0}^{\infty} p_k H_k < \infty$. *In addition, for* $x \ge 0$

$$\lim P\{Z_n/bn \le x \mid Z_n > 0\} = 1 - \exp(-x). \tag{17}$$

COMMENTS.

(1). Since the critical Galton-Watson process satisfies all conditions of Theorem 3 then we obtain a natural generalization of the classical Kolmogorov and Yaglom results (see [1] or [5]). The explanation is that the Borel-Cantelli lemma and the conditions $\sum p_n < \infty$ and $\sum r_n < \infty$ ensure that eventually immigration and emigration cease and hence long lived lines of descent behave like those of the simple branching process.

(2). The limit distribution (13) from Theorem 1 is an analog of ones proved by Foster [3] and Nagaev and Han [8]. Note that our process is nonhomogeneous while the models of Foster and Nagaev-Han are homogeneous Markov chains.

(3). The most interesting is Theorem 2. Since $\lim_{x\uparrow 1} P_1(x) = \lim_{x\downarrow 0} P_2(x)$ it follows that we obtain all "essential" nondegenerate sample paths of the process $\{Z_n\}$ which are two different types:

(i) $Z_n \sim n^{\xi_1}$ with probability $K/(1 + K)$, where $\xi_1 \in U(0, 1)$;
(ii) $Z_n \sim \xi_2 n$ with probability $1/(1 + K)$, where $\xi_2 \in \mathrm{Exp}(1/b)$.

(4). It is interesting to consider some particular cases of Theorem 2.

(i) If $L(n) \sim (\log n)^{\rho} L_1(\log n)$, where $\rho > -1$ and $L_1(n)$ is a s.v.f., then $M(n) \sim (\log n)^{\rho+1} L_1(\log n)/(\rho + 1)$. Now (14) is fulfilled with $K = \rho + 1$.

(ii) If $L(n) \sim L_1(\log\log n)/\log n$, where $L_1(n)$ is a s.v.f., then $M(n) \sim L_1(\log\log n)\log\log n$. Now, from (14) it follows that $K = 0$. Hence, from (16) we have

$P_2(x) = 1 - e^{-x}$, i.e. the classical Yaglom's limit theorem. On the other hand, $R_n \sim L_1(\log\log n)/bn$ and $A_n \sim L_1(\log\log n)\log\log n$, i.e. we do not obtain the Kolmogorov's asymptotics.

(iii) If $L(n) = \exp\{(\log\log n)^2\}$, it is not difficult to see that from (14) one obtains $K = \infty$. Obviously, this corresponds to the case $P_1(x) = x$ in (15).

(5). It is very unexpected that the obtained asymptotic results are similar to those in [6] and [7] for processes with decreasing state-dependent immigration.

**3. Preliminaries.** We will need the following well-known results for a critical Galton-Watson processes (see [1] or [5]):

$$0 < F_n(0) \le F_n(s) \le 1, \quad F_n(s) \uparrow 1, \tag{18}$$

uniformly for $0 \le s \le 1$;

$$Q_n(s) \equiv 1 - F_n(s) = (1 - s)(1 + \varepsilon_n(s))/(1 + bn(1 - s)), \tag{19}$$

where $\lim \varepsilon_n(s) = 0$ uniformly for $0 \le s \le 1$;

$$Q_n \equiv 1 - F_n(0) \sim 1/bn, \quad n \to \infty. \tag{20}$$

LEMMA 1. *Under conditions* (10) $U_k(n, s) = 1 + \alpha_n(s)$, $n \to \infty$, *where* $\alpha_n(s) = O(\sum_{j=0}^{n} p_{n-j}Q_j^2)$ *uniformly for* $k \le n$ *and* $0 \le s \le 1$.

PROOF. From (7) and (10) we have

$$1 - a_k(s) = \frac{p_k(1 - s)}{F(s)}\left(F(s)\,\frac{1 - G(s)}{m(1 - s)} - \frac{1 - F(s)}{1 - s}\right). \tag{21}$$

On the other hand, under conditions (10) for $0 \le s \le 1$

$$\begin{cases} 1 - s - b(1 - s)^2 \le 1 - F(s) \le 1 - s, \\ m(1 - s) - d(1 - s)^2 \le 1 - G(s) \le m(1 - s). \end{cases} \tag{22}$$

Now from (18), (21) and (22) it follows that

$$-(d + m)p_k(1 - s)^2/mF(0) \le 1 - a_k(s) \le bp_k(1 - s)^2/F(0).$$

Hence, for some positive constant $C$ and $0 \le s \le 1$

$$|1 - a_k(s)| \le Cp_k(1 - s)^2, \quad k \ge 0. \tag{23}$$

Relation (10) and (23) show that $a_{n-j}(F_j(s)) \to 1$ as $n \to \infty$ uniformly for $j \le n$ and $0 \le s \le 1$. Therefore, from (9) as $n \to \infty$ we obtain

$$\log U_k(n, s) = \textstyle\sum_{j=0}^{k} \log(1 - \{1 - a_{n-j}(F_j(s))\}) \sim -\sum_{j=0}^{k} \{1 - a_{n-j}(F_j(s))\}. \tag{24}$$

On the other hand, from (18) and (23) for $k \le n$ and $0 \le s \le 1$ it follows that

$$\left|\textstyle\sum_{j=0}^{k} \{1 - a_{n-j}(F_j(s))\}\right| \le C \sum_{j=0}^{n} p_{n-j}Q_j^2. \tag{25}$$

Now relations (24) and (25) prove the lemma.

LEMMA 2. *Under conditions* (10) *as* $n \to \infty$ *and* $0 \le s \le 1$

$$H_{n+1}(s) = 1 - \sum_{k=0}^{n} Q_k(s)p_{n-k}H_{n-k} + \alpha_n(s) + \beta_n(s), \tag{26}$$

*where*

$$\alpha_n(s) = O(\sum_{j=0}^{n} p_{n-j}Q_j^2), \quad \beta_n(s) = O(\alpha_n(s) \sum_{k=0}^{n} p_{n-k}Q_k(s)).$$

PROOF. It follows immediately from the representation (see (8))

$$H_{n+1}(s) = U_n(n, s) - \sum_{k=0}^{n} Q_k(s)U_k(n, s) \frac{1 - F_{k+1}(s)}{F_{k+1}(s)(1 - F_k(s))} p_{n-k}H_{n-k},$$

using Lemma 1 and the fact that

$$\lim(1 - F(F_k(s)))/(F_{k+1}(s)(1 - F_k(s))) = 1$$

as $k \to \infty$ uniformly for $0 \le s \le 1$.

LEMMA 3. *Assume* (10) *and* $p_n = O(1/\log n)$. *Then for* $0 \le s \le 1$

$$R_{n+1}(s) = \sum_{k=0}^{n} Q_k(s)p_{n-k}H_{n-k} + O(1/\log n). \tag{27}$$

PROOF. From Lemma 2 and (18) it will be sufficient to show that

$$\sum_{k=0}^{n} p_{n-k}Q_k = O(1), \quad \sum_{k=0}^{n} p_{n-k}Q_k^2 = O(1/\log n). \tag{28}$$

Indeed, since $p_n \le C/\log n$, $n \ge N$, then

$$\sum_{k=0}^{2n} p_{2n-k}Q_k \le (C/\log n) \sum_{k=0}^{n} Q_k + Q_n \sum_{k=0}^{n} p_k,$$

$$\sum_{k=0}^{2n} p_{2n-k}Q_k^2 \le (C/\log n) \sum_{k=0}^{n} Q_k^2 + Q_n^2 \sum_{k=0}^{n} p_k,$$

and we obtain (28) because of (20) and $\sum_{k=0}^{n} p_k = O(n/\log n)$.

LEMMA 4. *Suppose* (10) *and* $p_n = o(1/\log n)$. *Then* $\lim R_n = 0$.

PROOF. For each $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that $p_n \le \varepsilon/\log n$, $n \ge N$. Therefore, relation

$$0 \le \sum_{k=0}^{2n} Q_k p_{n-k} \le (\varepsilon/\log n) \sum_{k=0}^{n} Q_k + Q_n \sum_{k=0}^{n} p_k$$

shows that $\lim \sum_{k=0}^{n} Q_k p_{n-k} = 0$ because of (20) and $\sum_{k=0}^{n} p_k = o(n/\log n)$. The rest follows from Lemma 3.

LEMMA 5. *Under conditions* (10) *if additionally* $R_n \to \beta \ge 0$ *and for some* $s = s(n)$

$$\lim\{\min(1/\alpha_n(s), n) \sum_{k=0}^{n} Q_k(s)p_{n-k}\} = \infty, \tag{29}$$

*then as* $n \to \infty$

$$R_n(s) \sim (1 - \beta) \sum_{k=0}^{n} Q_k(s)p_{n-k}. \tag{30}$$

PROOF. From (26) it follows that

$$R_{n+1}(s) = (1-\beta)\textstyle\sum_{k=0}^{n} p_k Q_{n-k}(s) + \sum_{k=0}^{n}(\beta - R_k)p_k Q_{n-k}(s) - \alpha_n(s) - \beta_n(s). \tag{31}$$

For $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that $|R_n - \beta| < \varepsilon$, $n \ge N$. Therefore,

$$\left|\textstyle\sum_{k=0}^{n}(\beta - R_k)p_k Q_{n-k}(s)\right| \le 2Q_{n-N}\sum_{k=0}^{N} p_k + \varepsilon \sum_{k=N+1}^{n} p_k Q_{n-k}(s). \tag{32}$$

Now relation (30) follows from (31) using Lemma 2, (32) and (29).
Further we will use Lemma 5 with application of the following results.

LEMMA 6. *Under conditions* (10) *if additionally*
(i) $p_n \sim L(n)n^{-v}$, $0 \le v \le 1$, *then*

$$\alpha_n = \textstyle\sum_{j=0}^{n} p_{n-j}Q_j^2 = O(p_n); \tag{33}$$

(ii) $p_n = o(1/n)$, *then*

$$\alpha_n = o(1/n), \quad \gamma_n = \textstyle\sum_{k=0}^{n} p_{n-k}Q_k = o(\log n/n). \tag{34}$$

PROOF. (i) Since $L(n)$ is a s.v.f. then $L(nx)/L(n) \to 1$, $n \to \infty$, uniformly for $x$ belonging to every finite interval $0 < a_1 \le x \le a_2 < \infty$ (see Seneta [10], page 2). Hence, for every $\varepsilon > 0$ there exists $N < \infty$ such that for $n \ge N$ $(1-\varepsilon)L(n) \le L(nx) \le (1+\varepsilon)L(n)$, $x \in [1/2, 1]$. From here, (10) and (20) for $n \ge 2N$ we obtain

$$\begin{aligned}\alpha_n &= \textstyle\sum_{k=0}^{n} p_{n-k}Q_k^2 = \sum_{k\le n/2} + \sum_{n/2<k\le n}\\ &\le C(1+\varepsilon)(L(n)/n^v)\textstyle\sum_{k\le n/2} Q_k^2 + Q_{n/2}^2 \sum_{k\le n/2} p_k = O(p_n).\end{aligned}$$

(ii) For every $\varepsilon > 0$ there exists $N < \infty$ such that $p_k \le \varepsilon/k$, $k \ge N$. Hence, for $n \ge 2N$

$$\begin{aligned}\alpha_n &= \textstyle\sum_{k\le n/2} p_{n-k}Q_k^2 + \sum_{n/2<k\le n} p_{n-k}Q_k^2\\ &\le 2/n \textstyle\sum_{k\le n/2} Q_k^2 + Q_n^2\left(\sum_{k\le n/2} p_k\right) = o(1/n)\end{aligned}$$

and

$$\begin{aligned}\gamma_n &= \textstyle\sum_{k\le n/2} p_{n-k}Q_k + \sum_{n/2<k\le n} p_{n-k}Q_k\\ &\le (2\varepsilon/n)\textstyle\sum_{k\le n/2} Q_k + Q_n \sum_{k\le n/2} p_k = o(\log n/n).\end{aligned}$$

## 4. Proof of Theorem 1.

(i) Using (27) from Lemma 3 with $s = 0$ we can see that for some $0 < \delta < 1$

$$\begin{cases} H_{n+1} \le 1 - (\inf_{k\ge n(1-\delta)} H_k)\sum_{i\le n\delta} p_{n-i}Q_i + O(1/\log n),\\ H_{n+1} \ge 1 - (\sup_{k\ge n(1-\delta)} H_k)\sum_{i\le n\delta} p_{n-i}Q_i - \sum_{n\delta<i\le n} p_{n-i}Q_i + O(1/\log n)\end{cases} \tag{35}$$

On the other hand, for each $\varepsilon > 0$ and large enough $n$

$$\textstyle\sum_{k\le n\delta} p_{n-k}Q_k \le (K+\varepsilon)/(\log n(1-\delta))\sum_{k\le n\delta} Q_k \to (K+\varepsilon)/b,$$

$$\textstyle\sum_{k\le n\delta} p_{n-k}Q_k \ge ((K-\varepsilon)/\log n)\sum_{k\le n\delta} Q_k \to (K-\varepsilon)/b,$$

$$\textstyle\sum_{n\delta<k\le n} p_{n-k}Q_k \le Q_{[n\delta]}\sum_{k\le n(1-\delta)} p_k \sim \alpha/\log n.$$

Thus from (35) it follows $1 - \alpha \limsup H_n \le \liminf H_n \le \limsup H_n \le 1 - \alpha$ lin inf $H_n$. Hence, $(1 - \alpha)(\limsup H_n - \liminf H_n) \le 0$ and $\lim H_n = 1/(1 + \alpha)$.

The asymptotic behaviour of $A_n$ and $B_n$ follows immediately from (11) and (12) using Theorem 1 ([2], Chapter 8, Section 9).

(ii) By the same theorem

$$\sum_{k=0}^{n} p_k \sim np_n/(1 - \upsilon), \tag{36}$$

and for each $\delta \in (0, 1)$ and $s \in [0, 1]$

$$\sum_{k\le n\delta} p_k Q_{n-k}(s) \le Q_{[n\delta]} \sum_{k\le n\delta} p_k = o(p_n \log n). \tag{37}$$

On the other hand, if $L(u)$ is s.v.f. then (see Seneta [9], page 2)

$$L(ux)/L(u) \to 1, \quad u \to \infty, \tag{38}$$

uniformly in each finite interval $0 < a \le x \le b < \infty$.

Thus for each $\varepsilon > 0$ and large enough $n$ we have for $0 \le s \le 1$

$$\begin{aligned} \frac{(1-\varepsilon)^2 L(n)}{n^{\upsilon}} \sum_{k\le n(1-\delta)} Q_k(s) &\le \sum_{n\delta<k\le n} p_k Q_{n-k}(s) \\ &\le \frac{(1+\varepsilon)^2 L(n)}{(n\delta)^{\upsilon}} \sum_{k\le n(1-\delta)} Q_k(s). \end{aligned} \tag{39}$$

Now, from (37) and (39) with $s = 0$ it is not difficult to obtain (using (20)) that

$$\sum_{k=0}^{n} p_k Q_{n-k} \sim b^{-1} p_n \log n. \tag{40}$$

Hence, by Lemma 6 (29) is hold with $s = 0$ and from (30) we obtain that $R_n \sim b^{-1} p_n \log n$ because of Lemma 4 $\beta = 0$.

Now from (11) and (12) using Theorem 1 ([2], Chapter 8, Section 9) it is not difficult to see that $A_n \sim np_n/(1 - \upsilon)$ and

$$B_n \sim 2bnA_n - \sum_{k=0}^{n} kp_k H_k \sim 2bn^2 p_n/(1 - \upsilon)(2 - \upsilon).$$

(iii) To prove (13) it is sufficient to show that

$$\begin{aligned} S_n(u, x) &= \sum_{k=0}^{n} p_k Q_{n-k}(\exp(-un^{-x})) \\ &= \sum_{k\le n\delta} + \sum_{n\delta<k\le n} \sim (1 - x)p_n/(1 - \beta), \end{aligned} \tag{41}$$

Indeed, (41) follows from (37) and (39) using (19) and the fact that

$$\sum_{k\le n\delta} \{(1 - \exp(-un^{-x}))^{-1} + bk\}^{-1} \sim b^{-1}(1 - x)\log n, \quad n \to \infty.$$

Hence, by Lemma 6 (29) is fulfilled with $s(n) = \exp(-un^{-x})$, and by (30) (Lemma 5) and (41) we obtain

$$\begin{aligned} &\lim_{n\to\infty} E\{\exp(-uZ_n n^{-x}) \mid Z_n > 0\} \\ &\qquad = 1 - \lim_{n\to\infty} R_n(\exp(-un^{-x}))/R_n = x, \quad 0 < x < 1. \end{aligned} \tag{42}$$

Thus by the continuity theorem for Laplace transforms ([2], page 408) it follows from (42) that $\lim P\{Z_n n^{-x} \le y \mid Z_n > 0\} = x$ for each $y > 0$ which is equivalent to (13).

**5. Proof of Theorem 2.** From the conditions of the theorem and (19) it follows that for each $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that for $n \ge n_0$

$$(1-\varepsilon)n^{-1}L(n) \le p_n \le (1+\varepsilon)n^{-1}L(n)$$

$$(43)\qquad (1-\varepsilon)/\{(1-s)^{-1} + bn\} \le Q_n(s) \le (1+\varepsilon)/\{(1-s)^{-1}+bn\},$$
$$0 \le s \le 1.$$

Therefore, for some $N$ fixed, $n_0 \le N \le n - n_0$, we have

$$(44)\qquad \textstyle\sum_{k<N} p_k Q_{n-k} \le ((1+\varepsilon)/b(n-N)) \sum_{k<N} p_k = O(1/n),$$

$$(45)\qquad \textstyle\sum_{n-N<k\le n} p_k Q_{n-k} \le ((1+\varepsilon)L(n)/(n-N)) \sum_{k<N} Q_k = O(L(n)/n)$$

and

$$(46)\qquad b^{-1}(1-\varepsilon)^2 I_n \le \textstyle\sum_{N\le k\le n-N} p_k Q_{n-k} \le b^{-1}(1+\varepsilon)^2 I_n,$$

where

$$(47)\qquad I_n = \textstyle\sum_{k=N}^{n-N} (L(k)/k(n-k)) \sim n^{-1}(M(n) + L(n)\log n).$$

The last relation follows from representation

$$I_n = n^{-1}\left\{\sum_{k=N}^{n-N} \frac{L(k)}{k} + \sum_{N\le k\le n\delta} \frac{L(k)}{n-k} + \sum_{n\delta\le k\le n-N} \frac{L(k)}{n-k}\right\}, \quad 0<\delta<1,$$

because $\sum_{N\le k\le n\delta} L(k)/(n-k) \le \{n(1-\delta)\}^{-1} \sum_{N\le k\le n\delta} L(k) = O(L(n))$ and $\sum_{n\delta\le k\le n-N} L(k)/(n-k) \sim L(n)\log n$.

Relation (44)–(47) and Lemma 6 show that (29) is fulfilled with $s = 0$ and by (30) we obtain $R_n \sim (bn)^{-1}(M(n) + L(n)\log n)$.

The asymptotic behaviour of $A_n$ and $B_n$ follows from (11) and (12) using [2] (Ch. 8 Section 9) and the fact that $L(n) = o(M(n))$.

Now, from (43)–(45) it follows that for each $\varepsilon > 0$ there exists $n_1 = n_1(\varepsilon)$ such that for some fixed $N$, $n_1 \le N \le n - n_1$,

$$(48)\quad (1-\varepsilon)^2 V_n(u,x) \le \textstyle\sum_{k=0}^{n} p_k Q_{n-k}(\exp(-un^{-x})) \le (1+\varepsilon)^2 V_n(u,x) + \varepsilon,$$

where (similarly to (47)) for $u > 0$, $0 < x < 1$, and $n \to \infty$

$$(49)\qquad V_n(u,x) = \sum_{k=N}^{n-N} \frac{L(k)}{k\{b(n-k) + n^x u^{-1}\}} \sim (bn)^{-1}(M(n) + (1-x)L(n)\log n).$$

Therefore, (48) and (49) yield (29) with $s(n) = \exp(-un^{-x})$ and by (30) and (14) we obtain

$$(50)\qquad \lim R_n^{-1} R_n(\exp(-un^{-x})) = \{1 + (1-x)K\}/(1+K).$$

The conclusion (15) now follows from (42), (50) and the continuity theorem for Laplace transforms.

In the same way, it is not difficult to show that for $u > 0$

$$\sum_{k=0}^{n} p_k Q_{n-k}(\exp(-u/bn)) \sim \frac{uM(n)}{bn(1+u)} + \frac{u}{bn(1+u)} \sum_{k=N}^{n-N} \frac{L(k)}{(1+u^{-1})n-k}. \tag{51}$$

Since $\sum_{k=N}^{n-N} L(k)/((1+u^{-1})(n-k) \le u/n \sum_{k=N}^{n-N} L(k) = O(L(n))$ and $L(n) = o(M(n))$ it follows from (51), Lemma 5, Lemma 6 and (14) that

$$\lim R_n^{-1} R_n(e^{-u/bn}) = u/(1+u)(1+K).$$

Hence, $\lim E\{e^{uZ_n/bn} \mid Z_n > 0\} = K(1+K)^{-1} + \{(1+u)(1+K)\}^{-1}$, $u > 0$ and by the continuity theorem (16) follows.

**6. Proof of Theorem 3.** From conditions of the theorem it follows that $p_n = o(1/n)$ and by Lemma 4 $\lim H_n = 1$. Then from (11) and (12) $\lim A_n = \Delta$ and $B_n \sim 2b\Delta n$ because of $\sum_{i=0}^{n} iH_i p_i = \sum_{k=1}^{n} \sum_{j=k}^{\infty} H_j p_j - n \sum_{k=n+1}^{\infty} H_k p_k = o(n)$.

On the other hand, from Lemma 2 and Lemma 6 ((34)) one can find that

$$R_{n+1}(s) = \sum_{k=0}^{n} Q_{n-k}(s) p_k H_k + o(1/n). \tag{52}$$

For each $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that

$$\sum_{k=N+1}^{\infty} H_k p_k < \Delta\varepsilon. \tag{53}$$

Then for $n \ge N$ and $O \le s \le 1$ using (18) one obtains that

$$\begin{aligned} Q_n(s) \sum_{k=0}^{n} H_k p_k &\le W_n(s) \\ &= \sum_{k=0}^{n} H_k p_k Q_{n-k}(s) \le Q_{n-N}(s) \sum_{k=0}^{N} H_k p_k + (1-s) \sum_{k=N+1}^{n} p_k H_k. \end{aligned} \tag{54}$$

From here, putting $s = 0$ and using (53) it is not difficult to see that $W_n(0) \sim \Delta/bn$ and from (52) $R_n \sim \Delta/bn$.

On the other hand, from (54) putting $s = \exp(-u/bn)$, $u > 0$, and using (19) and (53) one can prove that

$$\begin{aligned} (1-\varepsilon)(u/(1+u)) &\le \liminf R_n^{-1} W_n(e^{-u/bn}) \le \limsup R_n^{-1} W_n(e^{-n/bn}) \\ &\le (1+\varepsilon)(u/(1+u)) + \varepsilon u. \end{aligned} \tag{55}$$

From (55) and (52) it follows that $\lim R_n(e^{-u/bn}) R_n^{-1} = u/(1+u)$, $u > 0$.

Therefore, $\lim E\{e^{-uZ_n/bn} \mid Z_n > 0\} = (1+u)^{-1}$, $u > 0$, which proves (17) by the continuity theorem for Laplace transforms ([2], page 408).

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## REFERENCES

[1] ATHREYA, K. and NEY, P. (1972). *Branching Processes.* Springer, Berlin.

[2] FELLER, W. (1966). *An Introduction to Probability Theory and its Applications 2.* Wiley, New York.

[3] FOSTER, J. H. (1971). A limit theorem for a branching process with state-dependent immigration. *Ann. Math. Statist.* **42** 1773–1776.

[4] HAN, L. V. (1980). Limit theorems for Galton-Watson branching processes with migration. *Siberian Math. J. XXI* **2** 183–194. (In Russian).

[5] KESTEN, H., NEY, P. and SPITZER, F. (1966). The Galton-Watson process with mean one and finite variance. *Theory Probab. Appl. XI* **4** 579–611.

[6] MITOV, K. V. and YANEV, N. M. (1982). Critical branching processes with decreasing state-dependent immigration. *Comptes rendus de l'Academie bulgare des Sciences, 36* **2** 193–196.

[7] MITOV, K. V. and YANEV, N. M. (1984). Critical Galton-Watson processes with decreasing state-dependent immigration. *J. Appl. Probab.* **21** 22–39.

[8] NAGAEV, S. V. and HAN, L. V. (1980). Limit theorems for critical Galton-Watson branching process with migration. *Theory Probab. Appl., XXV* **3** 523–534.

[9] PAKES, A. G. (1971). A branching process with a state-dependent immigration component. *Adv. Appl. Probab.* **3** 301–314.

[10] SENETA, E. (1976). Regularly Varying Functions. *Lecture Notes in Math. N. 508,* Springer, Berlin.

[11] SEVASTYANOV, B. A. and ZUBKOV, A. M. (1974). Controlled branching processes. *Theory Probab. Appl. XIX* **1** 15–25.

[12] VATUTIN, V. A. (1977). A critical Galton-Watson branching process with emigration. *Theory Probab. Appl. XXII* **3** 482–497.

[13] YANEV, N. M. (1975). Conditions of extinction of $\varphi$-branching processes with random $\varphi$. *Theory Probab. Appl. XX* **2** 433–440.

[14] YANEV, N. M. (1977). Controlled branching processes in random environment. *Math. Balkanica* **7** 137–156 (In Russian).

[15] YANEV, N. M. and MITOV, K. V. (1980). Controlled branching processes: The case of random migration. *Comptes rendus de l'Academie bulgare des Sciences 33* **4** 433–435.

[16] YANEV, N. M. and MITOV, K. V. (1983). The life periods of critical branching processes with random migration. *Theory Probab. Appl. XXVIII* **3** 458–467.

[17] YANEV, N. M. and MITOV, K. V. (1981). *Critical branching migration processes.* Math. and Education in Math. Proceedings of Tenth Spring Conference of the Union of Bulgarian Mathematicians, Sunny Beach. Bulgarian Academy of Sciences, Sofia 321–328 (In Russian).

[18] YANEV, N. M. and MITOV, K. V. (1984). Subcritical branching migration processes. PLISKA Stud. Math. Bulgar. **7** 75–82 (In Russian).

[19] YANEV, N. M. and MITOV, K. V. (1982). Limit theorems for controlled branching processes with non-homogeneous migration. *Comptes rendus de l'Academie bulgare des Sciences 35* **3** 299–301.

[20] YANEV, N. M. and MITOV, K. V. (1984). Controlled branching processes with non-homogeneous migration. PLISKA Stud. Math. Bulgar. **7** 90–96 (In Russian).

[21] YANEV, N. M. and MITOV, K. V. (1985). A critical branching process with decreasing migration. *Serdica* (Bulg. Math. Publ.), **11** 3.

[22] ZUBKOV, A. M. (1974). Analogues between Galton-Watson processes and $\varphi$-branching processes. *Theory Probab. Appl. XIX* **2** 319–339.

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