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CONVERGENCE OF QUADRATIC FORMS IN p-STABLE RANDOM VARIABLES AND θ_p-RADONIFYING OPERATORS

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Necessary and sufficient conditions are given for the almost sure convergence of the quadratic form $\sum \sum f_{jk} M_j M_k$ where (M_j) is a sequence of i.i.d. p-stable random variables. A connection is established between the convergence of the quadratic form and a radonifying property of the infinite matrix operator (f_{kj}).

1. Introduction. The aim of this paper is to study the convergence of the random quadratic forms of the form

$$\sum_k \sum_j f_{jk} M_j M_k \tag{1.1}$$

where (f_{jk}), j, $k = 1, 2, \cdots$, is a real infinite matrix and (M_j), $j = 1, 2, \cdots$, is a sequence of i.i.d. p-stable random variables with characteristic function $\exp(-|t|^p)$, $0 < p < 2$. Our results have obvious implications in the theory of double Wiener-type integrals of the form

$$\int \int f(x, y) M(dx) M(dy)$$

where $M(x)$ is a p-stable motion (cf. Corollary 3.2 and also Szulga and Woyczynski, 1983). We shall study them elsewhere.

We begin with a characterization of nonanticipating sequences (V_k) such that the "martingale" transform $\sum V_k M_k$ converges almost surely (Theorem 2.1). The necessary and sufficient condition turns out to be $\sum |V_k|^p < \infty$ a.s., and moreover the "martingale" transform converges a.s. exactly on the set $(\sum |V_k|^p < \infty)$. The sufficiency of the above condition could also be obtained from Kallenberg's (1975) results on stochastic integrals with respect to differential processes, but our proof of sufficiency is much more straightforward. This result applied to the sequence $V_k = \sum_{k=1}^{j-1} f_{jk} M_j$ shows that the convergence of the off-diagonal part of the iterated sum

$$\sum_k \left(\sum_{j<k} f_{jk} M_j\right) M_k \tag{1.2}$$

is equivalent to the almost sure convergence in ℓ^p of the random vector series

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$\sum x_j M_j$ where $x_j = \sum_{k=j+1}^{\infty} f_{jk} e_k \in \ell^p$ and (e_k) is the standard basis in the dual of ℓ^p (Theorem 2.2). Theorem 2.3 shows, in turn, that the series $\sum x_j M_j$ converges a.s. if and only if

$$(1.3) \qquad N_p(F) = \sum_k \sum_j |f_{jk}|^p \left(1 + \log^+ \frac{|f_{jk}|^p \sum_j \sum_k |f_{jk}|^p}{\sum_l |f_{jl}|^p \sum_l |f_{lk}|^p}\right) < \infty,$$

where $F = (f_{jk}) = (\langle x_j, e_k \rangle)$. The most important ingredient in the proof of this characterization was shown to us by Gilles Pisier and is included here with his permission. Thus (1.3) provides a necessary and sufficient condition for the a.s. convergence of the off-diagonal part (1.2), and an analysis of the interplay between the diagonal and off-diagonal parts of (1.1) permits a full characterization of matrices (f_{jk}) for which the quadratic form (1.1) converges a.s. (Theorem 3.2).

Our results should be compared with the case where the M_j's are i.i.d. Gaussian random variables, i.e., when $p = 2$, in which case the convergence of the series $\sum_k f_{kk}$ and $\sum_{jk} f_{jk}^2$ is necessary and sufficient for the almost sure and quadratic mean convergence of (1.1) (see Sjörgen, 1982, and Varberg, 1966). We would also like to mention here that the infinite quadratic forms satisfy a fairly general 0–1 law (cf. de Acosta, 1976).

In Section 5 we include some auxiliary results, and we conclude the introduction by sketching how our results on convergence of random vector series can be interpreted in the theory of θ_p-radonifying operators.

A connection with θ_p-radonifying operators. Let $1 < p \le 2$, and let θ_p be the canonical p-stable measure on ℓ^q, $1/p + 1/q = 1$, generated by $\mathbb{M} = \{M_i: i = 1, 2, \cdots\}$. The characteristic functional of θ_p is given by $\int_{\ell^q} \exp(i \langle x, y \rangle) \theta_p(dy) = \exp(-\|x\|_p^p)$, $x \in \ell^p$. For a Banach space E, a linear operator $F: \ell^q \to E$ is said to be θ_p-radonifying if $\theta_p \circ F^{-1}$ extends to a Radon measure. We denote by $\mathscr{R}_p(\ell^q, E)$ the class of all such operators.

In the Gaussian case, i.e., $p = 2$, the class $\mathscr{R}_2(\ell^2, E)$ has been extensively studied and compared with the classes Π_p of p-absolutely summing operators. The main result is that $\Pi_2(\ell^2, E) = \mathscr{R}_2(\ell^2, E)$ if and only if E is of cotype 2 (cf. Chobanjan and Tarieladze, 1977). For $1 < p < 2$ it has been proved that $\Pi_p(\ell^q, E) = \mathscr{R}_p(\ell^q, E)$ if and only if E is of stable type p and is isomorphic to a subspace of a quotient of some L^p (cf. Linde, Mandrekar, Weron, 1980, and also Thang and Tien, 1982, for other characterizations).

A straightforward application of Itô and Nisio's Theorem (1968) gives the equivalence of the following three properties of the operator $F: \ell^q \to E$:

(a) $F \in \mathscr{R}_p(\ell^q, E)$.

(b) There exists an E valued strongly measurable random vector $(F\mathbb{M})$ such that

$$\langle F^* y^*, \mathbb{M} \rangle = \langle y^*, F\mathbb{M} \rangle \quad \text{a.s.,} \quad y^* \in E^*.$$

(c) The series $\sum F(e_j) M_j$ converges a.s. in E, where (e_j) is the standard basis in ℓ^q.

The above equivalences are heuristically better understood if one keeps in mind the following "identities":

$$E \exp(i \langle F^*y^*, \mathbb{M}\rangle) = E \exp(i \textstyle\sum_k M_k \langle F^*y^*, e_k\rangle)$$
$$= \exp(-\textstyle\sum_k |\langle F^*y^*, e_k\rangle|^p) = \exp(-\| F^*y^* \|_p^p).$$

An explicit characterization of operators in $\mathscr{R}_p(\ell^q, \ell^p)$ was one of the most interesting open problems in the theory of θ_p-radonifying operators. Theorem 2.3, in conjunction with the equivalence of (a) and (c), solves this problem and shows that $F \in \mathscr{R}_p(\ell^q, \ell^p)$ if and only if $N_p(F^*) < \infty$.

Finally, let us remark that since ℓ^p, $0 < p \le 1$, is a quasi-normed space with separating dual, the above discussion need not be restricted to the case $p > 1$ (one only has to replace ℓ^q by $(\ell^p)^*$, cf. Marcus and Woyczynski, 1979).

2. Convergence of stable triangular quadratic forms and of p-stable random series in ℓ^p.

THEOREM 2.1. *Let $V_k = V_k(M_1, \cdots, M_{k-1})$, $k = 2, 3, \cdots$, be a nonanticipating sequence of random variables. Then almost surely*

$$\{\textstyle\sum V_k M_k \text{ converges}\} = \{\textstyle\sum |V_k|^p < \infty\}.$$

PROOF. Let $\mathscr{F}_k = \sigma(M_1, \cdots, M_k)$, and

$$G(x) = P\{|M_1| > x\} \sim cx^{-p},$$

as $x \to \infty$, $c > 0$. Also let us set

$$A = \{\textstyle\sum V_k M_k \text{ converges}\}, \qquad B = \{\textstyle\sum |V_k|^p < \infty\}$$

and

$$C_n = \{|V_n M_n| > 1\}, \quad n = 2, 3, \cdots.$$

Then we have that

$$A \subset \{C_n \text{ i.o.}\}^c = \{\textstyle\sum P(C_n | \mathscr{F}_{n-1}) < \infty\} \text{ a.s.}$$

where the equality is implied by a conditional form of a Borel-Cantelli lemma (cf. Breiman, 1968, page 96). Since

$$P(C_n | \mathscr{F}_{n-1}) = G(|V_n|^{-1}),$$

we get that $G(|V_n|^{-1}) \to 0$ a.s. on A as $n \to \infty$. Therefore, for sufficiently large $N = N(\omega)$,

$$\textstyle\sum_{n=N}^{\infty} |V_n|^p \le 2/c \textstyle\sum_{n=N}^{\infty} G(|V_n|^{-1}) < \infty$$

a.s. on A. This shows that $P(A \backslash B) = 0$.

Now we shall prove that $P(B \backslash A) = 0$. Let $\pi_n = \pi_n(\cdot, \omega)$ be a regular conditional distribution of $V_n M_n$ given $\mathscr{F}_{n-1}$. We have that

$$\pi_n(E, \omega) = P(V_n M_n \in E | M_1, \cdots, M_{n-1}) = \mu_p(V_n^{-1}(\omega)E)$$

where μ_p is the distribution of M_1. By a theorem of Hill (1982), $\sum V_n(\omega) M_n(\omega)$

converges for almost all ω's for which the series $\sum X_n^{(\omega)}$ converges in probability P', where $\{X_n^{(\omega)}(\omega')\}$ is a sequence of independent r.v.'s (defined on another probability space (Ω', P')) with distributions $\{\pi_n(\cdot, \omega)\}$. In our case the $X_n^{(\omega)}$'s have characteristic functions $\exp(-|V_n(\omega)|^p t^p)$. Thus $P(B\backslash A) = 0$ which completes the proof of the theorem. □

Theorem 2.1 enables us to translate the problem of a.s. convergence of quadratic forms into a more tractable problem of the a.s. convergence of series of independent random vectors with values in ℓ^p, whose standard basis is denoted by (e_k), $k = 1, 2, \cdots$.

THEOREM 2.2. *Let* $(f_{jk}: 1 \le j \le k - 1, k \ge 2)$ *be a triangular matrix of real numbers, and* $x_j =_{\text{def}} (0, \cdots, 0, f_{j,j+1}, f_{j,j+2}, \cdots) = \sum_{k=j+1}^{\infty} f_{jk} e_k$, $j = 1, 2, \cdots$. *Then*

$$\sum_{k=2}^{\infty} \left(\sum_{j=1}^{k-1} f_{jk} M_j\right) M_k$$

converges a.s. if and only if for each $j = 1, 2, \cdots$, $x_j \in \ell^p$ *and the vector random series* $\sum x_j M_j$ *converges a.s. in* ℓ^p.

PROOF. Setting $V_k = \sum_{j=1}^{k-1} f_{jk} M_j$, $k = 2, 3, \cdots$, $V_1 = 0$, by Theorem 2.1 we have that the series $\sum V_k M_k$ converges a.s. if and only if $\sum |V_k|^p < \infty$ a.s., which is equivalent to the a.s. convergence in ℓ^p of the series $\sum e_k V_k$.

Now assume that $S = \sum e_k V_k$ converges a.s. in ℓ^p. Since

$$\sum_{k=1}^{n+1} e_k V_k = \sum_{j=1}^{n} \left(\sum_{k=j+1}^{n+1} f_{jk} e_k\right) M_j,$$

Proposition 4.1 applied to $x_{jn} = \sum_{k=j+1}^{n+1} f_{jk} e_k$, and $Y_j = M_j$, gives that $x_{jn} \to x_j = \sum_{k=j+1}^{\infty} f_{jk} e_k \in \ell^p$ as $n \to \infty$, and that the series $\sum x_j M_j$ converges a.s. in ℓ^p.

Conversely, assume that $x_j = \sum_{k=j+1}^{\infty} f_{jk} e_k \in \ell^p$ and $\sum x_j M_j$ converges a.s. in ℓ^p. The operator

$$\ell^p \ni x \to R_n(x) = \sum_{k=n+2}^{\infty} \langle x, e_k \rangle e_k \in \ell^p, \quad p > 0,$$

is continuous and linear and $R_n(x) \to 0$ *in* ℓ^p for every x as $n \to \infty$. Thus a.s.

$$0 \leftarrow R_n\left(\sum_{j=1}^{\infty} x_j M_j\right) = \sum_{j=1}^{\infty} R_n(x_j) M_j = \sum_{j=1}^{n} \left(\sum_{k=n+2}^{\infty} f_{jk} e_k\right) M_j + \sum_{j=n+1}^{\infty} x_j M_j.$$

Hence,

$$\sum_{j=1}^{n} \left(\sum_{k=n+2}^{\infty} f_{jk} e_k\right) M_j \to 0$$

a.s in ℓ^p as $n \to \infty$, and thus

$$\sum_{k=1}^{n+1} V_k e_k = \sum_{j=1}^{n} x_j M_j - \sum_{j=1}^{n} \left(\sum_{k=n+2}^{\infty} f_{jk} e_k\right) M_j$$

converges a.s. in ℓ^p. □

The next step is to characterize the almost surely convergent p-stable series in ℓ^p, $0 < p < 2$. For $p \ge 1$, Theorem 2.3 can also be deduced from (and is essentially equivalent to) a result sketched in the appendix of Giné and Zinn (1983), and attributed by the authors to Gilles Pisier. However, our proof is more direct.

THEOREM 2.3. *Let $0 < p < 2$ and let $F = (f_{jk}\colon j, k \geq 1)$ be a matrix of real numbers such that for every $j \geq 1$, $x_j = \sum_{k=1}^{\infty} f_{jk} e_k \in \ell^p$. Then the series $\sum x_j M_j$ converges almost surely in ℓ^p if and only if $N_p(F) < \infty$ (cf. (1.3)).*

PROOF. Let

$$\| F \|_p^p = \sum_j \sum_k | f_{jk} |^p = \sum_j \| x_j \|_p^p.$$

Observe that if $N_p(F) < \infty$ then $\| F \|_p < \infty$ and, on the other hand, if $\sum x_j M_j$ converges a.s. then also $\| F \|_p < \infty$. The latter fact is an immediate corollary to the Borel-Cantelli lemma since $\infty > \sum_j P(\| x_j M_j \|_p > 1) = \sum_j P(| M_1 | > \| x_j \|_p^{-1})$ and the convergence of the last series is equivalent to the convergence of $\sum_j \| x_j \|_p^p$ in view of the tail behavior of stable distribution.

By the representation for stable processes obtained in Proposition 1.5 of Marcus and Pisier (1984) and used here in the discrete parameter case (in the case of $1 \leq p < 2$, Corollary 3 of LePage, Woodroofe and Zinn, 1981, would suffice, and it was Gilles Pisier's idea that we use it here), $\sum x_j M_j$ converges in ℓ^p a.s. if and only if $\sum \varepsilon_j Y_j \Gamma_j^{-1/p}$ converges a.s. in ℓ^p, where (ε_j) are independent Bernoulli r.v.'s, (Y_j) are i.i.d. random elements in ℓ^p with law

$$\mathscr{L}(Y_j) = \| F \|_p^{-p} \sum_k \| x_k \|_p^p \, \delta_{x_k / \| x_k \|_p}, \quad j = 1, 2, \cdots,$$

and $\Gamma_j = X_1 + \cdots + X_j$ where $X_1, X_2 \cdots$ are i.i.d. with $P(X_1 > u) = e^{-u}$. Moreover the sequences (ε_j), (Y_j), (Γ_j) are independent of each other. By the law of large numbers, $\Gamma_j / j \to 1$ a.s. Applying twice Fubini's theorem and the comparison principle in quasi-normed spaces obtained in Theorem 4.4 of Marcus and Woyczynski (1979), we get that the series $\sum x_j M_j$ converges in ℓ^p a.s. if and only if $\sum \varepsilon_j Y_j j^{-1/p}$ does. Since $\sup_j \| \varepsilon_j Y_j j^{-1/p} \|_p \leq 1$, by Theorem 4.2 of Marcus and Woyczynski (1979), the series $\sum \varepsilon_j Y_j j^{-1/p}$ converges a.s. in ℓ^p if and only if it converges in ℓ^p in the pth mean.

We show first that if $N_p(F) < \infty$ then

$$\lim_{n,m \to \infty} E \| \textstyle\sum_{j=m}^{n} j^{-1/p} \varepsilon_j Y_j \|_p^p = 0. \tag{2.1}$$

Let $Y_j = \sum_{k=1}^{\infty} Y_{jk} e_k$. Notice that for each k, $(Y_{jk}, j = 1, 2, \cdots)$ is a sequence of i.i.d random variables and

$$\begin{aligned} &E | Y_{1k} |^p \left(1 + \log^+ \frac{| Y_{1k} |^p}{E | Y_{1k} |^p} \right) \\ &\quad = \| F \|_p^{-p} \sum_{j=1}^{\infty} | f_{jk} |^p \left(1 + \log^+ \frac{| f_{jk} |^p \| F \|_p^p}{\sum_{\ell=1}^{\infty} | f_{j\ell} |^p \sum_{\ell=1}^{\infty} | f_{\ell k} |^p} \right) =_{\text{def}} a_k < \infty, \end{aligned} \tag{2.2}$$

so that, by Proposition 4.2, for each k, the series of real r.v.'s $\sum j^{-1/p} \varepsilon_j Y_{jk}$ converges a.s. and in the pth mean, i.e.,

$$\limsup_{n,m \to \infty} E | \textstyle\sum_{j=m}^{n} j^{-1/p} \varepsilon_j Y_{jk} |^p = 0. \tag{2.3}$$

Since $\sum_k a_k = N_p(F) \| F \|_p^p$, by our assumption, for each $\varepsilon > 0$ there exists an integer k_0 such that $\sum_{k=k_0}^{\infty} a_k < \varepsilon$. Now, by (2.3), symmetry of Y_{jk}'s, Proposition

4.2 and (2.2),

$$\begin{aligned}
&\limsup_{n,m\to\infty} E\,\| \textstyle\sum_{j=m}^{n} j^{-1/p}\varepsilon_j Y_j \|_p^p \\
&= \limsup_{n,m\to\infty} \textstyle\sum_{k=1}^{\infty} E\,| \sum_{j=m}^{n} j^{-1/p}\varepsilon_j Y_{jk}|^p \\
&= \limsup_{n,m\to\infty} \textstyle\sum_{k=k_0}^{\infty} E\,| \sum_{j=m}^{n} j^{-1/p}\varepsilon_j Y_{jk}|^p \\
&\le 2 \textstyle\sum_{k=k_0}^{\infty} E| \sum_{j=1}^{\infty} j^{-1/p}\varepsilon_j Y_{jk}|^p \le 2c \sum_{k=k_0}^{\infty} a_k < 2c\varepsilon
\end{aligned}$$

which proves (2.1).

Now, assume that the series $\sum j^{-1/p}\varepsilon_j Y_j$ converges in ℓ^p in the pth mean. Then, by (2.2) and Proposition 4.2,

$$\begin{aligned}
N_p(F) &= \|F\|_p^p \textstyle\sum_{k=1}^{\infty} E\,|Y_{1k}|^p\left(1 + \log^+ \dfrac{|Y_{1k}|^p}{E\,|Y_{1k}|^p}\right) \\
&\le c\, \|F\|_p^p \textstyle\sum_{k=1}^{\infty} E\,| \sum_{j=1}^{\infty} j^{-1/p}\varepsilon_j Y_{jk}|^p \\
&= c\, \|F\|_p^p\, E\, \| \textstyle\sum_{j=1}^{\infty} j^{-1/p}\varepsilon_j Y_j \|_p^p < \infty
\end{aligned}$$

which completes the proof of the theorem. □

COROLLARY 2.1. $\sum_j |\sum_k f_{jk} M_k|^p < \infty$ a.s. *if and only if* $N_p(F) < \infty$.

REMARK 2.1. The condition

$$\sum_k \sum_j |f_{jk}|^p (1 + |\log \textstyle\sum_l |f_{lk}|^p|) < \infty \tag{2.4}$$

is clearly slightly stronger than $N_p(F) < \infty$, but it may sometimes be easier to check, and it is very easy to establish. Indeed by applying Proposition 4.3 with $W_k = |\sum_j f_{jk} M_j|$ it follows immediately that (2.4) implies that $\sum x_j M_j$ converges a.s. in ℓ^p, when $x_j = \sum_k f_{jk} e_k \in \ell^p$.

3. Convergence of general quadratic forms in p-stable random variables.

Let

$$Q_n = \sum_{k=1}^{n} \sum_{j=1}^{n} f_{jk} M_j M_k, \quad n = 1, 2, \cdots,$$

where (M_j), $j = 1, 2, \cdots$, are as above, i.e., i.i.d. with $E \exp(itM_j) = \exp(-|t|^p)$, $0 < p < 2$. Denote the diagonal and off-diagonal parts of Q_n respectively by

$$D_n = \sum_{k=1}^{n} f_{kk} M_k^2 \quad \text{and} \quad R_n = \sum_{k,j=1;k\ne j}^{n} f_{jk} M_j M_k.$$

The diagonal part, being a series of independent random variables, is easy to handle.

THEOREM 3.1. *The sequence* (D_n) *converges a.s. as* $n \to \infty$ *if and only if*

$$\sum_{k=1}^{\infty} |f_{kk}|^{p/2} < \infty.$$

PROOF. Let us observe that $\sum P(|f_{kk}| M_k^2 > 1) < \infty$ if and only if $\sum |f_{kk}|^{p/2} < \infty$. Kolmogorov's three series theorem gives now the "only if" part of the claim.

The proof of the "if" part follows directly from Proposition 4.3 because for $p < 2$ and $f_{kk} \neq 0$,

$$E\,|f_{kk}|M_k^2 I(|f_{kk}|M_k^2 \le 1)$$
$$= \int_0^1 P(M_k^2 > t\,|f_{kk}^{-1}|)\,dt \le c\,|f_{kk}|^{p/2} \int_0^1 t^{-p/2}\,dt = \text{Const }|f_{kk}|^{p/2}. \ \square$$

The above result and the results of previous section give the following.

THEOREM 3.2. *The sequence of quadratic forms* (Q_n) *converges a.s. as* $n \to \infty$ *if and only if*

$$\textstyle\sum_{k=1}^\infty |f_{kk}|^{p/2} < \infty$$

and

$$\sum_{k=2}^\infty \sum_{j=1}^{k-1} |f_{jk}^s|^p\left(1 + \log^+ \frac{|f_{jk}^s|^p}{\sum_{l=j+1}^\infty |f_{jl}^s|^p \sum_{l=1}^{k-1} |f_{lk}^s|^p}\right) < \infty$$

where $f_{jk}^s = (f_{jk} + f_{kj})/2$.

PROOF. The above result is a straightforward corollary to Theorems 3.1, 2.2, and 2.3 (the second condition of the theorem is clearly equivalent to the condition $N_p(F^s) < \infty$ where $F^s = (f_{jk}^s; j < k)$ since if either condition is satisfied necessarily $\|F^s\|_p < \infty$) and to the fact that the sequence (Q_n) converges a.s. if and only if both sequences (D_n) and (R_n) converge a.s. We prove the only if part of this latter assertion.

Let a, b be reals with $a \neq 0$, and set $\varepsilon = \operatorname{sgn}(b/a)$. Then for $M = M_1$, we have

$$
\begin{aligned}
P\{|aM^2 + bM| > 1\} &= P\{|a(\varepsilon M)^2 + b(\varepsilon M)| > 1\}\\
&= P\{||a|M^2 + |b|M| > 1\}\\
&\ge P\{||a|M^2 + |b|M| > 1, M > 0\} \qquad (3.1)\\
&= P\{|a|M^2 + |b|M > 1, M > 0\}\\
&\ge P\{M > |a|^{-1/2}\} = \tfrac{1}{2} P\{|M| > |a|^{-1/2}\}.
\end{aligned}
$$

Assume that (Q_n) converges a.s. We have $Q_n = \sum_{k=1}^n (V_k + f_{kk}M_k)M_k$ where $V_k = \sum_{j=1}^{k-1} (f_{jk} + f_{kj})M_j$, $k \ge 2$, $V_1 = 0$. By the conditional Borel-Cantelli lemma (cf. Breiman, 1968, page 96), the a.s. convergence of (Q_n) implies that

$$\textstyle\sum_{k=1}^\infty P\{|V_kM_k + f_{kk}M_k^2| > 1 \mid M_1, \cdots, M_{k-1}\} < \infty \text{ a.s.}$$

It follows by (3.1) that $\sum P\{|f_{kk}M_k^2| > 1\} < \infty$, where the sum extends over all k for which $f_{kk} \neq 0$, which implies that $\sum_{k=1}^\infty |f_{kk}|^{p/2} < \infty$, in view of the tail behavior of the M_k's. Now, by Theorem 3.1, D_n converges a.s. and by the assumption $R_n = Q_n - D_n$ converges a.s. as well. $\square$

In view of Theorem 4.1 below and of Theorem 2.1 we have the following corollary on the a.s. convergence of the off-diagonal part of the quadratic form.

COROLLARY 3.1. *If* $0 < r < p$ *and*

$$\sum_{k=2}^{\infty} \left(\sum_{j=1}^{k-1} |f_{jk} + f_{kj}|^p\right)^{r(1\wedge 1/p)} < \infty$$

then the sequence $R_n = \sum_{k\neq j}^{n} f_{jk}M_jM_k$ *converges a.s.*

The following result on the convergence in L^r, $0 < r < p$, of the off-diagonal part of the quadratic form, which is equivalent to Corollary 3.1 when $1 \le p < 2$, and weaker when $0 < p < 1$, has a simple direct proof.

THEOREM 3.3. *If* $0 < r < p$ *and*

$$\sum_{k=2}^{\infty} \left(\sum_{j=1}^{k-1} |f_{jk} + f_{kj}|^p\right)^{r/p} < \infty$$

then the sequence $R_n = \sum_{k\neq j}^{n} f_{jk}M_jM_k$ *converges in* L^r *(and a.s. when* $1 \le r < p < 2$*).*

PROOF. We have, with $V_k = \sum_{j=1}^{k-1} (f_{jk} + f_{kj})M_j$, $k \ge 2$, $V_1 = 0$,

$$\begin{aligned} E|R_n - R_{m+1}|^r &= E\left|\sum_{k=m}^{n} V_kM_k\right|^r \le \text{Const} \sum_{k=m}^{n} E|V_kM_k|^r \\ &= \text{Const} \sum_{k=m}^{n} E|V_k|^r E|M_k|^r \\ &= \text{Const} \sum_{k=m}^{n} \left(\sum_{j=1}^{k-1} |f_{jk} + f_{kj}|^p\right)^{r/p} \end{aligned}$$

which proves the conclusion. When $1 \le r < p < 2$, R_n is a martingale and as such converges also a.s. □

The following characterization of functions of two variables which take countably many values on rectangles and for which the double stochastic integral with respect to p-stable motion M exists is obtained immediately from Theorem 3.2:

COROLLARY 3.2. *Let* $f(s, t) = \sum_{j<k} b_{jk} I_{A_j \times A_k}(s, t)$ *where* $A_j = [a_j, a_{j+1})$, $j = 1, 2, \cdots$, $0 = a_1 < a_2 < \cdots \le 1$. *Then the iterated integral*

$$\int_0^1 \int_0^t f(s, t)M(ds)M(dt) \tag{3.2}$$

exists if and only if

$$\int_0^1 \int_0^t |f(s,t)|^p \left(1 + \log^+ \frac{|f(s,t)|^p}{\int_s^1 |f(s,u)|^p\,du \int_0^t |f(u,t)|^p\,du}\right) ds\,dt < \infty. \tag{3.3}$$

PROOF. Put $g_{jk} = b_{jk}|A_j|^{1/p}|A_k|^{1/p}$. Then the integral (3.2) becomes the quadratic form $\sum_{j<k} g_{jk}M_jM_k$ where (M_j) are i.i.d. standard stable, and, from Theorem 3.2, one directly obtains (3.3). □

REMARK 3.1. The natural conjecture here, which is currently being established, is that (3.3) is a necessary and sufficient condition for the iterated integral (3.2) to exist for general f.

4. Auxiliary results. The proof of Theorem 2.2 relies on the following technical property which, roughly speaking, justifies the change of the order of summation in the series $\sum_j (\sum_{k>j} f_{jk} e_k) M_j$.

PROPOSITION 4.1. *Let E be a complete metric linear space, $x_{jn} \in E$, $n, j = 1, 2, \cdots$, and (Y_j) a sequence of nonzero independent symmetric real random variables. If*

$$S_n = \sum_{j=1}^{n} x_{jn} Y_j \to S$$

in probability as $n \to \infty$ then there exists a sequence $(x_j) \subset E$ such that for each j, $x_{jn} \to x_j$ as $n \to \infty$ and the series $\sum x_j Y_j$ converges a.s. to S.

PROOF. Let $\| \cdot \|$ be a monotonic F-norm on E, i.e., $\| ax \| \le \| x \|$ for $| a | \le 1$ (which always exists by Rolewicz, 1972, page 16, Theorem I.2.2). Fix $j \ge 1$ and let $a, b > 0$ be such that $P\{| Y_j | > a\} > b$. Let $\varepsilon > 0$. For $r \ge n \ge j$ we set $c(r, n) = 1$ if $\| a(x_{jr} - x_{jn}) \| > \varepsilon$ and $c(r, n) = 0$ otherwise. By the monotonicity of $\| \cdot \|$ we have

$$\begin{aligned} b\, c(r, n) &\le P\{\| Y_j(x_{jr} - x_{jn}) \| > \varepsilon,\ | Y_j | > a\} \\ &\le P\{\| Y_j(x_{jr} - x_{jn}) \| > \varepsilon\} \\ &\le 2P\{\| \tfrac{1}{2} \textstyle\sum_{k=1}^{n} Y_k(x_{kr} - x_{kn}) \| > \varepsilon/2\} \\ &\le 4P\{\| \tfrac{1}{4} (S_r - S_n) \| > \varepsilon/4\} \end{aligned}$$

where the last two inequalities are justified by the fact that for independent symmetric random vectors X, Y, $P\{\| X \| > \varepsilon\} \le 2P\{\| (X + Y)/2 \| > \varepsilon/2\}$. It then follows from the assumption that $c(r, n) \to 0$ as $r, n \to \infty$, i.e., $\{x_{jr}\}$ is a Cauchy sequence for every j, and by the completeness of E there exists $x_j \in E$ such that $x_{jn} \to x_j$ as $n \to \infty$.

Let now $\varepsilon > 0$ and let N be such that

$$P\{\| \tfrac{1}{2}(S_r - S_n) \| > \varepsilon/2\} \le \varepsilon/2$$

for every $r \ge n \ge N$. By the symmetry argument used above

$$P\{\| \textstyle\sum_{j=1}^{n} Y_j(x_{jr} - x_{jn}) \| \ge \varepsilon\} \le \varepsilon.$$

Keeping n fixed and letting $r \to \infty$ we get

$$P\{\| \textstyle\sum_{j=1}^{n} x_j Y_j - S_n \| \ge \varepsilon\} \le \varepsilon.$$

Thus $\sum_{j=1}^{n} x_j Y_j \to S$ in probability and, since the Y's are independent, also a.s. □

The next proposition was used in the crucial step in the proof of Theorem 2.3.

PROPOSITION 4.2. *Let $0 < p < 2$. If $X, X_1, X_2, \cdots$ are i.i.d. symmetric random variables, then there exists a constant $c = c(p)$ such that*

$$c^{-1} E | X |^p \left(1 + \log^+ \frac{| X |^p}{E | X |^p}\right) \le E \left| \textstyle\sum_{j=1}^{\infty} j^{-1/p} X_j \right|^p \le c E | X |^p \left(1 + \log^+ \frac{| X |^p}{E | X |^p}\right).$$

PROOF. To obtain the upper estimate we proceed as follows: Let $m_p = (E|X|^p)^{1/p}$.

$$E|\sum_j j^{-1/p}X_j|^p \le 2^p E|\sum_j j^{-1/p}X_j I(|X_j| \le m_p j^{1/p})|^p + 2^p E|\sum_j j^{-1/p}X_j I(|X_j| > m_p j^{1/p})|^p = 2^p(I_1 + I_2).$$

The first term, in view of orthogonality of truncations, is estimated as follows:

$$\begin{aligned} I_1 &\le (E|\sum_j j^{-1/p}X_j I(|X_j| \le m_p j^{1/p})|^2)^{p/2} \\ &= (\sum_j j^{-2/p} E|X|^2 I(|X| \le m_p j^{1/p}))^{p/2} \\ &= [\sum_j j^{-2/p} \sum_{i=1}^{j} E|X|^2 I(m_p(i-1)^{1/p} < |X| \le m_p i^{1/p})]^{p/2} \\ &= [\sum_{i=1}^{\infty} (\sum_{j=i}^{\infty} j^{-2/p}) E|X|^2 I(m_p(i-1)^{1/p} < |X| \le m_p i^{1/p})]^{p/2} \\ &\le \text{Const } m_p^p [\sum_{i=1}^{\infty} iP(i-1 < |X/m_p|^p \le i)]^{p/2} \\ &\le \text{Const } m_p^p [E|X/m_p|^p + 1]^{p/2} \le \text{Const } E|X|^p. \end{aligned}$$

For the second term we obtain by Khinchine's inequality,

$$\begin{aligned} I_2 &\le \text{Const } \sum_j j^{-1} E|X|^p I(|X| > m_p j^{1/p}) \\ &= \text{Const } \sum_{j=1}^{\infty} j^{-1} \sum_{i=j}^{\infty} E|X|^p I(m_p i^{1/p} < |X| \le m_p(i+1)^{1/p}) \\ &\le \text{Const } m_p^p \sum_{i=1}^{\infty} (\sum_{j=1}^{i} j^{-1})(i+1)P(i^{1/p} < |X/m_p| \le (i+1)^{1/p}) \\ &\le \text{Const } m_p^p \sum_{i=1}^{\infty} (1 + \log i)(i+1)P(i < |X/m_p|^p \le i+1) \\ &\le \text{Const } m_p^p E|X/m_p|^p(1 + \log^+|X/m_p|^p) \\ &= \text{Const } E|X|^p\left(1 + \log^+ \frac{|X|^p}{E|X|^p}\right). \end{aligned}$$

Now the upper estimate follows.

To obtain the lower estimate notice that by Khinchine's inequality

$$E|\sum j^{-1/p}X_j|^p \ge \text{Const } E(\sum j^{-2/p}X_j^2)^{p/2} \ge \text{Const } E \sup_j j^{-1}|X_j|^p.$$

Since for any positive a_i, $1 - \prod(1 - a_i) \ge 1 - \exp(-\sum a_i) \ge \sum a_i/(1 + \sum a_i)$, we have for independent Z_j's and $\delta = \inf\{t > 0: \sum_j P(|Z_j| > t) \le 1\}$ that

$$\begin{aligned} E \sup|Z_j| &= \left(\int_0^{\delta} + \int_{\delta}^{\infty}\right)(1 - \prod_{j=1}^{\infty}(1 - P(|Z_j| > t)))\,dt \\ &\ge 2^{-1}\left[\delta + \sum_{j=1}^{\infty} \int_{\delta}^{\infty} P(|Z_j| > t)\,dt\right]. \end{aligned}$$

This is part of Lemma 3.2 of Giné and Zinn (1983) with their proof. Since $1 \ge \sum_{j=1}^{\infty} P(|Z_j| > \delta)$,

$$E \sup|Z_j| \ge 2^{-1}\max[\delta, \sum E|Z_j|\, I(|Z_j| > \delta)].$$

Now let $Z_j = j^{-1}|X_j|^p$ so that $E|X|^p/2 \le \delta \le E|X|^p$. By the above estimate

$$E\ \sup_j j^{-1}|X_j|^p \ge 2^{-1}\max\{2^{-1}E|X|^p, \textstyle\sum_j j^{-1}E|X|^p I(|X|^p > jE|X|^p)\}$$
$$\ge 2^{-1}\max\left\{2^{-1}E|X|^p, E|X|^p \int_1^{\max\{|X|^p/E|X|^p,\ 1\}} u^{-1}\,du\right\}$$
$$\ge 8^{-1}E|X|^p\left(1 + \log^+ \frac{|X|^p}{E|X|^p}\right)$$

which gives the desired lower estimate. □

The following elementary proposition (cf. Szulga and Woyczynski, 1983) is used in Remark 2.1.

PROPOSITION 4.3. *If for a sequence* (W_k) *of random variables* $\sum_k P(|W_k| > 1) < \infty$, *and* $\sum_k E|W_k I(|W_k| \le 1)| < \infty$, *then* $\sum |W_k| < \infty$ *a.s.*

PROOF. Indeed, let $Y_k = W_k I(|W_k| \le 1)$, $Z_k = W_k - Y_k$. Then $|W_k| = |Y_k| + |Z_k|$. $\sum |Y_k|$ converges a.s. since $E(\Sigma |Y_k|) < \infty$ by the second assumption, and $\sum |Z_n|$ converges a.s. by the Borel-Cantelli lemma. □

We now establish a more precise criterion for summability of stable r.v.'s which is used in Section 3. For a symmetric p-stable r.v. X the quantity c_X is defined by $E\exp(itX) = \exp(-c_X|t|^p)$ and satisfies

$$\|X\|_{L^r} = (E|X|^r)^{1/r} = \text{Const}(r, p)c_X^{1/p}, \quad 0 < r < p,$$

and for independent symmetric p-stable r.v.'s (X_k),

$$c_{\sum_k a_k X_k} = \textstyle\sum_k |a_k|^p c_{X_k}. \tag{4.1}$$

DEFINITION 4.1. *The r.v.'s* (X_k) *are jointly symmetric p-stable if for every sequence* (a_k) *with a finite number of nonzero elements the r.v.* $\sum_k a_k X_k$ *is symmetric p-stable.*

THEOREM 4.1. *Let* (X_k) *be jointly symmetric p-stable r.v.'s with* $0 < p < 2$ *and let* $r > 0$. *Then a necessary and sufficient condition for*

$$\textstyle\sum_{k=1}^\infty |X_k|^r < \infty \text{ a.s.}$$

is that for some $0 < s < p$,

$$E\left(\textstyle\sum_{k=1}^\infty |X_k|^r\right)^{s(1\wedge 1/r)} < \infty.$$

PROOF. Assume $X = (X_k) \in \ell^r$ a.s. and define $\Phi: \Omega \to \ell^r$ by $\Phi(\omega) = (X_k(\omega)) = X(\omega)$ if $(X_k(\omega)) \in \ell^r$ and $\Phi(\omega) = 0$ otherwise. Then Φ induces a symmetric p-stable measure $\mu = P \circ \Phi^{-1}$ on ℓ^r. For $x = (x_k) \in \ell^r$ define $q(x) = \sum_k |x_k|^r$ when $0 < r \le 1$ and $q(x) = (\sum_k |x_k|^r)^{1/r}$ when $r > 1$. Then q is a measurable seminorm on ℓ^r (a norm when $r \geq 1$), and by Theorem 3.2 in de Acosta (1975) we have for $0 < s < p$,

$$E\{q(X(\omega))\}^s = \int_\Omega q^s(X(\omega))\, dP(\omega) = \int_{\ell^r} q^s(x)\, d\mu(x) < \infty.$$

The converse is clear. □

When $1 < r < p < 2$ we can take $s = r$ and the necessary and sufficient condition becomes

$$\sum_{k=1}^\infty E\,|X_k|^r < \infty \quad \text{or} \quad \sum_{k=1}^\infty c_{X_k}^{r/p} < \infty.$$

When $r = p$, Theorem 4.1 gives

$$\sum_{k=1}^\infty |X_k|^p < \infty \quad \text{a.s.} \quad \text{if and only if}$$
$$E(\textstyle\sum_{k=1}^\infty |X_k|^p)^{s(1\wedge 1/p)} < \infty \quad \text{for some} \quad 0 < s < p. \tag{4.2}$$

In the case when X_k's are independent, by Schwartz' theorem (cf., e.g., Woyczynski (1978), page 277) this necessary and sufficient condition simplifies to the condition

$$\sum_{k=1}^\infty c_{X_k}(1 + |\log c_{X_k}|) < \infty. \tag{4.3}$$

When X_k's are of the form $X_k = \sum_j f_{jk} M_j$, where M_j's are independent, (4.3) is replaced by the condition $N_p(F) < \infty$ (Corollary 2.1), and in this context Corollary 2.1 can be seen as an extension of Schwartz' theorem.

Since every sequence of jointly symmetric p-stable r.v.'s (X_k) is of the form $X_k = \int_0^1 f_k(t)\, dM(t)$, $k = 1, 2, \cdots$, where $M(t)$, $0 \leq t \leq 1$, is a stable motion (i.e., has independent stationary symmetric p-stable increments) and $\int_0^1 |f_k(t)|^p\, dt < \infty$, $k = 1, 2, \cdots$, (cf. Kuelbs, 1973), the methods of the present paper permit to show that a necessary and sufficient condition for (4.2) is

$$\sum_k \int_0^1 |f_k(t)|^p \left(1 + \log^+ \frac{|f_k(t)|^p}{\int_0^1 |f_k(u)|^p\, du \sum_l |f_l(t)|^p}\right) dt < \infty.$$

This is an "integral analog" of Corollary 2.1.

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