The Annals of Probability
1985, Vol. 13, No. 4, 1358–1363

DOMINATION OF LAST EXIT DISTRIBUTIONS

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We first show by an example that, unlike the domination of hitting distributions, the process with smaller last exit distributions may not be obtainable from the one with bigger last exit distributions by killing and time-change. Then we prove that, under a fairly general condition, the last exit distributions of a transient Hunt process cannot nontrivially dominate those of another transient Hunt process.

Section 1. In the recent development of Markov process theory, last exit times and last exit distributions have become increasingly important. Since the last exit times of a Markov process can be viewed as the hitting times of the reversed process, there is a strong "duality" between "hitting" and "last exit." For example, it is well known that if two processes have the same hitting distributions, then they are essentially the same process up to a time-change, i.e., any one of them can be obtained from the other by a time change (see [1] Chap. 5). The dual version of this result is also true, i.e., if two transient Hunt processes have the same last exit distributions, then they are essentially the same process up to a time change (see Glover [2]). Recall that, given two Markov processes X and Y, we say that Y can be obtained from X by a time-change if there is a continuous additive functional A_t of X, i.e., a continuous strictly increasing process $A_t \geq 0$ adapted to the σ fields generated by X and satisfying

$$\forall\, t < \zeta \quad \text{and} \quad s > 0, \qquad A_{t+s} = A_t + A_s \circ \theta_t,$$

where ζ is the lifetime of X, such that Y_t and $X(\tau_t)$ are identical in law. Here τ_t is the inverse of A_t, i.e.,

$$\forall\, t > 0, \qquad \tau_t = \inf\{s > 0;\, A_s > t\}.$$

However, "last exit" is essentially different from "hitting": it is not always true that a result proved for hitting distributions holds for last exit distributions as well. This is because, in general, we cannot reverse a process that is conditioned to start at a certain point to obtain a process with initial distribution also concentrated on the same point.

Recall a result of Shih. Let X and Y be two Hunt processes on the same state space E and with semigroups P_t and Q_t, respectively. We will use D, with or without subscript, for any relatively compact open set. Shih studied the following question. Suppose the hitting distributions of X dominate those of Y, i.e., if

$$\forall\, x \in E \text{ and } D, \qquad P_D(x, \cdot) \geq Q_D(x, \cdot), \tag{1}$$

Received September 1984; revised November 1984.

AMS 1980 *subject classifications.* Primary 60J40; Secondary 60J45.

Key words and phrases. Markov processes, hitting times, hitting distributions, last exit times, last exit distributions.

then what is the relation between X and Y? He proved in [3] that (1) implies that Y can be obtained from X by first a killing, then a time-change.

In this paper we study a similar problem involving last exit distributions. We first introduce some notation.

For a Borel subset B of E, we use γ_B to denote the last exit time from B for either X or Y according to the context, i.e.,

$$\gamma_B = \sup\{t > 0;\ X_t \in B\} \tag{2}$$

or with X replaced by Y. Here we use the convention that the sup of an empty set is 0. Let L_B^X be the last exit distribution of B for X, i.e.,

$$\begin{gathered}\forall\, x \in E \quad \text{and Borel set} \quad C \subset E,\\ L_B^X(x, C) = L_B^X 1_C(x) = P^x\{X(\gamma_B -) \in C;\ \gamma > 0\}.\end{gathered} \tag{3}$$

Similarly define the last exit distribution L_B^Y of B for Y. Observe that we have used the left limits X_{t-} and Y_{t-} to define the last exit distributions.

From now on, we assume that both X and Y are transient Hunt processes. By "transient," we mean:

$$\forall\, D, \qquad \gamma_D < \infty \quad \text{almost surely.} \tag{4}$$

REMARK. L_D^X and L_D^Y, as measures, are concentrated on $\overline{D}$ whereas for a compact set K, L_K^X and L_K^Y may not be concentrated on K. So it is easier to work with the last exit distributions of open sets. By [2], we see that if $\forall\, D$, $L_D^X = L_D^Y$, then X and Y are essentially the same process up to a time-change.

Section 2. The following example shows that, in general, the process with smaller last exit distributions is not obtainable from the one with bigger last exit distributions by killing and time-change.

EXAMPLE. Let $E = (-\infty, \infty)$ and let Y_t be the uniform motion to the right (with unit speed), B_t be the one-dimensional Brownian motion, and $X_t = B_t + Y_t$, i.e., Brownian motion plus drift. Let D be a bounded open subset of $(-\infty, \infty)$ and let $b = \sup\{y \in D\}$. It is clear that if $x < b$, then

$$L_D^X(x, \cdot) = L_D^Y(x, \cdot) = \delta_b,$$

the unit mass at b. If $x \geq b$, then $L_D^Y(x, \cdot) = 0$ and $L_D^X(x, \cdot) = c\delta_b$ for some constant $c > 0$. Hence L_D^X nontrivially dominates L_D^Y. But Y cannot be obtained from X by killing and time-change. This is because first Y obviously is not a time-change of X, and if Z is a process obtained by killing X, then Z must have killing inside E, i.e.,

$$\exists\, x \in E, \qquad P^x\{Z(\zeta -) \in E\} > 0. \tag{5}$$

But Y has no killing inside E.

However, Y can be obtained from X by a "generalized" time-change. Let

$$A_t = \max\{s > 0;\ X_s = t\} \quad \text{and} \quad \tau_t = \inf\{s > 0;\ X_s > t\}.$$

Then X_{τ_t} is a uniform motion. A_t defined above is a continuous increasing process. Though it is not an additive functional of X, it satisfies:

$$\text{(6)} \qquad \begin{aligned} &\text{If} \quad t \mapsto A_t \quad \text{is strictly increasing at} \quad t = u, \quad \text{then} \\ &\forall\, s > 0, \qquad A_{u+s} = A_u + A_s \circ \theta_u. \end{aligned}$$

In general, given a Hunt process X, if A_t is a continuous increasing process adapted to the σ fields generated by X and satisfying (6), then X_{τ_t} is a standard process, which can be viewed as obtained from X by a generalized time-change. This can be proved by adapting the arguments in [1, Chap. 5].

Section 3. Observe that if we kill a Hunt process, we obtain a process with smaller hitting distributions, but this is not true for last exit distributions. In general, the last exit distributions of a killed process are not comparable with the last exit distributions of the original process. In fact, we will prove, under a fairly general condition, that the last exit distributions of a transient Hunt process cannot nontrivially dominate those of another transient Hunt process.

As usual, we use T_B to denote the hitting time of B for either X or Y and for $x \in E$, let $T_x = T_{\{x\}}$. Consider the following condition, which is weaker than requiring points to be polar:

$$\text{(7)} \qquad \forall\, x \in E, \qquad P^x\{T_x < \infty\} = 0.$$

Proposition 1. *Assume* (7). *If* $\forall\, D$, $L_D^X \geq L_D^Y$, *then* $\forall\, D$, $L_D^X = L_D^Y$, *i.e.*, X *and* Y *are essentially the same process up to a time-change.*

Proof. Since the total mass of $L_D^X(x, \cdot)$ is $P_D 1(x)$ and $\forall\, x \in D$, $P_D 1(x) = 1 = Q_D 1(x)$, we have

$$\text{(8)} \qquad \forall\, x \in D, \qquad L_D^X(x, \cdot) = L_D^Y(x, \cdot).$$

For $x \notin \overline{D}$, let B_a be the open ball around x with radius $a > 0$ so that its closure $\overline{B}_a$ is disjoint from $\overline{D}$. Let $D_a = D \cup B_a$ and let γ and γ_a be the last exit times from D and D_a, respectively. Since $x \in D_a$, $L_{D_a}^X(x, \cdot) = L_{D_a}^Y(x, \cdot)$. Observe $\gamma_a > 0$ P^x-a.e. and Q^x-a.e. We have

$$P^x\{X(\gamma_a -) \in dz\} = Q^x\{Y(\gamma_a -) \in dz\}.$$

Since $\gamma_a = \gamma$ on $\{X(\gamma_a -) \in \overline{D}\}$,

$$\text{(9)} \qquad P^x\{\gamma > 0\} = P^x\{X(\gamma_a -) \in \overline{D}\} + P^x\{X(\gamma_a -) \in \overline{B}_a, \gamma > 0\}.$$

By transience and quasi-left continuity of X, we have

$$\text{(10)} \qquad \lim_{a \to 0} P^x\{X(\gamma_a -) \in \overline{B}_a, \gamma > 0\} \leq P^x\{T_x < \infty\} = 0.$$

It follows from (9) and (10),

$$\begin{aligned} L_D^X(x, E) = P^x\{\gamma > 0\} &= \lim_{a \to 0} P^x\{X(\gamma_a -) \in \overline{D}\} \\ &= \lim_{a \to 0} L_{D_a}^X(x, \overline{D}) = \lim_{a \to 0} L_{D_a}^Y(x, \overline{D}) \\ &= L_D^Y(x, E). \end{aligned}$$

Now for $x \in \partial D$, choose $D_n \uparrow D$ such that $x \notin \overline{D}_n$. Since $P_D 1(x) = \lim_n P_{D_n} 1(x)$ and $Q_D 1(x) = \lim_n Q_{D_n} 1(x)$,

$$\forall\, x \in E, \qquad L_D^X(x, E) = L_D^Y(x, E).$$

Hence we must have $L_D^X(x, \cdot) = L_D^Y(x, \cdot)$ because $L_D^X(x, \cdot) \geq L_D^Y(x, \cdot)$. □

REMARK. It might be interesting to notice that, without assuming (7), (9) yields

$$(11) \qquad P^x\{\gamma_D > 0\} = P^x\{\gamma_x < \gamma_D > 0\} + P^x\{\gamma_x = \gamma_{D \cup \{x\}} > 0\},$$

where $\gamma_x = \gamma_{\{x\}}$.

Section 4. Now we consider a modified version of last exit distributions. Define M_D^X by

$$(12) \qquad \begin{gathered} \forall x \in E \quad \text{and Borel set} \quad C \subset E, \\ M_D^X(x, C) = M_D^X 1_C(x) = P^x\{X(\gamma_D -) \in C;\, 0 < \gamma_D < \zeta\}. \end{gathered}$$

Similarly we define M_D^Y with X replaced by Y.

M_D^X is the part of last exit distribution that is not obtained from the killing of X. Since $P_D 1(x) = Q_D 1(x)$ for any $x \in D$, this forces $L_D^X(x, \cdot) = L_D^Y(x, \cdot)$ for any $x \in D$, which is the key point in the proof of Proposition 1. But this does not hold for M_D^X and M_D^Y if X and Y have killing inside E. However, the conclusion of Proposition 1 holds also for the modified last exit distributions M_D^X and M_D^Y.

PROPOSITION 2. *Assume* (7). *If* $\forall\, D$, $M_D^X \geq M_D^Y$, *then* $\forall\, D$, $L_D^X = L_D^Y$; *hence* $M_D^X = M_D^Y$.

PROOF. By the proof of Proposition 1, it is enough to show (8). Fix $x \in E$. Let

$$(13) \qquad \mu(dz) = P^x\{X(\zeta -) \in dz\} \quad \text{and} \quad \nu(dz) = Q^x\{Y(\zeta -) \in dz\}.$$

STEP 1. $\nu \geq \mu$. If not, for some compact K, $\mu(K) > \nu(K)$. Choose $D_n \downarrow K \cup \{x\}$ with $\mu(\partial D_n) = \nu(\partial D_n) = 0$. By (7), $\mu(\{x\}) = \nu(\{x\}) = 0$, so for sufficiently large n, $\mu(D_n) > \nu(D_n)$. Let $D' = D_n$ for such an n. We have, with $\gamma = \gamma_{D'}$,

$$P^x\{\gamma = \zeta\} = P^x\{X(\zeta -) \in D'\} > Q^x\{Y(\zeta -) \in D'\} = Q^x\{\gamma = \zeta\}.$$

On the other hand, $x \in D'$,

$$P^x\{\gamma > 0\} = 1 = Q^x\{\gamma > 0\};$$

hence

$$P^x\{0 < \gamma < \zeta\} < Q^x\{0 < \gamma < \zeta\}.$$

This contradicts $M_{D'}^X 1 \geq M_{D'}^Y 1$.

STEP 2. $\nu = \mu$. If not, for some compact K, $\nu(K) > \mu(K)$. Let

$$(14) \qquad \delta = \nu(K) - \mu(K) > 0.$$

Choose $D \supset K$ so that

$$(15) \qquad \mu(\overline{D}) - \mu(K) \le \frac{\delta}{6} \quad \text{and} \quad \nu(\overline{D}) - \nu(K) \le \frac{\delta}{6}.$$

By (7) and the quasi-left continuity of X, $\mu(\{x\}) = 0$. We may assume $x \notin K$, hence $x \notin \overline{D}$, by choosing D properly. Since

$$P^x\{X(\zeta -) \in \overline{D} - K\} \le \frac{\delta}{6},$$

by choosing first $D_1 \supset \underline{D}$ sufficiently large, then $D_2 \supset \overline{D}_1 \cup \{x\}$ sufficiently large, and putting $G = D_2 - \overline{D}$, we will have

$$(16) \qquad |P^x\{X(\gamma_G -) \in D_1, 0 < \gamma_G < \zeta\} - P^x\{X(\zeta -) \in K\}| \le \frac{2\delta}{6}.$$

Similarly,

$$(17) \qquad |Q^x\{Y(\gamma_G -) \in D_1, 0 < \gamma_G < \zeta\} - Q^x\{Y(\zeta -) \in K\}| \le \frac{2\delta}{6}.$$

Combining (14), (16), and (17), we obtain,

$$M_G^Y(x, D_1) - M_G^X(x, D_1) \ge \frac{2\delta}{6}.$$

This contradicts the assumption that $M_G^Y \le M_G^X$.

STEP 3. Now we show that for $x \in D$, $L_D^X f(x) = L_D^Y f(x)$ for any bounded continuous function f. This will imply (8). First assume

$$P^x\{X(\zeta -) \in \partial D\} = 0 = Q^x\{Y(\zeta -) \in \partial D\}.$$

Then

$$\begin{aligned} L_D^X f(x) &= M_D^X f(x) + P^x\{f(X(\zeta -)); \zeta = \gamma_D\} \\ &= M_D^X f(x) + \int_D f(z)\mu(dz), \end{aligned}$$

$$L_D^Y f(x) = M_D^Y f(x) + \int_D f(z)\nu(dz).$$

Since $M_D^X \ge M_D^Y$ and $\mu = \nu$, $L_D^X f(x) \ge L_D^Y f(x)$. But $L_D^X 1(x) = 1 = L_D^Y 1(x)$, so $L_D^X f(x) = L_D^Y f(x)$. Now for arbitrary D, choose $D_n \uparrow D$ such that

$$\forall\, n, \qquad P^x\{X(\zeta -) \in \partial D_n\} = 0 \quad \text{and} \quad Q^x\{Y(\zeta -) \in \partial D_n\} = 0.$$

Then $\gamma_{D_n} \uparrow \gamma_D$ and

$$\begin{aligned} L_D^X f(x) &= P^x\{f(X(\gamma_D -))\} = \lim_n P^x\{f(X(\gamma_{D_n} -))\} \\ &= \lim_n L_{D_n}^X f(x) = \lim_n L_{D_n}^Y f(x) \\ &= L_D^Y f(x). \end{aligned}$$

□

Acknowledgment. I wish to express my gratitude to J. Glover who suggested this problem and the example in Section 1.

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