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# AN IMPROVED SUBADDITIVE ERGODIC THEOREM[1]

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A new version of Kingman's subadditive ergodic theorem is presented, in which the subadditivity and stationarity assumptions are relaxed without weakening the conclusions. This result applies to a number of situations that were not covered by Kingman's original theorem. The proof involves a rather simple reduction to the additive case, where Birkhoff's ergodic theorem can be applied.

**1. The result.** Subadditive ergodic theory is one of the major achievements in probability theory of the past twenty years. The development of this theory began with Hammersley and Welsh (1965) and was most fully realized by Kingman (1968). An extensive discussion of the theory, which includes a treatment of many examples and applications, has been given by Kingman (1973, 1976) and by Hammersley (1974). Other examples occur in Smythe and Wierman (1978) and Steele (1978). A generalization to multiparameter processes was carried out by Smythe (1976).

In order to put our discussion in context, we will begin by stating Kingman's theorem. Suppose $\{X_{m,n}\}$ is a collection of random variables indexed by integers satisfying $0 \le m < n$. Kingman's assumptions are:

(1.1) $$X_{l,n} \le X_{l,m} + X_{m,n} \quad \text{whenever } 0 \le l < m < n.$$

(1.2) The joint distributions of $\{X_{m+1,n+1}, 0 \le m < n\}$ are the same as those of $\{X_{m,n}, 0 \le m < n\}$.

(1.3) For each $n$, $E|X_{0,n}| < \infty$ and $EX_{0,n} \ge -cn$ for some constant $c$.

His conclusions are:

(1.4) $$\gamma = \lim_{n\to\infty} \frac{1}{n} EX_{0,n} = \inf_n \frac{1}{n} EX_{0,n}.$$

(1.5) $$X = \lim_{n\to\infty} \frac{X_{0,n}}{n} \text{ exists a.s. and in } L_1.$$

(1.6) $$EX = \gamma.$$

The original proof of this result was given by Kingman (1968, 1976). Other proofs have been given by Burkholder (1973), Derriennic (1975), Del Junco (1977), Smeltzer (1977), Steele (1980), and Katznelson and Weiss (1982). A generalization of this theorem in which a small error term is allowed on the right side of (1.1) was obtained by Derriennic (1983).

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As was pointed out above, there are many examples that come from a variety of areas which satisfy Kingman's assumptions (1.1), (1.2), and (1.3). However, many situations that appear at first glance to be subadditive turn out on closer examination not to satisfy (1.1) and/or (1.2). Examples of this can be found in the contexts of age-dependent branching processes (see Section 2 and Note 9 of Hammersley, 1974, and Section 3.3 of Kingman, 1976), first passage percolation (see Definition 4.3 and Theorem 5.7 of Smythe and Wierman, 1978), and contact processes (see Theorem 2.1 and the remark following its proof in Durrett, 1980). In order to treat such examples, the theory of superconvolutive distributions was developed by Kesten (1973) and Hammersley (1974). This theory is described in Chapter 3 of Kingman (1976). There are several drawbacks to the superconvolutive theory. One is that in order for it to be applicable, the process must have something like independent increments. Another is that one can only conclude from it convergence in probability in (1.5) or almost sure convergence along a sequence of times which grows exponentially rapidly. In order to obtain almost sure convergence along the full sequence, it is then necessary to have a separate argument, which often uses some type of monotonicity.

The foregoing comments are intended to explain why a weakening of the assumptions in Kingman's theorem is desirable. By looking at some of the examples mentioned above, one finds that often (1.1) is satisfied for $l = 0$ even if it fails for $l \geq 1$. Thus our replacement for (1.1) is

$$X_{0,n} \leq X_{0,m} + X_{m,n} \quad \text{whenever } 0 < m < n. \tag{1.7}$$

Of course (1.2) and (1.7) together imply (1.1), so now we must weaken (1.2). Our replacement for (1.2) has two parts:

(1.8) The joint distributions of $\{X_{m+1,m+k+1}, k \geq 1\}$ are the same as those of $\{X_{m,m+k}, k \geq 1\}$ for each $m \geq 0$,

and

(1.9) For each $k \geq 1$, $\{X_{nk,(n+1)k}, n \geq 1\}$ is a stationary process.

THEOREM 1.10. *Suppose that* (1.7), (1.8), (1.9), *and* (1.3) *are satisfied. Then* (1.4), (1.5), *and* (1.6) *are true as well. If the stationary processes in* (1.9) *are ergodic, then* $X = \gamma$ *a.s.*

This theorem provides an alternate approach to many of the situations which have been treated using the superconvolutive theory. The weaknesses in that theory which were mentioned earlier are not present in Theorem 1.10. As in Kingman's case, one could replace (1.3) by the assumption that $EX_{0,1}^{+} < \infty$, provided that one allows $\gamma = -\infty$ in (1.4) and does not claim $L_1$ convergence in (1.5). This extension of Theorem 1.10 is obtained by a simple truncation argument as in the proof of Theorem 1.8 of Kingman (1976), and will be omitted.

The proof of Theorem 1.10 will be presented in the next section. An example that satisfies (1.7), (1.8), and (1.9), but for which (1.1) and (1.2) fail is described in Section 3.

**2. The proof.** Let $X_n = X_{0,n}$, $\gamma_n = EX_n$,

$$\overline{X} = \limsup_{n\to\infty} \frac{1}{n} X_n,$$

and

$$\underline{X} = \liminf_{n\to\infty} \frac{1}{n} X_n.$$

The proof will be broken up into four steps:

$$\gamma = \lim_{n\to\infty} \frac{1}{n}\gamma_n = \inf_{n\geq 1} \frac{1}{n}\gamma_n \in (-\infty, \infty). \tag{2.1}$$

(2.2) $E\overline{X} \leq \gamma$, and if the stationary processes in (1.9) are ergodic, then $\overline{X} \leq \gamma$ a.s.

$$E\underline{X} \geq \gamma. \tag{2.3}$$

$$\lim_{n\to\infty} E\left|\frac{1}{n}X_n - X\right| = 0, \tag{2.4}$$

where $X$ is the common value of $\underline{X}$ and $\overline{X}$.

The proofs of (2.1), (2.2), and (2.4) are elementary, and are the same as in Kingman's case. In fact, Kingman (1976) observed on page 178 that (1.9) was sufficient in order to prove (2.2). These proofs are included here for completeness. The proof of (2.3) (this is the "difficult half" according to Kingman) is new, but is based on the proof given by Durrett (1980) of the strong law for the edge of a one-dimensional contact process.

PROOF OF (2.1). By (1.7) and (1.8),

$$\gamma_{m+n} \leq \gamma_m + \gamma_n. \tag{2.4a}$$

Define $\gamma$ by

$$\gamma = \inf_{n\geq 1} \frac{1}{n}\gamma_n,$$

which is finite by (1.3). Fix an $m \geq 1$ and write $n = km + l$, where $0 \leq l < m$. By (2.4a),

$$\gamma_n \leq k\gamma_m + \gamma_l.$$

As $n \to \infty$, $n/k \to m$, so that

$$\limsup_{n\to\infty} \frac{1}{n}\gamma_n \leq \frac{1}{m}\gamma_m.$$

Since $m$ is arbitrary, we conclude that

$$\gamma \leq \liminf_{n\to\infty} \frac{1}{n}\gamma_n \leq \limsup_{n\to\infty} \frac{1}{n}\gamma_n \leq \gamma.$$

PROOF OF (2.2). Fix $k \geq 1$ and use (1.7) repeatedly to write

$$X_{kn} \leq \sum_{j=1}^{n} X_{k(j-1),kj}. \tag{2.5}$$

By Birkhoff's ergodic theorem and (1.9),

$$\frac{1}{n} \sum_{j=1}^{n} X_{k(j-1),kj} \tag{2.6}$$

converges a.s. and in $L_1$ to a random variable with mean $\gamma_k$. Therefore

$$E \limsup_{n \to \infty} \frac{X_{kn}}{kn} \leq \frac{1}{k} \gamma_k. \tag{2.7}$$

Using (1.7) again,

$$X_{kn+j} \leq X_{kn} + X_{kn,kn+j}. \tag{2.8}$$

By (1.8) the distribution of $X_{kn,kn+j}$ depends only on $j$. This distribution has a finite first moment by (1.3). Therefore by the Borel–Cantelli lemma,

$$\lim_{n \to \infty} \frac{X_{kn,kn+j}}{n} = 0 \quad \text{a.s.}$$

for each $j$. Hence by (2.7) and (2.8),

$$E\overline{X} \leq \frac{1}{k} \gamma_k.$$

Letting $k \to \infty$, we obtain $E\overline{X} \leq \gamma$. If the stationary processes in (1.9) are ergodic, then the a.s. limit of (2.6) is $\gamma_k$, so that $\overline{X} \leq \gamma$ a.s.

PROOF OF (2.3). Let $U_n$ be a random variable which is independent of all the $X_{l,m}$, and which is uniformly distributed on $\{1, 2, \dots, n\}$, and let

$$Y_k^n = X_{k+U_n} - X_{k+U_n-1}.$$

Then

$$\begin{aligned} EY_k^n &= \frac{1}{n} \sum_{l=1}^{n} E[X_{k+l} - X_{k+l-1}] \\ &= \frac{1}{n} E[X_{k+n} - X_k] = \frac{1}{n}(\gamma_{k+n} - \gamma_k), \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} E(Y_k^n)^+ &= \frac{1}{n} \sum_{l=1}^{n} E[X_{k+l} - X_{k+l-1}]^+ \leq \frac{1}{n} \sum_{l=1}^{n} EX_{k+l-1,k+l}^+ \\ &\leq EX_1^+ \end{aligned} \tag{2.10}$$

by (1.7) and (1.8). Therefore by (2.9), (2.10), and (2.1),

$$\sup_n E|Y_k^n| < \infty, \tag{2.11}$$

and

$$\lim_{n\to\infty} EY_k^n = \gamma \quad \text{for each } k \geq 1. \tag{2.12}$$

By (2.11), there is a subsequence $n_i$ so that the joint distributions of $\{Y_k^{n_i}, k \geq 1\}$ converge to those of some collection $\{Y_k, k \geq 1\}$. For any bounded continuous $f$ on $R^\infty$ which depends on only finitely many coordinates,

$$Ef(Y_1, Y_2, \ldots) = \lim_{i\to\infty} \frac{1}{n_i} \sum_{l=1}^{n_i} Ef(X_{l+1} - X_l, X_{l+2} - X_{l+1}, \ldots). \tag{2.13}$$

From this it is easy to see that $\{Y_k, k \geq 1\}$ is a stationary sequence. By (1.7) and (1.8),

$$Y_1^n = X_{U_n+1} - X_{U_n} \leq X_{U_n, U_n+1} =_{\mathrm{d}} X_1$$

where $=_{\mathrm{d}}$ denotes equality in distribution. Therefore by (1.3), $\{(Y_1^n)^+, n \geq 1\}$ is uniformly integrable. By Fatou's lemma and (2.12), it then follows that

$$EY_1 \geq \limsup_{n\to\infty} EY_1^n = \gamma.$$

By Birkhoff's ergodic theorem,

$$Y = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} Y_k$$

exists a.s. and $EY = EY_1 \geq \gamma$. It remains to prove that $\underline{X}$ is stochastically larger than $Y$. For this, it is enough to show that

$$(Y_1, Y_1 + Y_2, Y_1 + Y_2 + Y_3, \ldots) \leq_{\mathrm{d}} (X_1, X_2, X_3 \ldots), \tag{2.14}$$

where $\leq_{\mathrm{d}}$ denotes (joint) stochastic monotonicity. But this is an easy consequence of (1.7) and (1.8), since by (2.13)

$$\begin{aligned} &Ef(Y_1, Y_1 + Y_2, Y_1 + Y_2 + Y_3, \ldots) \\ &= \lim_{i\to\infty} \frac{1}{n_i} \sum_{l=1}^{n_i} Ef(X_{l+1} - X_l, X_{l+2} - X_l, X_{l+3} - X_l, \ldots) \\ &\leq \lim_{i\to\infty} \frac{1}{n_i} \sum_{l=1}^{n_i} Ef(X_{l,l+1}, X_{l,l+2}, X_{l,l+3}, \ldots) \\ &= Ef(X_1, X_2, X_3, \ldots) \end{aligned}$$

for any increasing bounded continuous function on $R^\infty$ which depends on only finitely many coordinates.

PROOF OF (2.4). Let $X$ be the common value of $\underline{X}$ and $\overline{X}$, which agree by (2.2) and (2.3). Then of course $EX = \gamma$. By (2.5) with $k = 1$, $\{(1/n)X_n^+, n \geq 1\}$ is uniformly integrable. Therefore

$$\lim_{n\to\infty} E\left[\frac{1}{n} X_n - X\right]^+ = 0.$$

Conclusion (2.4) follows from this and (2.1).

A final comment about the proof of (2.3) is in order. In Kingman's original proof, he constructed a stationary process $\{Y_k,\ k \geq 1\}$ with $EY_k = \gamma$ such that

$$Y_{k+1} + \cdots + Y_m \leq X_{k,m} \quad \text{a.s.} \tag{2.15}$$

whenever $k < m$. By (2.14), the random variables $Y_k$ and $X_k$ can be constructed on a common probability space in such a way that (2.15) holds for $k = 0$. It is not hard to check that under Kingman's stronger hypotheses, our construction can be used to obtain (2.15) for all $k$.

**3. An example.** In this section, we describe an example which satisfies (1.7), (1.8), and (1.9), but not (1.1) and (1.2). It is rather typical of the application of Theorem 1.10. The example is from oriented percolation. [See Durrett (1984) for much more on this model.] Let

$$I = \{(m,n) \in Z^2\colon n \geq 0 \text{ and } m+n \text{ is even}\}.$$

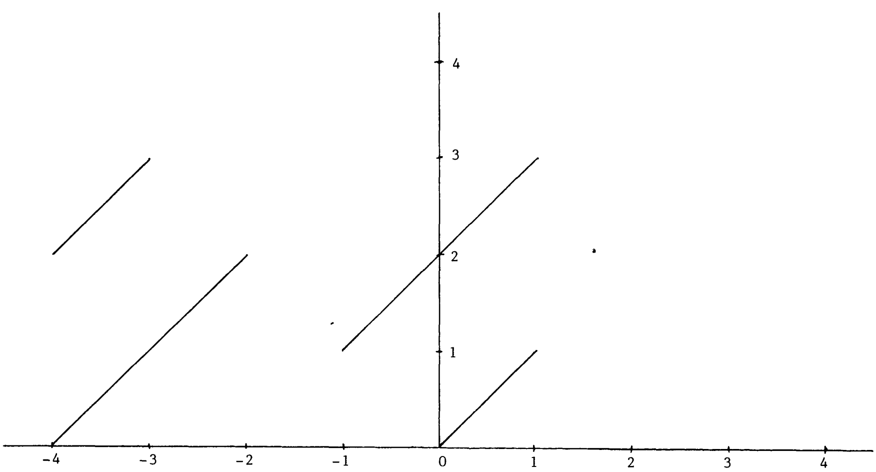
A path in $I$ is a sequence $(m_1, n_1), \dots, (m_k, n_k)$ such that for each $i$, $n_{i+1} = n_i + 1$ and $m_{i+1} = m_i \pm 1$. Each edge joining an $(m,n)$ to $(m+1, n+1)$ or to $(m-1, n+1)$ is independently labelled open or closed with probability $p$ or $1-p$, respectively. A path is said to be active if all the edges joining successive points in that path are open. For $n \geq 0$, let

$$X_n = \max\{m\colon \exists \text{ an active path from } (l,0) \text{ to } (m,n) \text{ for some } l \leq 0\}.$$

For $0 \leq m < n$, let

$$X_{m,n} = \max\{k\colon \exists \text{ an active path from } (l,m) \text{ to } (k,n) \text{ for some } l \leq X_m\} - X_m.$$

Then $X_{0,n} = X_n$, and (1.7) and (1.8) are easily verified. To check (1.9), it suffices to note that for each $k$, the random variables $\{X_{nk,(n+1)k}, n \geq 0\}$ are independent and identically distributed. A realization of the process which illustrates how (1.1) can fail is given in the figure below. The edges which are drawn in are open; the ones which are not are closed. This shows also that (1.2) fails, since as



mentioned earlier, (1.2) and (1.7) imply (1.1). In this realization, $X_1 = 1$, $X_2 = -2$, $X_{1,2} = -1$, $X_{1,3} = 0$, and $X_{2,3} = -1$, so that $X_{1,3} > X_{1,2} + X_{2,3}$.

Of course, this example does not necessarily satisfy assumption (1.3). It clearly does satisfy $EX_{0,1}^{+} < \infty$, and does satisfy (1.3) for $p$ sufficiently close to one (see Durrett, 1984).

## REFERENCES

BURKHOLDER, D. L. (1973). Discussion of Prof. Kingman's paper. *Ann. Probab.* **1**, 900–902.

DEL JUNCO, A. (1977). On the Decomposition of a Subadditive Stochastic Process. *Ann. Probab.* **5** 298–302.

DERRIENNIC, Y. (1975). Sur le Theoreme Ergodique Sous-Additif. *C.R. Acad. Sci. Paris Ser. A* **281** 985–988.

DERRIENNIC, Y. (1983). Un Theoreme Ergodique Presque Sous-Additif. *Ann. Probab.* **11** 669–677.

DURRETT, R. (1980). On the Growth of One Dimensional Contact Processes. *Ann. Probab.* **8** 890–907.

DURRETT, R. (1984). Oriented Percolation in Two Dimensions. *Ann. Probab.* **12** 999–1040.

HAMMERSLEY, J. M. and WELSH, J. A. D. (1965). First-Passage Percolation, Subadditive Processes, Stochastic Networks and Generalized Renewal Theory. *Bernoulli–Bayes–Laplace Anniversary Volume*, ed. J. Neyman and L. M. Le Cam. Springer, Berlin.

HAMMERSLEY, J. M. (1974). Postulates for Subadditive Processes. *Ann. Probab.* **2** 652–680.

KATZNELSON, Y. and WEISS, B. (1982). A Simple Proof of Some Ergodic Theorems. *Israel J. Math.* **42** 291–296.

KESTEN, H. (1973). Discussion of Prof. Kingman's Paper. *Ann. Probab.* **1** 903.

KINGMAN, J. F. C. (1968). The Ergodic Theory of Subadditive Stochastic Processes. *J. Roy. Statist. Soc. Ser. B* 30 499–510.

KINGMAN, J. F. C. (1973). Subadditive Ergodic Theory. *Ann. Probab.* **1** 883–909.

KINGMAN, J. F. C. (1976). Subadditive Processes. *Lecture Notes in Mathematics* **539** 167–223. Springer, Berlin.

SMELTZER, M. D. (1977). Subadditive Stochastic Processes. *Bulletin of the A.M.S.* 83 1054–1056.

SMYTHE, R. T. (1976). Multiparameter Subadditive Processes. *Ann. Probab.* **4** 772–782.

SMYTHE, R. T. and WIERMAN, J. C. (1978). First-Passage Percolation on the Square Lattice. *Lecture Notes in Mathematics* **671** Springer, Berlin.

STEELE, J. M. (1978). Empirical Discrepancies and Subadditive Processes. *Ann. Probab.* **6** 118–127.

STEELE, J. M. (1980). Kingman's Subadditive Ergodic Theorem. Unpublished manuscript.

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