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CORRECTION TO "WEAK AND L^p-INVARIANCE PRINCIPLES FOR SUMS OF B-VALUED RANDOM VARIABLES"

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The proofs of Theorems 1 and 2 of [11] contain several gaps which will be corrected in this note.

1. Correction of the proof of Theorem 2. Let us start with Section 2 of [11], containing the proof of Theorem 2. In Theorem 2 it was tacitly assumed that the limit law G is nondegenerate. This has to be added to the hypotheses.

In 1982 an error was discovered in Theorem 5 of the paper [5] of Dudley and Kanter which, as an immediate consequence, invalidates [11, Lemma 2.2]. For $1 < \alpha \leq 2$ Dudley could easily fix the argument. However, for $0 < \alpha < 1$, Marcus [9] disproved both [5, Theorem 5] and [11, Lemma 2.2] by an elegant counterexample. A second gap in the proof of Theorem 2 is caused by the fact that the proof of [6, Lemma 1.7] contains a flaw. This invalidates the proof of [11, (2.5)]. However, both [6, Lemma 1.7] and [11, (2.5)] are correct as they stand, as can be seen by [11, (1.14)] and Karamata's theorem.

We now correct these two gaps. We follow [11, Section 2] until the end of page 73, dropping Lemma 2.2 and its proof. In Lemma 2.3 relation (2.4) and its proof are correct. We drop [11, (2.5)] and its proof and show directly that

$$a(n)^{-1}a(n\varepsilon_n) \to 0. \tag{1}$$

Inspection of pages 74 and 75 shows that this is all that is missing in the proof of [11, (1.14)].

To prove (1) we fix $k \in \mathbb{N}$. Then by [11, (2.1), (2.3), and (2.4)]

$$\lim_{n\to\infty} a(n)^{-1}a(n/k) = k^{-1/\alpha}. \tag{2}$$

Hence there is a sequence $\{r(n),\ n \geq 1\}$ with $r(n) \to \infty$ and $r(n) \leq n/(2k)$ for $n \geq n_0$ and some n_0 such that

$$a(n)^{-1}a(r(n)) \to 0. \tag{3}$$

We also can assume $\varepsilon_n \leq 1/(2k)$ for $n \geq n_0$. Hence we can define $\theta_n \geq 0$ for $n \geq n_0$ by

$$n\theta_n + [r(n)] + [n\varepsilon_n] = [n/k].$$

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Let

$$X_n := a(n\theta_n)^{-1}\Big(\sum_{\nu \le n\theta_n} f(x_\nu) - f(b_{n\theta_n})\Big),$$

$$Y_n := a(n\varepsilon_n)^{-1}\Bigg(\sum_{\nu = n\theta_n + [r(n)] + 1}^{[n/k]} f(x_\nu) - f\big(b_{[n\varepsilon_n]}\big)\Bigg),$$

$$Z_n := a(n/k)^{-1}\Big(\sum_{\nu \le n/k} f(x_\nu) - f\big(b_{[n/k]}\big)\Big),$$

and

$$\gamma_n := a(n/k)^{-1} a(n\theta_n), \qquad \delta_n := a(n/k)^{-1} a(n\varepsilon_n).$$

Since by (3)

$$a(n)^{-1}\Bigg(\sum_{\nu = n\theta_n + 1}^{n\theta_n + [r(n)]} f(x_\nu) - f\big(b_{[r(n)]}\big)\Bigg) \to 0 \quad \text{in prob.}$$

we have by a well-known [7, Theorem 17.2.1] mixing inequality uniformly for all $|t| \le M$, $M > 0$

$$\big|\,|E\exp(it\gamma_n X_n)|\,|E\exp(it\delta_n Y_n)| - |E\exp(itZ_n)|\,\big|$$

$$\le 16\alpha(r(n)) + E\left|\exp\left(ita(n/k)^{-1}\sum_{\nu = n\theta_n + 1}^{n\theta_n + [r(n)]} \cdots\right) - 1\right| \to 0. \tag{4}$$

Now X_n, Y_n, and Z_n all converge in law to $G \circ f^{-1}$ by [7, Theorem 18.1.1]. Thus $|E\exp(itX_n)| \to \exp(-c|t|^\alpha)$ for some $c > 0$ and similarly for Y_n and Z_n. Let $\{\gamma_{n'}^\alpha + \delta_{n'}^\alpha,\ n' \in \mathbb{N}\}$ be any subsequence of $\{\gamma_n^\alpha + \delta_n^\alpha,\ n \ge 1\}$. Then there is a subsequence n'' such that $(\gamma_{n''}, \delta_{n''}) \to (\gamma, \delta)$. Then by (4) $\gamma < \infty$, $\delta < \infty$, and $\gamma^\alpha + \delta^\alpha = 1$. Hence $\gamma_n^\alpha + \delta_n^\alpha \to 1$ as $n \to \infty$. So

$$\limsup_{n \to \infty} a(n/k)^{-1} a(n\varepsilon_n) \le 1.$$

This together with (2) implies (1) and hence [11, (1.14)].

To close the second gap in the proof of Theorem 2 we show that the only limit laws G which can occur as limits of sums [11, (1.13)] are the stable ones. We need the following simple fact.

LEMMA 1. *Let X_n and X_n' be B-valued random variables with $\mathscr{L}(X_n) \to \lambda$ and $\mathscr{L}(X_n') \to \mu$. Suppose that the σ-fields $\mathscr{F}_n$ and $\mathscr{G}_n$ generated by X_n and X_n', respectively, satisfy*

$$\alpha(n) := \sup\{|P(C \cap D) - P(C)P(D)| : C \in \mathscr{F}_n, D \in \mathscr{G}_n\} \to 0.$$

Then

$$\mathscr{L}(X_n + X_n') \to \lambda * \mu.$$

PROOF. For $B = \mathbb{R}$ this follows at once from

$$|E\exp(it(X_n + X_n')) - E\exp(itX_n)E\exp(itX_n')| \leqq 16\alpha(n), \qquad t \in \mathbb{R},$$

(see [7, Theorem 17.2.1]) and letting $n \to \infty$. For general B we first note that $\{\mathscr{L}(X_n + X_n'),\ n \geqq 1\}$ is tight and thus relatively compact. The result follows since $\mathscr{L}(f(X_n + X_n')) \to (\lambda \circ f^{-1}) * (\mu \circ f^{-1}) = (\lambda * \mu) \circ f^{-1}$ for each $f \in B^*$. $\square$

Suppose now that [11, (1.13)] holds. Fix $k \geqq 1$ and write $\tau(n) := [\log n]$. We define blocks H_j and I_j $(1 \leqq j \leqq k)$ of consecutive integers of length n and $\tau(n)$, respectively, leaving no gaps between them, i.e.,

$$H_j := ((n + \tau(n))(j-1),\ nj + \tau(n)(j-1)],$$

$$I_j := (nj + \tau(n)(j-1),\ (n + \tau(n))j],$$

$$S_j(n) := \sum_{\nu \in H_j} x_\nu, \qquad T_j(n) := \sum_{\nu \in I_j} x_\nu,$$

$$X_n^{(j)} := a(n)^{-1}\big(S_j(n) - b_n\big).$$

By (1.14) $a(\tau(n))/a((n + \tau(n))k) \to 0$. Hence by (1.13) with $\tau(n)$ instead of n

$$\mathscr{L}\Big(a((n + \tau(n))k)^{-1}\big(T_j(n) - b_{\tau(n)}\big)\Big) \to \delta_0, \qquad 1 \leqq j \leqq k,$$

the point mass at 0. Thus we can discard the sums over the short blocks I_j, $1 \leqq j \leqq k$. By [11, (1.13)] $\mathscr{L}(X_n^{(j)}) \to G$, $1 \leqq j \leqq k$. We apply Lemma 1 $k-1$ times and obtain

$$\mathscr{L}\Big(\sum_{j \leqq k} X_n^{(j)}\Big) \to G * \cdots * G.$$

On the other hand, since the short blocks can be discarded $\sum_{j \leq k} X_n^{(j)}$ can be approximated by a properly centered multiple of $(S_{(n+\tau(n))k} - b_{(n+\tau(n))k})/a((n + \tau(n))k)$. We apply [11, (1.13)] and the convergence of types theorem and obtain the result (see [2, page 200]).

2. Correction of the proof of Theorem 1. The proof of [11, Theorem 1] contains a serious flaw, kindly pointed out to me by H. Dehling. It is the result of an error in the calculation yielding relation [11, (3.6)]. This should read

$$\max_{\nu \leqq n_k} a(\nu)/a(t_k) \leqq 2\varepsilon^{4/\alpha}.$$

As a consequence in [11, (3.12)] and [11, (3.13)] the exponents $6/\alpha$ will have to be replaced by $2/\alpha$ yielding the bound ε^2 in Lemma 3.2, a bound far too large for completion of the argument on [11, page 79] (see [11, (3.31)] and [11, (3.32)]).

For $\alpha = 2$, the Gaussian case, the gap is easily filled because instead of [11, (3.1)] we can use the Fernique–Landau–Shepp theorem and obtain

$$G\{x\colon \|x\| \geqq \lambda\} \leqq c_1\exp(-c_2\lambda^2)$$

for some positive constants c_1 and c_2. Hence subject to the above corrections Lemma 3.1 and the rest of the argument remain valid as they stand.

In the case of Gaussian limit laws this method has been employed in many papers to prove invariance principles. It is based on estimates of the Prohorov distance of the laws of the properly normalized partial sums and the limit law combined with an application of Strassen's theorem on joint laws with given marginals. In an effort to repair the proof of [11, Theorem 1] for $\alpha < 2$ Dabrowski, Dehling, and I refined this method, just mentioned. This refined method was presented in a recent paper [4] where it was used to give a much simpler proof of a theorem of de Acosta [1]. Here, however, I shall not present the proof of [11, Theorem 1] via the refined method since it is now more economical to apply de Acosta's theorem [1] directly. Moreover, this new proof is also interesting from a methodological point of view since for a long period of time it was unknown whether or not [11, Theorem 1] could be derived from de Acosta's theorem [1].

We follow the proof of [11, Theorem 1] in Section 3 until the end of the proof of Lemma 3.2 replacing ε^4 by ε^{-4} on the right side of the display just above Lemma 3.1 and making the corrections mentioned in the first paragraph of this section. In addition, the factors $\frac{1}{2}$ and $\frac{1}{4}$ in (3.12) and the factor $\frac{1}{2}$ in the last line of (3.13) should read $\frac{1}{4}$, $\frac{1}{8}$, and $\frac{1}{4}$, respectively. Also in (3.7) t_{k-1} should read t_{k+1}. Also note that (3.5) and (3.7) remain valid if we replace the exponent on the right side of these two inequalities by $18/\alpha$.

We first construct two sequences $\{x_j,\ j \geqq 1\}$ and $\{y_j,\ j \geqq 1\}$ of independent identically distributed random variables with the proper law and depending perhaps on ε such that

$$P\left\{\max_{j \leqq n} \left\| a(n)^{-1} S_j - n^{-1/\alpha} T_j \right\| > 6\varepsilon\right\} \ll \varepsilon. \tag{5}$$

For this purpose we consider the triangular array $\{a(n_k)^{-1} x_{t_k+j},\ 1 \leqq j \leqq n_k,\ k = 1, 2, \dots\}$. By de Acosta's main result [1] there exists without loss of generality (in the sense of Strassen) a triangular array $\{y_{kj},\ 1 \leqq j \leqq n_k,\ k = 1, 2, \dots\}$ of independent identically distributed stable random variables of index α such that

$$P\left\{\max_{1 \leqq h \leqq n_k} \left\| a(n_k)^{-1} \sum_{j \leqq h} x_{t_k+j} - n_k^{-1/\alpha} \sum_{j \leqq h} y_{kj} \right\| \geqq \varepsilon^6\right\} \leqq \varepsilon^6 \tag{6}$$

for all $k \geqq k_0$. Of course, we can assume also without loss of generality that the $2n_k$-dimensional vectors $\{(x_{t_k+1}, \dots, x_{t_{k+1}}, y_{k1}, \dots, y_{kn_k}),\ k = 1, 2, \dots\}$ are independent. Each integer ν can be represented uniquely in the form $\nu = t_k + j$ with $1 \leqq j \leqq n_k$. We now define the desired sequence $\{y_\nu,\ \nu \geqq 1\}$ by setting $y_\nu = y_{kj}$. Then (6) can be rewritten as

$$P\left\{\max_{1 \leqq h \leqq n_k} \left\| a(n_k)^{-1} \sum_{j \leqq h} x_{t_k+j} - n_k^{-1/\alpha} \sum_{j \leqq h} y_{t_k+j} \right\| \geqq \varepsilon^6\right\} \leqq \varepsilon^6. \tag{7}$$

Let $n \geqq t_{M_1}$ with $M_1 = k_0 + s$, define M by $t_{M-1} \leqq n < t_M$, and put $m := M - s$; here s is defined in [11, (3.4)]. We replace [11, (3.28)] by (as in [11] we write

$S(j) := S_j$ and $T(j) := T_j$)

$$
\begin{aligned}
&P\left\{\max_{j \leqq n} \left\| a(n)^{-1} S(j) - n^{-1/\alpha} T(j) \right\| \geqq 6\varepsilon \right\} \\
&\leqq P\left\{ \max_{j \leqq t_{m-1}} \left\| a(n)^{-1} S(j) \right\| \geqq 2\varepsilon \right\} + P\left\{ \max_{j \leqq t_{m-1}} \left\| n^{-1/\alpha} T(j) \right\| \geqq \varepsilon \right\} \\
&\quad + \sum_{m-1 \leqq k < M} P\left\{ \max_{h \leqq n_k} \left\| a(n_k)^{-1} \sum_{j \leqq h} x_{t_k + j} - n_k^{-1/\alpha} \sum_{j \leqq h} y_{t_k + j} \right\| \geqq \varepsilon / s \right\} \\
&\quad + \sum_{m-1 \leqq k < M} P\left\{ \left| a(n_k) a(n)^{-1} - (n_k/n)^{1/\alpha} \right| \max_{h \leqq n_k} \left\| \sum_{j \leqq h} y_{t_k + j} \right\| \geqq 2\varepsilon n_k^{1/\alpha} / s \right\} \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} \quad \text{(say)},
\end{aligned}
$$

since by [11, (3.7) and (3.5)] $a(n) \geqq a(n_k)$ for $M - s \leqq k < M$. By [11, (3.4)] $t_{m-1} \leq n_M$ and thus by [11, Lemma 3.2]

$$
\mathrm{I} \leqq P\left\{ \max_{j \leqq n_M} \| S(j) \| \geqq \varepsilon a(t_M) \right\} \ll \varepsilon, \qquad \mathrm{II} \ll \varepsilon.
$$

Since

$$
s \leqq \varepsilon^{-5} \tag{8}
$$

we obtain from (7) above III $\ll \varepsilon$. Finally, by [11, (3.5) and (3.7)] $|a(n_k)a(n)^{-1} - (n_k/n)^{1/\alpha}| \ll \varepsilon^{18/\alpha}$. Hence by Ottaviani's inequality (see the argument at the beginning of the proof of Lemma 3.2) and by (8) we also obtain IV $\ll \varepsilon$. This proves (5).

The sequences $\{x_\nu, \nu \geqq 1\}$ and $\{y_\nu, \nu \geqq 1\}$ constructed may still depend on ε. In [11], this was overlooked, as was kindly pointed out to me by de Acosta. But using a device of Major [8], also used in [10] in a similar context, we easily can construct universal sequences of $\{x_\nu, \nu \geqq 1\}$ and $\{y_\nu, \nu \geqq 1\}$.

For given $p \geqq 1$ choose sequences $\{x_\nu^{(p)}, \nu \geqq 1\}$ and $\{y_\nu^{(p)}, \nu \geqq 1\}$ of independent random variables with partial sums $S_k^{(p)}$ and $T_k^{(p)}$, respectively, such that for all $n \geqq n_0(p)$

$$
P\left\{ \max_{k \leqq n} \left\| a(n)^{-1} S_k^{(p)} - n^{-1/\alpha} T_k^{(p)} \right\| \geqq 2^{-p} \right\} \leqq 2^{-p}. \tag{9}
$$

Moreover, we assume without loss of generality that the x sequences are independent from one another and that the y sequences are independent from one another. Put $r(p) = \sum_{q \leqq p} n_0(q)$. We now define

$$
x_\nu = x_{\nu - r(p)}^{(p)} \quad \text{and} \quad y_\nu = y_{\nu - r(p)}^{(p)} \quad \text{if } r(p) < \nu \leqq r(p+1). \tag{10}
$$

Then $\{x_\nu, \nu \geqq 1\}$ and $\{y_\nu, \nu \geqq 1\}$ are sequences of independent random variables with common distribution F and G, respectively, satisfying [11, (1.7)]. Indeed, let $\varepsilon > 0$ be given and let p_0 be such that $2^{-p_0} < \varepsilon$. Let N_0 be so large that

$$
P\left\{ a(n)^{-1} \max_{k \leqq r(p_0)} \|S_k\| > \varepsilon \right\} < \varepsilon
$$

and

$$P\left\{n^{-1/\alpha} \max_{k \leqq r(p_0)} \|T_k\| > \varepsilon\right\} < \varepsilon$$

for all $n \geqq N_0$. Now let $n \geqq \max(N_0, n_0(p_0))$ be given and choose M such that $r(M) < n \leqq r(M+1)$. Then by (9) and (10)

$$\begin{aligned}\max_{k \leqq n} \left\| a(n)^{-1} S_k - n^{-1/\alpha} T_k \right\| &\leqq a(n) \max_{k \leqq r(p_0)} \|S_k\| + n^{-1/\alpha} \max_{k \leqq r(p_0)} \|T_k\| \\ &\quad + \sum_{p=p_0+1}^{M-1} \max_{r(p) \leqq j \leqq r(p+1)} \left\| \sum_{\nu=r(p)+1}^{j} \left(a(n)^{-1} x_\nu - n^{-1/\alpha} y_\nu \right) \right\| \\ &\quad + \max_{k \leqq n-r(M)} \left\| a(n)^{-1} S_k^{(M)} - n^{-1/\alpha} T_k^{(M)} \right\| \\ &< 2\varepsilon + \sum_{p=p_0}^{M} 2^{-p} < 4\varepsilon\end{aligned}$$

except on a set of probability $< 4\varepsilon$.

3. Remarks on Theorem 4. In Theorem 4 of [11] only the Gaussian case can be salvaged. However, Major's trick no longer can be applied. Modifying the argument, Dabrowski [3] has been able to repair the proof of Theorem 4 in the Gaussian case.

Samur [12] gives a counterexample to Theorem 4, non-Gaussian part, and comments about modifications possibly needed in the Gaussian case.

We conclude with a remark.

As noted in [11], page 71, Theorem 2 of Simons and Stout [13] is contained in [11], Theorem 1, as a special case. However, it might be interesting to observe that their proof can be augmented so as to yield their result in the stronger form [11, (1.7)]. Indeed, in their proof the sequences $\{x_j,\ j \geqq 1\}$ and $\{y_j,\ j \geqq 1\}$ are constructed in such a way that $\{(x_j, y_j),\ j \geqq 1\}$ is a sequence of independent random vectors in $\mathbb{R}^2$. Hence a combination of the conclusion of their theorem with Ottaviani's maximal inequality applied to the sequence $\{a(n)^{-1}x_j - n^{-1/\alpha}y_j,\ 1 \leqq j \leqq n\}$ yields their result in stronger form [11, (1.7)]. It is already observed in their paper that the case $\alpha = 2$ cannot be treated with their method since it would contradict the central limit theorem.

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