The Annals of Probability
1986, Vol. 14, No. 3, 1080–1084

ON CHANGING TIME FOR TWO-PARAMETER STRONG MARTINGALES: A COUNTEREXAMPLE

BY PETER IMKELLER[1]

Ohio State University

A two-parameter Wiener-measurable strong martingale is constructed for which no time change by means of stopping domains into the Wiener sheet exists.

Introduction. In one-parameter martingale theory, Brownian motion owes part of its significance to the fact that any continuous martingale may be considered as a Brownian motion running with a different clock. This means that by a suitable time change any continuous martingale can be transformed into Brownian motion. Since its multiparameter analog, the Wiener sheet, is a strong martingale and the strong martingale property is preserved under transformations by stopping domains (see Nualart and Sanz [4], page 151), it could only be expected to play a similarly central role among strong martingales. Nualart and Sanz [4], pages 153–156, exhibits a class of strong martingales for which time changes exist which transform them into the Wiener sheet. In this paper we construct a strong martingale which is measurable w.r.t. the Wiener sheet's filtration, but allows no transformation into the Wiener sheet by changing time with stopping domains. Cairoli and Walsh [2] give a counterexample for another type of time change.

0. Preliminaries, notation, and definitions. The parameter space $\mathbb{R}_+^2$ is endowed with the usual partial ordering, i.e., coordinatewise linear ordering, $\leq$, with respect to which intervals are defined in the usual way. Coordinates of points in parameter space are marked by lower indices, for example $t = (t_1, t_2)$. If $f\colon \mathbb{R}_+^2 \to \mathbb{R}$ is a function and $J =]s, t] \subset \mathbb{R}_+^2$ an interval, we denote by

$$\Delta_J f = f(t) - f((t_1, s_2)) - f((s_1, t_2)) + f(s)$$

the increment of f over J. If $(B, \mathfrak{B})$ and $(C, \mathfrak{C})$ are measurable spaces, then $\mathscr{M}(\mathfrak{B}, \mathfrak{C})$ is the linear space of all measurable functions from B to C. Our basic probability space $(\Omega, \mathfrak{F}, P)$ is supposed to be complete, the basic filtration $(\mathfrak{F}_t)_{t \in \mathbb{R}_+^2}$ to be the filtration of a Wiener sheet W which lives on $(\Omega, \mathfrak{F}, P)$. Furthermore, each $\mathfrak{F}_t$ is assumed to be augmented by the zero-sets of $\mathfrak{F}$. For $t \in \mathbb{R}_+^2$ let $\mathfrak{F}_{t_1}^1 = \bigvee_{s_2 \in \mathbb{R}_+} \mathfrak{F}_{(t_1, s_2)}$ and $\mathfrak{F}_{t_2}^2 = \bigvee_{s_1 \in \mathbb{R}_+} \mathfrak{F}_{(s_1, t_2)}$. The σ-field $\mathfrak{P}$ of "previsible sets" is, as usual, generated by the rectangles $F \times]s, t]$, where $]s, t] \subset \mathbb{R}_+^2$, $F \in \mathfrak{F}_s$.

Received October 1984.

[1] On leave from Mathematisches Institut der Ludwig-Maximilians-Universitaet Muenchen.

AMS 1980 *subject classifications*. Primary 60G40, 60G44; secondary 60H05.

Key words and phrases. Two-parameter strong martingales, Wiener sheet, stopping domains, changing time.

A stochastic process X is always assumed to belong to $\mathcal{M}(\mathfrak{F} \otimes \mathfrak{B}(\mathbb{R}_+^2), \mathfrak{B}(\mathbb{R}))$. X is said to be "adapted," if $X_t \in \mathcal{M}(\mathfrak{F}_t, \mathfrak{B}(\mathbb{R}))$ for $t \in \mathbb{R}_+^2$, "progressively measurable," if $X|_{\Omega \times [0,t]} \in \mathcal{M}(\mathfrak{F}_t \otimes \mathfrak{B}([0,t]), \mathfrak{B}(\mathbb{R}))$ for $t \in \mathbb{R}_+^2$, "previsible," if $X \in \mathcal{M}(\mathfrak{P}, \mathfrak{B}(\mathbb{R}))$. An adapted process M such that M_t is integrable for each $t \in \mathbb{R}_+^2$, is called a "strong martingale" if for each interval $J =]s, t]$ we have $E(\Delta_J M | \mathfrak{F}_{s_1}^1 \vee \mathfrak{F}_{s_2}^2) = 0$. Given $Y \in \mathcal{M}(\mathfrak{P}, \mathfrak{B}(\mathbb{R}))$ such that $\int_{\Omega \times]0,t]} Y^2 d(P \otimes \lambda) < \infty$ for each $t \in \mathbb{R}_+^2$ (λ being the Lebesgue measure on $\mathfrak{B}(\mathbb{R}_+^2)$), the stochastic integral process $\int_{\Omega \times]0, \cdot]} Y dW$ is a strong martingale (see Cairoli and Walsh [1]).

Let $\mathcal{S}$ be the set of all closed subsets S of $\mathbb{R}_+^2$ such that $t \in S$ implies $[0, t] \subset S$. A "stopping domain" is a mapping D: $\Omega \to \mathcal{S}$ such that $(\omega, t) \to 1_{D(\omega)}(t)$ is progressively measurable. In particular, 1_D is adapted. Stopping domains and associated stopping lines are considered to be the multiparameter counterparts of stopping times (cf. Merzbach [3]).

1. The counterexample. Let $Y \in \mathcal{M}(\mathfrak{P}, \mathfrak{B}(\mathbb{R}))$ be such that

$$\int_{\Omega \times]0,t]} Y^2 d(P \otimes \lambda) < \infty \quad \text{for each } t \in \mathbb{R}_+^2, \tag{1.1}$$

and let $M = \int_{]0, \cdot]} Y dW$ be the strong martingale associated with it. Nualart and Sanz [4], page 153, proved that if there exists a family $(D_t)_{t \in \mathbb{R}_+^2}$ of stopping domains such that

$$\int_{\mathbb{R}_+^2} 1_{D_t} Y^2 d\lambda = t_1 t_2 \quad \text{a.s.,} \tag{1.2}$$

$$D_s \cap D_t = D_{s \wedge t}, \tag{1.3}$$

for $s, t \in \mathbb{R}_+^2$, are satisfied, then $N_t = \int 1_{D_t} dM = \int 1_{D_t} Y dW$, $t \in \mathbb{R}_+^2$, is a Brownian sheet. Of course, (1.2) is also necessary for N to be a Brownian sheet. We will exhibit the existence of a process $Y \in \mathcal{M}(\mathfrak{P}, \mathfrak{B}(\mathbb{R}))$ satisfying (1.1),

$$\begin{aligned} &\int_{]0, t_1] \times \mathbb{R}_+} Y^2 d\lambda = \infty \quad \text{a.s.,} \\ &\int_{\mathbb{R}_+ \times]0, t_2]} Y^2 d\lambda = \infty \quad \text{a.s. for } t > 0, \end{aligned} \tag{1.4}$$

such that no family of stopping domains fulfills (1.2). This means that the corresponding strong martingale M cannot, by any time change of the above form, be transformed into the Wiener sheet.

Let us first construct Y. For $k, l \in \mathbb{Z}$, consider the dyadic rectangles

$$A_{k,l} = \begin{cases}](l+1)2^{-2k}, (l+2)2^{-2k}] \times]2^{-2k}, 2^{-2k+1}], & \text{if } l \geq 1, \\ ]2^{-2k}, 2^{-2k+1}] \times]2^{-2k}, 2^{-2k+1}], & \text{if } l = 0, \\ ]2^{-2k}, 2^{-2k+1}] \times](-l+1)2^{-2k}, (-l+2)2^{-2k}], & \text{if } l \leq -1, \end{cases}$$

$$B_{k,l} = \begin{cases}]l2^{-2k}, (l+1)2^{-2k}] \times]2^{-2k-1}, 2^{-2k}], & \text{if } l \geq 1, \\ ]2^{-2k-1}, 2^{-2k}] \times]2^{-2k-1}, 2^{-2k}], & \text{if } l = 0, \\ ]2^{-2k-1}, 2^{-2k}] \times]-l2^{-2k}, (-l+1)2^{-2k}], & \text{if } l \leq -1. \end{cases}$$

Figure 1 may help to visualize the situation.

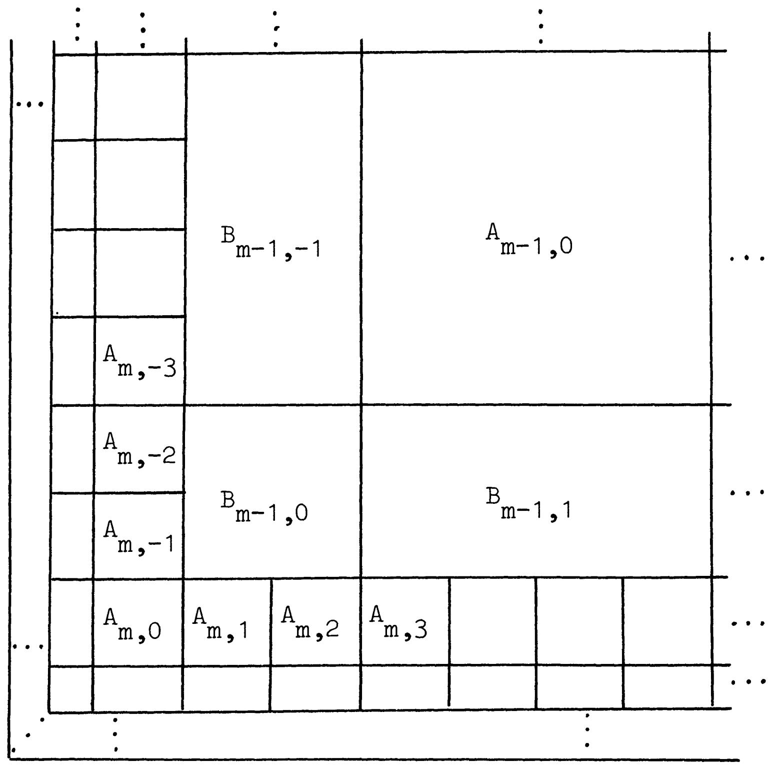

FIG. 1

Put $Y = \sum_{k,l \in \mathbb{Z}} 1_{A_{k,l}} 1_{\{\Delta_{B_{k,l}} W > 0\}}$. By construction, $Y \in \mathscr{M}(\mathfrak{P}, \mathfrak{B}(\mathbb{R}))$. Also, it is easy to see that (1.1) and (1.4) hold true. Now assume that there is a family of stopping domains $(D_t)_{t \in \mathbb{R}_+^2}$ such that (1.2) is fulfilled. To get a contradiction, it is actually enough to consider one time point. Therefore, pick $t \in \mathbb{R}_+^2$ such that $t_1 \cdot t_2 = 1$. We will show that

$$\int_{\mathbb{R}_+^2} 1_{D_t} Y^2 \, d\lambda < 1 \tag{1.5}$$

with positive probability, which is in contradiction to (1.2). For $s \in \mathbb{R}_+^2$ put

$$T_{s_1}^1 := \inf\{u \in \mathbb{R}_+ : (s_1, u) \notin D_t\}, \qquad S_{s_1}^1 := \int_{[0, T_{s_1}^1[} Y^2(s, \cdot)\, ds_2,$$

$$T_{s_2}^2 := \inf\{u \in \mathbb{R}_+ : (u, s_2) \notin D_t\}, \qquad S_{s_2}^2 := \int_{[0, T_{s_2}^2[} Y^2(s, \cdot)\, ds_1,$$

where, as usual, $\inf \varnothing = \infty$. Since D_t is a stopping domain, we have

$$T_{s_1}^1, S_{s_1}^1 \in \mathscr{M}\left(\mathfrak{F}_{s_1}^1, \mathfrak{B}(\mathbb{R})\right), \qquad T_{s_2}^2, S_{s_2}^2 \in \mathscr{M}\left(\mathfrak{F}_{s_2}^2, \mathfrak{B}(\mathbb{R})\right).$$

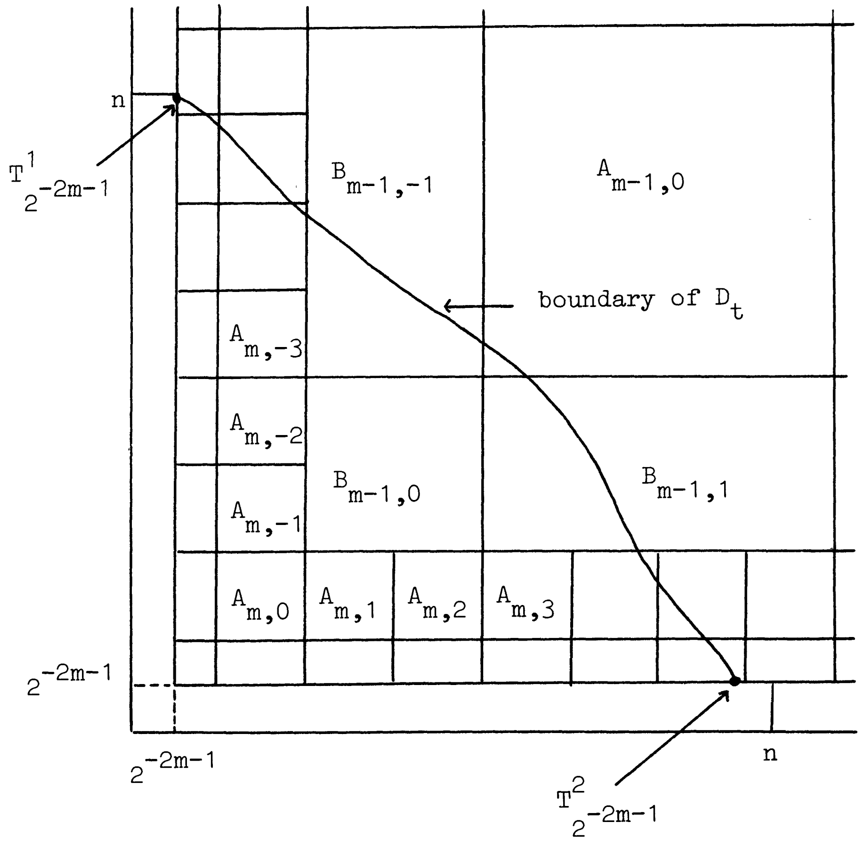

FIG. 2

For $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ let now

$$C_{m,n} := \Big\{\omega \in \Omega\colon \int_0^{2^{-2m-1}} S^1_{s_1}(\omega)\,ds_1 < \tfrac{1}{2}, \int_0^{2^{-2m-1}} S^2_{s_2}(\omega)\,ds_2 < \tfrac{1}{2},$$
$$T^1_{2^{-2m-1}}(\omega) \le n, T^2_{2^{-2m-1}}(\omega) \le n\Big\}.$$

Observe that $T^1_\cdot$, $T^2_\cdot$ are decreasing by definition of stopping domains. Hence, by (1.4), we have T^1_u, T^2_u finite a.s. for $u > 0$. It is therefore possible to ensure $P(C_{m,n}) > 0$ by consecutively choosing m and n large enough. Pick such m and n and look at Figure 2 for illustration.

According to our construction, there are only finitely many $A_{k,l}$'s which satisfy

$$A_{k,l} \cap]2^{-2m-1}, n] \times]2^{-2m-1}, n] \neq \varnothing.$$

Let the corresponding $B_{k,l}$'s be enumerated by $E_1, \dots, E_p$ and consider

$$F_{m,n} = C_{m,n} \cap \{\Delta_{E_1} W \le 0, \dots, \Delta_{E_p} W \le 0\}.$$

Since $C_{m,n} \in \mathfrak{F}^1_{2^{-2m-1}} \vee \mathfrak{F}^2_{2^{-2m-1}}$, W has independent increments, and the Gauss distribution is symmetric, we can conclude

$$(1.6)\qquad \begin{aligned} P(F_{m,n}) &= P(C_{m,n})P(\Delta_{E_1}W \le 0, \dots, \Delta_{E_p}W \le 0) \\ &= P(C_{m,n})\left(\tfrac{1}{2}\right)^p > 0. \end{aligned}$$

Pick $\omega \in F_{m,n}$. By definition of stopping domains, $D_t(\omega) \subset [0, n] \times [0, n]$. Hence

$$(1.7)\qquad \begin{aligned} \int_{\mathbb{R}^2_+} 1_{D_t(\omega)} Y^2(\omega, \cdot)\, d\lambda &\le \int_{\mathbb{R}^2_+} 1_{D_t(\omega)} Y^2(\omega, \cdot) 1_{]0, 2^{-2m-1}] \times \mathbb{R}_+}\, d\lambda \\ &\quad + \int_{\mathbb{R}^2_+} 1_{D_t(\omega)} Y^2(\omega, \cdot) 1_{\mathbb{R}_+ \times]0, 2^{-2m-1}]}\, d\lambda \\ &= \int_0^{2^{-2m-1}} S^1_{s_1}(\omega)\, ds_1 + \int_0^{2^{-2m-1}} S^2_{s_2}(\omega)\, ds_2 < \tfrac{1}{2} + \tfrac{1}{2} = 1. \end{aligned}$$

(1.5) follows from (1.6) and (1.7).

Our example shows in particular, that by means of stopping domains, passages of the quadratic variation of strong martingales through fixed levels cannot be stopped in general.

REFERENCES

[1] CAIROLI, R. and WALSH, J. B. (1975). Stochastic integrals in the plane. *Acta Math.* **134** 111–183.

[2] CAIROLI R. and WALSH, J. B. (1977). On changing time. *Lecture Notes in Math.* **581** 349–355. Springer, Berlin.

[3] MERZBACH, E. (1979). Processus stochastiques à indices partiellement ordonnés. Rapp. Interne 55, Centre de Mathématiques Appliquées, Ecole Polytechnique, Palaiseau.

[4] NUALART, D. and SANZ, M. (1981). Changing time for two-parameter strong martingales. *Ann. Inst. H. Poincaré* **17** 47–163.

MATHEMATISCHES INSTITUT DER LUDWIG-MAXIMILIANS-UNIVERSITAET MUENCHEN
D-8000 MUENCHEN 2
THERESIENSTRASSE 39
FEDERAL REPUBLIC OF GERMANY