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RANDOM SETS WITHOUT SEPARABILITY

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Suppose $\mathscr{V}$ and $\mathscr{F}$ are sets of subsets of X, for some fixed X. We apply König's lemma from infinitary combinatorics to prove that if $\mathscr{V}$ and $\mathscr{F}$ satisfy some simple closure properties, and T is a Choquet capacity on X, then there is a probability measure on $\mathscr{F}$ such that for every $V \in \mathscr{F}$, $\{F \in \mathscr{F}: F \cap V \neq \varnothing\}$ is measurable with probability $T(V)$. This extends the well-known case when $\mathscr{F}$ and $\mathscr{V}$ are the closed (respectively, open) subsets of a second countable Hausdorff space X. The result enables us to define a general notion of "random measurable set"; for example, we can build a point process with Poisson distribution on any infinite (possibly nontopological) measure space.

1. Introduction. A *random element* of a collection $\mathscr{F}$ is an $\mathscr{F}$-valued random variable $\mathscr{U}$, or, equivalently, a probability measure on $\mathscr{F}$. It may happen that $\mathscr{F} \subseteq \mathscr{P}(X)$, where X is some other set; in this case, the random element $\mathscr{U}$ of $\mathscr{F}$ is called a *random set.*

If $\mathscr{V}$ is another subset of $\mathscr{P}(X)$, we may require that the event "$\mathscr{U}$ hits V" be measurable for each V in $\mathscr{V}$. If T is a sufficiently well-behaved real-valued function defined on $\mathscr{V}$, we may in addition require that the probability that $\mathscr{U}$ hits V is equal to $T(V)$. The assertion that a random set exists meeting these requirements is called a "Choquet theorem." Matheron (1975) and Kendall (1973) have proved Choquet theorems in a variety of topological and pseudotopological settings, requiring in all cases some sort of countability hypothesis. (The referee has kindly pointed out that the basic constructions here appeared first in Revuz (1955) and Huneycut (1971).)

With the aid of König's lemma from set theory, we prove the Choquet theorem in a much more general context. First, we note that the argument in [4] only requires that $\mathscr{F}$ contain the singletons from X; we generalize this hypothesis somewhat and prove (Theorem 3.4) the existence of a *finitely* additive probability measure P satisfying the above requirements.

Our main result, Theorem 4.1, is that when $\mathscr{F}$ is closed under countable intersections, P can be extended to a countably additive measure. We give applications in Section 5; for example, if $\mathscr{V} = \mathscr{F}$ is a σ-algebra, the hypotheses are satisfied automatically, and we can thus produce a "random measurable set" with respect to any measure space.

2. Notation. Suppose $\mathscr{G} \subseteq \mathscr{P}(X)$. Call $\mathscr{G}$ a $\cap f$*-paving* (respectively, $\cup f$, $\cap c$, $\cup c$, or $\neg$-paving) of X if $\phi \in \mathscr{G}$, $X \in \mathscr{G}$, and $\mathscr{G}$ is closed under finite intersec-

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tions (respectively, finite unions, countable intersections, countable unions, or complementation). For example, $\mathscr{G}$ is an algebra on X if it is a $\cup f\neg$-paving

If $\{x\} \in \mathscr{G}$ for every $x \in X$, say that $\mathscr{G}$ *contains the singletons from* X. Fix arbitrary pavings $\mathscr{V}$ and $\mathscr{F}$ of X; by "the usual hypotheses" we will mean that

(i) $\mathscr{V}$ is a $\cup f$-paving, and
(ii) either $\mathscr{V} \subseteq \mathscr{F}$, or $\mathscr{F}$ is a $\cup f$-paving containing the singletons of X.

For $A, B_0, \ldots, B_n \in \mathscr{P}(X)$, write $\mathscr{G}^A_{B_0 \cdots B_n} = \{F \in \mathscr{F}\colon F \cap A = \phi, F \cap B_i \neq \phi$ for $i \leq n\}$. Let $\mathscr{A}_{\mathscr{V}} = \{\mathscr{F}^A_{B_0 \cdots B_n}\colon A, B_0, \ldots, B_n \in \mathscr{V}\}$ and let $\mathscr{A}^f_{\mathscr{V}}$ be the closure of $\mathscr{A}_{\mathscr{V}}$ under finite unions. Write $\mathscr{F}^A$ for $\mathscr{F}^A_X$ and $\mathscr{F}_A$ for $\mathscr{F}^{\phi}_A$.

During this and the next section we will omit the proofs of propositions when they are very easy or are similar to those in ([4], Section 2-2).

PROPOSITION 2.1. *Under the usual hypotheses, $\mathscr{A}^f_{\mathscr{V}}$ is an algebra on $\mathscr{F}$. Moreover, every element of $\mathscr{A}^f_{\mathscr{V}}$ can be written as a finite disjoint union of elements of $\mathscr{A}$.*

A given element A of $\mathscr{A}_{\mathscr{V}}$ may have two different representations, e.g., $\mathscr{F}^A_B = \mathscr{F}^A_{B,B}$. Call a representation $\mathscr{F}^A_{B_0 \cdots B_n}$ *reduced* if for all $i < j \leq n$, $B_i \cap A = \phi$ and neither $B_i \subseteq B_j$ nor $B_j \subseteq B_i$. (This differs somewhat from the definition in [4].)

PROPOSITION 2.2. *If $\mathscr{V}$ is an algebra then every element of $\mathscr{A}_{\mathscr{V}}$ has a reduced representation.*

PROPOSITION 2.3. *Suppose $\mathscr{V}$ is an algebra, and the usual hypotheses hold. Then every nonempty element of $\mathscr{A}_{\mathscr{V}}$ has a reduced representation which is unique up to permutation of the lower sets.*

PROOF. When $\mathscr{F}$ contains singletons this is [4] (Lemma 2-2-1). Suppose that $\mathscr{V} \subseteq \mathscr{F}$, and that $\mathscr{F}^A_{B_0 \cdots B_n} = \mathscr{F}^D_{E_0 \cdots E_m}$ are reduced. Since $A^c \in \mathscr{F}^A_{B_0 \cdots B_n}$, $A^c \in \mathscr{F}^D_{E_0 \cdots E_m}$, so $D \subseteq A$. Similarly $A \subseteq D$, so $A = D$. Now fix $j \leq n$, and let $E = (\bigcup_{i=0}^m E_i)/B_j$. Since $E \notin \mathscr{F}^A_{B_0 \cdots B_n}$, $E \notin \mathscr{F}^D_{E_0 \cdots E_m}$, so $E_i \subseteq B_j$ for some $i \leq m$. In other words, every B_j contains some E_i. Similarly, every E_i contains some B_j. Since the representations are reduced, $m = n$ and $\{B_0, \ldots, B_n\} = \{E_0, \ldots, E_m\}$. □

3. Capacities. Given a function $T\colon \mathscr{P}(X) \to [0,1]$, define functions S_n on $\mathscr{P}(X)^{n+1}$ by

$$S_0(A) = 1 - T(A),$$
$$S_{n+1}(A|B_0 \cdots B_n) = S_n(A|B_0 \cdots B_{n-1}) - S_n(A \cup B_n|B_0 \cdots B_{n-1}).$$

Call T a *Choquet capacity* on X provided:

(i) $T(\phi) = 0, T(X) = 1$;
(ii) $S_n \geq 0$ for each n; and
(iii) If $\{A_n\colon n \in \mathbb{N}\}$ is an increasing sequence of sets, then $T(\bigcup_n A_n) = \lim_{n \to \infty} T(A_n)$.

In practice, we will only require that T be defined on a σ-algebra containing $\mathscr{V}$. The reader is warned that this use of the term "Choquet capacity" is nonstandard. Shafer (1979) uses the term "continuous upper probability function" for a function satisfying (i)–(iii).

PROPOSITION 3.1. *Suppose that* $\mathscr{V}$ *is an algebra and that the usual hypotheses hold. If* $\mathscr{F}^A_{B_0 \cdots B_n}$ *is the reduced representation for* $\mathscr{F}^D_{E_0 \cdots E_m}$, *then* $S_{n+1}(A|B_0 \cdots B_n) = S_{m+1}(D|E_0 \cdots E_m)$.

Thus we may define a function $P\colon \mathscr{A}_{\mathscr{V}} \to \mathbb{R}$ unambiguously by $P(\mathscr{F}^A_{B_0 \cdots B_n}) = S_{n+1}(A|B_0 \cdots B_n)$, where of course we put $P(\phi) = 0$.

When $\mathscr{V}$ is not necessarily an algebra, but $\mathscr{F}$ contains the singletons from X, both this fact and the next proposition follow from the work in [4].

PROPOSITION 3.2. *Under the usual hypotheses, suppose* $F = G_0 \cup \cdots \cup G_n$, *where* $\{F, G_0, \ldots, G_n\} \subseteq \mathscr{A}_{\mathscr{V}}$ *and the union is disjoint. Then* $P(F) = P(G_0) + \cdots + P(G_n)$.

PROOF. As just noted, we need only consider the case where $\mathscr{V}$ is an algebra. Induct on n. When $n = 0$ this is just Proposition 3.1. We may thus suppose that $n \geq 1$ and $F \neq \phi$. Let $F = \mathscr{F}^A_{B_0 \cdots B_m}$. Arguing as in Proposition 2.3, we can show that one of the G_i's, say G_0, has the form $\mathscr{F}^A_{D_0 \cdots D_k}$. A similar argument shows that for some $D \in \mathscr{V}$ and some $i \leq k$ with $D_i \subseteq D$, G_1 has the form $\mathscr{F}^{A \cup D}_{E_0 \cdots E_l}$. Let $F_0 = \mathscr{F}^D$, $F_1 = \mathscr{F}_D$. The definition of the functions S_n guarantees that $P(G) = P(G \cap F_0) + P(G \cap F_1)$ for all $G \in \mathscr{A}_{\mathscr{V}}$. Note that $F \cap F_0 = \bigcup_{i \leq n} G_i \cap F_0 = \bigcup_{i \neq 0} G_i \cap F_0$ since $G_0 \cap F_0 = \phi$; similarly, $F \cap F_1 = \bigcup_{i \neq 1} G_i \cap G_0$. Thus

$$P(F) = P(F \cap F_0) + P(F \cap F_1) = \sum_{i \neq 0} P(G_i \cap F_0) + \sum_{i \neq 1} P(G_i \cap F_1)$$

$$= \sum_{i \leq n} P(G_i \cap F_0) + P(G_i \cap F_1) = \sum_{i \leq n} P(G_i),$$

as desired. □

PROPOSITION 3.3. *Under the same hypotheses, suppose* $F_0 \cup \cdots \cup F_m = G_0 \cup \cdots \cup G_n$, *where* $\{F_0, \ldots, F_m, G_0, \ldots, G_n\} \subseteq \mathscr{A}_{\mathscr{V}}$ *and the unions are disjoint. Then* $P(F_0) + \cdots + P(F_m) = P(G_0) + \cdots + P(G_n)$.

PROOF. When $m = 0$ or $n = 0$ this is just Proposition 3.2. Thus,

$$\sum_{i \leq m} P(F_i) = \sum_{i \leq m} \sum_{j \leq n} P(F_i \cap G_j) = \sum_{j \leq n} \sum_{i \leq m} P(F_i \cap G_j) = \sum_{j \leq n} P(G_j). \ \square$$

The following is now immediate.

THEOREM 3.4. *Under the usual hypotheses, let* T *be a Choquet capacity on* X; *then there is a finitely additive set function* $P\colon \mathscr{A}^f_{\mathscr{V}} \to [0,1]$ *such that* $P(\mathscr{F}^A_{B_0 \cdots B_n}) = S_{n+1}(A|B_0 \cdots B_n)$.

The following corollary follows now from the Caratheodory Extension Theorem ([6]).

COROLLARY 3.5. *Suppose P is the set function from Theorem 3.4, and suppose that whenever $\{F_n: n \in \mathbb{N}\}$ is a sequence from $\mathscr{A}_{\mathscr{V}}^{f}$ decreasing to ϕ, $\lim_{n\to\infty} P(F_n) = 0$. Then P can be extended to a probability measure on the smallest σ-algebra containing $\mathscr{A}_{\mathscr{V}}^{f}$.*

4. $\cap$ c-Pavings.

THEOREM 4.1. *In addition to the usual hypotheses, suppose that $\mathscr{F}$ is a $\cap c$-paving and that $\{V^c: V \in \mathscr{V}\} \in \mathscr{F}$. Then the conclusion of Corollary 3.5 holds.*

The proof proceeds in several stages. Suppose $\{F_n: n \in \mathbb{N}\}$ is a sequence from $\mathscr{A}_{\mathscr{V}}^{f}$ decreasing to ϕ. We may write $F_n = F_n^0 \cup \cdots \cup F_n^{k_n}$, where for all i, j, and n, if $F_{n+1}^i \cap F_n^j \neq \phi$ then $F_{n+1}^i \subseteq F_n^j$. Call a function τ: $\mathbb{N} \to \mathbb{N}$ a *branch* if $\{F_n^{\tau(n)}: n \in \mathbb{N}\}$ is a maximal subset of $\{F_n^i: n \in \mathbb{N}, i \leq k_n\}$ which is linearly ordered by reverse inclusion. Let $\mathbb{T}$ be the set of all branches. If $\tau, \sigma \in \mathbb{T}$, write $\sigma \sim_n \tau$ if $\sigma(i) = \tau(i)$ for all $i \leq n$.

LEMMA 4.2. *Suppose $\mathbb{T}' \subseteq \mathbb{T}$, that θ: $T' \to \mathbb{N}$, and that whenever $\tau \sim_{\theta(\sigma)} \sigma$, $\theta(\tau) = \theta(\sigma)$. Then θ has finite range.*

PROOF. This is an easy consequence of König's lemma ([3], Theorem 4.7). □

For each branch τ and each $n \in \mathbb{N}$, we may write $F_n^{\tau(n)} = \mathscr{F}_{B_0^\tau \cdots B_{k_{\tau,n}}^\tau}^{A_n^\tau}$, where for fixed τ, $A_n^\tau \subseteq A_{n+1}^\tau$ and $k_{\tau,n} \leq k_{\tau,n+1}$.

LEMMA 4.3. *Let τ be a branch. Then $\lim_{n\to\infty} P(F_n^{\tau(n)}) = 0$.*

PROOF. Let $A = (\bigcup_{n=0}^{\infty} A_n^\tau)^c \in \mathscr{F}$. Since $\bigcap_{n=0}^{\infty} F_n = \phi$, $A \notin \bigcap_{n=0}^{\infty} F_n^{\tau(n)}$, so A^c contains some B_i^τ. It follows that for n sufficiently large,

$$P(F_n^{\tau(n)}) = S_{k_{\tau,n}+1}(A_n^\tau | B_0^\tau, \dots, B_{k_{\tau,n}}^\tau) \leq S_1(A_n^\tau | B_{k_{\tau,n}}^\tau) = S_0(A_n^\tau) - S_0(A_n^\tau \cup B_i^\tau),$$

which goes to 0 as $n \to \infty$. □

In the proof of Lemma 4.3, say that τ is *annihilated by* B_i^τ. There is a countable collection $\mathscr{E} = \{E_n: n \in \mathbb{N}\}$ such that every branch is annihilated by some E_i.

Let $\mathbb{T}_i = \{\tau \in \mathbb{T}: \tau$ is annihilated by $E_i\}$ and $[\mathbb{T}_i]_n = \mathscr{F}_{E_i}^{A}$, where $A = \bigcap_{\tau \in \mathbb{T}_i} A_n^\tau$.

LEMMA 4.4. *For any $i \in \mathbb{N}$, there is an $N \in \mathbb{N}$ such that $\bigcup_{\tau \in \mathbb{T}_i} F_n^{t(n)} \subseteq [\mathbb{T}_i]_n$ for $n \geq N$.*

PROOF. Define θ on $\mathbb{T}_\iota$ by $\theta(\tau) =$ least n with $E_i \in \{B_0^\tau, \dots, B_{k_{\tau,n}}^\tau\}$. The hypotheses of Lemma 4.2 are easily verified, so for some $N \in \mathbb{N}$, $\theta(\tau) \le N$ for all $\tau \in \mathbb{T}_\iota$. If $n \ge N$ and $\tau \in \mathbb{T}_\iota$, $F_n^{\tau(n)} \subseteq \mathscr{F}_{E_\iota}^{A_n^\tau} \subseteq [\mathbb{T}_i]_n$, as desired. □

LEMMA 4.5. *For any* $i \in \mathbb{N}$, $\lim_{n\to\infty} P([\mathbb{T}_i]_n) = 0$.

PROOF. Since $[\mathbb{T}_\iota]_n \in \mathscr{A}_{\mathscr{V}}$, it suffices by the argument of Lemma 4.3 to show that $\bigcap_{n=0}^\infty [\mathbb{T}_\iota]_n = \phi$.

Suppose not; then there is an $x \in E_i \setminus \bigcup_{n=0}^\infty \bigcap_{\tau\in\mathbb{T}_\iota} A_n^\tau$. For $\tau \in \mathbb{T}_i$ let $\theta(\tau)$ be the least n with $E_i \in \{B_0^\tau, \dots, B_{k_{\tau,n}}^\tau\}$ and $x \in A_n^\tau$. (Such an n exists since E_i annihilates τ.) By Lemma 4.2 find $N \in \mathbb{N}$ with $\theta(\tau) \le N$ for all $\tau \in \mathbb{T}_i$. Then $x \in \bigcap_{\tau\in\mathbb{T}_\iota} A_N^\tau$, a contradiction. □

PROOF OF THEOREM 4.1. By Corollary 3.5, it suffices to show that $\lim_{n\to\infty} P(F_n) = 0$. Fix $\varepsilon > 0$, and for $n \in \mathbb{N}$ let $\varepsilon_n = \varepsilon \cdot 2^{-(n+1)}$. Define θ on $\mathbb{T}$ by $\theta(\tau) =$ least n such that for some $i \in \mathbb{N}$ and $\sigma \in \mathbb{T}_i$, $\sigma \sim_n \tau$ and $P([\mathbb{T}_i]_n) < \varepsilon_i$. By Lemma 4.2, there is an $N \in \mathbb{N}$ with $\theta(\tau) \le N$ for all $\tau \in \mathbb{T}$. Let I be a finite subset of $\mathbb{N}$ such that for each $\tau \in \mathbb{T}$ there exists an $i \in I$ and $\sigma \in \mathbb{T}_\iota$ with $P([\mathbb{T}_\iota]_N) < \varepsilon_\iota$ and $\sigma \sim_{\theta(\tau)} \tau$. Then for $n > N$, $P(F_n) \le P(\bigcup_{i\in I} [\mathbb{T}_\iota]_N) \le \sum_{i\in I} \varepsilon_i \le \varepsilon$, as desired. □

5. Applications. We conclude with three examples where the hypotheses of Theorem 4.1 are satisfied.

EXAMPLE 5.1. Suppose X is a topological space, $\mathscr{V}$ consists of the open sets of X, and $\mathscr{F}$ consists of the closed sets of X. Theorem 4.1 holds for any Choquet capacity T; the probability space that results is a *random closed set.*

(Our notion of a random closed set differs from that in [4]. The space X there must be second countable; ours need not. On the other hand, the probability measure P is defined there on a wider class of measurable sets. The two notions are equivalent when X is second countable.)

EXAMPLE 5.2. Suppose $\mathscr{V}$ is an algebra, and $\mathscr{F}$ is a σ-algebra containing $\mathscr{V}$. Again, Theorem 4.1 holds; the random element of $\mathscr{F}$ is a *random measurable set.*

Suppose in this example that $\underline{X}$ is a topological space, with open sets $\overline{\mathscr{V}}$ and closed sets $\overline{\mathscr{F}}$, and suppose that $\overline{\mathscr{V}} \subseteq \mathscr{V}$. The map Λ: $\mathscr{F} \to \overline{\mathscr{F}}$, where $\Lambda(E)$ is the closure of E, is evidently $\mathscr{A}_{\mathscr{V}}^{\mathscr{I}} - \mathscr{A}_{\overline{\mathscr{V}}}^{\mathscr{I}}$ measurable, and in fact measure-preserving. This illustrates the connection between random measurable sets and random closed sets.

EXAMPLE 5.3. As a special case of Example 5.2, suppose $(X, \mathscr{F}, m)$ is an infinite measure space, $\mathscr{V} = \mathscr{F}$, $\lambda > 0$, and T is defined on $\mathscr{F}$ by

$$T(A) = \begin{cases} 1 - \exp(-\lambda \cdot m(A)) & \text{if } m(A) < \infty \\ 1 & \text{otherwise.} \end{cases}$$

A standard computation verifies that T is a capacity on X. The probability measure produced is called a *generalized Poisson process*.

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