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THE SYMMETRY GROUP AND EXPONENTS OF OPERATOR STABLE PROBABILITY MEASURES

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There exist exponents of an operator stable measure which commute with every operator in the measure's symmetry group. These exponents together with a new norm lead to some simplifications in the representation of the Lévy measure.

0. Introduction. An operator-stable (OS) probability measure μ on a finite-dimensional real vector space V is the limit distribution of operator normed and centered sums of a sequence of i.i.d. random vectors in V. The classical stable laws on $\mathbb{R}^1$ are a special case. If μ is full and operator stable, then μ is infinitely divisible so if $\hat{\mu}$ is the ch.f. of μ, then for $t > 0$, $\hat{\mu}^t$ is the ch.f. of an infinitely divisible measure μ^t. The role of the index in the one-dimensional case is played by an invertible linear operator B on V called the exponent of μ. If we define $t^B = \exp\{(\mathrm{Ln}\, t)B\} = \sum_{j=0}^{\infty}(\mathrm{Ln}\, t)^j B^j/j!$, then B is an *exponent* for μ if

$$(1) \qquad \mu^t = t^B\mu * \delta(b(t)), \qquad t > 0,$$

where $\delta(b(t))$ is the unit mass at $b(t) \in V$ and $t^B\mu = \mu t^{-B}$. In [7] it was proved that full OS distributions always have at least one exponent.

An exponent of a full OS law μ determines much of its structure. (See [2] and [7] for the results which are now described.) In general μ has both a Gaussian component μ_g and a Poisson component μ_p. These components are concentrated on independent subspaces determined by the exponent B. To be precise let $f(x)$ denote the minimal polynomial of B. Then $f(x) = g(x)h(x)$ where the roots of g have real parts equal to $\frac{1}{2}$ while those of h have real parts greater than $\frac{1}{2}$. The Gaussian component μ_g is concentrated on $V_g = \mathrm{kernel}(g(B))$ while μ_p is concentrated on $V_p = \mathrm{kernel}(h(B))$. Furthermore, $V = V_g \oplus V_p$, μ_g and μ_p are full and OS on V_g and V_p, respectively. The exponents of μ_g and μ_p are the restrictions of B to V_g and V_p, respectively. Now let M denote the Lévy measure of μ. The exponent determines a major part of the structure of M. From (1) upon noting that $t \cdot M$ is the Lévy measure of μ^t and that $t^BM = Mt^{-B}$ is the Lévy measure of $t^B\mu$, one sees that $t \cdot M = t^BM$. This fact can be used to show that if

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A is a Borel subset of V_p, then

$$M(A) = \int_L M_x(A)K(dx), \tag{2}$$

where K is a finite measure on a Borel subset L of the unit sphere U in V_p and M_x is concentrated on the single orbit $\{t^B x\colon t > 0\}$ determined by x. The Lévy measure M_x also satisfies the condition that $t \cdot M_x = t^B M_x$ and as a result,

$$M_x\{t^B x\colon t > s\} = 1/s, \qquad s > 0$$

[i.e., $M_x(A) = \int_0^\infty I_A(t^B x)t^{-2}\,dt$]. From (2) it follows that the support of M is the union of orbits of t^B. Each orbit begins at the origin and extends to infinity [i.e., $\lim_{t\to 0} t^B x = 0$ and $\lim_{t\to\infty}\|t^B x\| = \infty$]. The shape of these orbits is determined by the exponent B. In particular cases orbits can be straight lines ($B = \lambda I$), half of a parabola [$B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $V = \mathbb{R}^2$], or spirals [e.g., $B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = I + Q$ where $Q + Q^* = 0$ so t^Q is a rotation]. The expression for M_x above shows that the tail behavior of M along orbits is determined by B. The measure K assigns weights to the orbits and determines which orbits are included in the support of M. Together B and K determine M. But, in general B and K are not unique. Is there a reasonable way to choose a particular exponent and measure K? The set of exponents depends on the amount of symmetry possessed by μ. Call a linear operator A on V a symmetry of μ if for some $a \in V$, $\mu = A\mu * \delta(a)$. It is natural to expect that a symmetry of μ should take orbits into orbits while leaving K invariant. (See Theorem 7 below.) In particular, if $BA = AB$, then $At^B x = t^B Ax$ (since t^B is a power series in B) so orbits are taken by A into orbits. Furthermore, the requirement that B commutes with every symmetry tends to pick out exponents with nice properties whenever possible. (See Theorems 4 and 5.)

EXAMPLE. Suppose that μ is the standard Gaussian measure on $\mathbb{R}^d$. If X and Y are i.i.d. μ, the measure corresponding to $X + Y$ is $\mu * \mu = 2^{1/2}\mu$. One suspects (and easily verifies) that $\frac{1}{2}I$ is an exponent for μ. Suppose that S is a skew operator, that is, that $S + S^* = 0$. For each $t > 0$, t^S is orthogonal and so $t^S\mu = \mu$, i.e., t^S is a symmetry of μ. It follows that $\frac{1}{2}I + S$ is also an exponent for μ, for *any* skew operator S (see Theorem 1). Thus operator stable measures may have many exponents; the number of exponents depends on the size of the collection of symmetries of μ. Does an operator stable measure have a "simplest" exponent?

A lemma of Schur's ([6], page 173) suggests a possible answer. This lemma says: "Let F be a family of linear operators on a Hilbert Space H and suppose that the only closed subspaces which are invariant under every operator in F are $\{0\}$ and H. If A is a self-adjoint linear operator on H that commutes with every operator in F, then $A = cI$ for some scalar c." (As usual, I denotes the identity operator.) Schur's lemma suggests that the "simplest" exponent would be one which commutes with a large collection of operators. In this example, $\frac{1}{2}I$ is the only exponent of μ which commutes with *every* symmetry of μ. We will show below that there is always an exponent of μ which commutes with all the symmetries of μ (Theorem 2).

Our results on commuting exponents are applied to simplify the representation of the Lévy measure of an OS law in Section 3. There we define a new norm. The unit sphere relative to this norm plays the role of L above. The corresponding mixing measure K does not depend on the choice of an exponent (Theorem 6). This representation provides a simple relationship between the symmetries of μ and those of K. These results complement those of Kucharczak [5], Jurek [3], and Hudson and Mason [2].

1. Preliminaries. Let μ be a full OS probability measure on a finite dimensional real vector space V. $GL(V)$ denotes the set of all invertible operators on V. For $A \in GL(V)$, we define $A\mu = \mu \circ A^{-1}$. Two groups of interest in connection with μ are the *symmetry group*

$$S(\mu) = \{A \in GL(V) \colon A\mu * \delta(a) = \mu \text{ for some } a \in V\}$$

and

$$G = \{A \in GL(V) \colon \text{for some } t > 0, \text{for some } a \in V, \mu^t = A\mu * \delta(a)\}.$$

It is known that $S(\mu)$ is a compact, normal subgroup of G. For any closed group H, TH will denote the *tangent space* of H. Thus $A \in TH$ if and only if $A = \lim_{n \to \infty}(H_n - I)/d_n$ where $\{H_n\} \subset H$ and $\{d_n\}$ is a real null sequence. We recall that the exponential maps TH onto the connected component of I in H. CH will denote the *center* of H, that is, those elements of H which commute with every element of H. Recall that CH is a subgroup of H.

The collection of exponents of μ, denoted $E(\mu)$, is the set of all operators for which (1) holds. The following result gives a basic fact about exponents.

THEOREM 1. *Let $B \in E(\mu)$. Then*

(i) *every eigenvalue of B has real part $\geq \frac{1}{2}$,*
(ii) $E(\mu) = B + TS(\mu)$.

For a proof of this result see [1] and [7].

2. Commuting exponents. In this section we investigate the existence of an exponent which commutes with *every* operator in $S(\mu)$. Such exponents will be called *commuting* and the collection of commuting exponents will be denoted by $E_c(\mu)$.

PROPOSITION 1. *Let $A \in S(\mu)$ and $B \in E(\mu)$. Then $ABA^{-1} \in E(\mu)$. Moreover, if $S(\mu)$ is discrete, the unique exponent B is commuting.*

PROOF. We have $A\mu = \mu * \delta(a)$ and

$$(A\mu)^t = A\mu^t = A(t^B\mu * \delta(b(t))) = At^B\mu * \delta(Ab(t)) = t^{ABA^{-1}}(A\mu) * \delta(Ab(t)).$$

Hence

$$\mu^t = t^{ABA^{-1}}\mu * \delta\big(Ab(t) - ta + t^{ABA^{-1}}a\big)$$

and $ABA^{-1} \in E(\mu)$. Now if $S(\mu)$ is discrete, $TS(\mu) = 0$ and B is the unique exponent. Thus $ABA^{-1} = B$ and B is commuting. □

The following example shows that not all exponents are commuting.

EXAMPLE. Let μ be the symmetric Cauchy distribution on R^2. Then $I \in E(\mu)$ and $S(\mu)$ is the full orthogonal group. Hence $TS(\mu)$ consists of the skew symmetric operators. By Theorem 1, $E(\mu) = I + TS(\mu)$ so $B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ is an exponent. Also $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in S(\mu)$. A direct computation shows that $AB \neq BA$. Furthermore, A does not map orbits into other orbits.

The main result of this section is that commuting exponents always exist.

THEOREM 2. *$E_c(\mu)$ is nonempty.*

PROOF. Let H be a Haar probability measure on the compact group $S(\mu)$, and let $B \in E(\mu)$. Define

$$M = \int_{S(\mu)} sBs^{-1}\, dH(s).$$

Since $E(\mu)$ is closed and convex by Theorem 1 and closed under conjugation by elements of $S(\mu)$ by Proposition 1, $M \in E(\mu)$. If $A \in S(\mu)$, then by the invariance property of Haar measure

$$\begin{aligned} AMA^{-1} &= \int_{S(\mu)} AsBs^{-1}A^{-1}\, dH(s) \\ &= \int_{S(\mu)} (As)B(As)^{-1}\, dH(s) \\ &= \int_{S(\mu)} sBs^{-1}\, dH(s) = M. \end{aligned}$$

Thus $M \in E_c(\mu)$. □

The collection of all commuting exponents is characterized in our next result.

THEOREM 3. *Suppose $B \in E_c(\mu)$. Then $E_c(\mu) = B + TCS(\mu)$.*

PROOF. Assume $B \in E_c(\mu)$. Using the relation between groups and their tangent spaces one readily verifies the equivalence of the following statements.

(i) $\tilde{B} \in E_c(\mu)$,
(ii) $\tilde{B} - B \in TS(\mu)$ and $\tilde{B} - B$ commutes with every element of $S(\mu)$.
(iii) For all t, $\exp\{t(\tilde{B} - B)\} \in CS(\mu)$, and
(iv) $\tilde{B} - B \in TCS(\mu)$. □

COROLLARY. *$E_c(\mu) = E(\mu)$ if and only if $TS(\mu) = TCS(\mu)$.*

We now examine the extent to which the structure of a commuting exponent is determined by the "size" of $S(\mu)$.

THEOREM 4. *Let $B \in E_c(\mu)$. If the only proper subspace of V invariant under $S(\mu)$ is 0, then $B = \lambda I + WQW^{-1}$, where W is positive definite and Q is skew-symmetric. Furthermore, either $Q = 0$ or $Q^2 = WQ^2W^{-1} = -\beta^2 I$ for some $\beta > 0$.*

PROOF. Since $S(\mu)$ is compact, there is a positive definite operator W and a closed subgroup G of the orthogonal group such that

$$S(\mu) = WGW^{-1}.$$

It follows that $S(W^{-1}\mu) = G$. Since $B \in E_c(\mu)$, $B_0 \equiv W^{-1}BW \in E_c(W^{-1}\mu)$. Write $B_0 = B_1 + B_2$ where $B_1 = \frac{1}{2}(B_0 + B_0^*)$ is self-adjoint and $B_2 = \frac{1}{2}(B_0 - B_0^*)$ is skew-symmetric. Since $B_0 \in E_c(W^{-1}\mu)$, $AB_0 = B_0A$ for $A \in G$. Take adjoints to see that $B_0^*A^* = A^*B_0^*$ for $A \in G$. But every operator in G is orthogonal so $G = \{A^*\colon A \in G\}$. Thus

$$AB_0^* = B_0^*A, \qquad A \in G.$$

It follows that every operator in G commutes with B_1 which is self-adjoint. Now by hypothesis the only proper subspace of V invariant under $S(\mu)$ and hence under G is 0. By Schur's lemma, $B_1 = \lambda I$ for some real number λ. Now consider B_2. Since B_2 is skew-symmetric, it is normal and thus its minimal polynomial is the product $p_1(x), \dots, p_k(x)$ of distinct irreducible polynomials. If $k > 1$, then $\ker p_1(B_2)$ is a proper subspace of V which is invariant under G contrary to our hypothesis. Thus $k = 1$ and the minimal polynomial of B_2 is either x or $x^2 + \beta^2$ for some $\beta > 0$. (A skew-symmetric operator has purely imaginary eigenvalues.) If it is x, then $B_2 = 0$; otherwise, $B_2^2 = -\beta^2 I$. From $B_0 = B_1 + B_2 = \lambda I + B_2$, we obtain upon setting $Q = B_2$,

$$W^{-1}BW = \lambda I + Q$$

or $B = \lambda I + WQW^{-1}$. Finally $B \in E(\mu)$ so the real part of every eigenvalue of B is not less than $\frac{1}{2}$, i.e., $\lambda \geq \frac{1}{2}$. □

COROLLARY. *If in addition to the hypothesis of the theorem, either B is diagonalizable or* $\dim V$ *is odd, then $B = \lambda I$.*

PROOF. First suppose B is diagonalizable. Let v be an arbitrary eigenvector of B so $Bv = \lambda_0 v$ for some real λ_0. By Theorem 4, $B = \lambda I + WQW^{-1}$ so v is an eigenvector of WQW^{-1}. In particular, $WQW^{-1}v = (\lambda_0 - \lambda)v$. Hence $(WQW^{-1})^2 v = (\lambda_0 - \lambda)^2 v$. But $WQW^{-1} = 0$ or $(WQW^{-1})^2 = -\beta^2 I$. In either case it follows that $\lambda_0 = \lambda$. Since B is diagonalizable, $B = \lambda I$. Now suppose $\dim V$ is odd. Since Q is skew-symmetric,

$$\det Q = \det Q^* = \det(-Q) = -\det Q,$$

so Q is singular. Hence $Q^2 \neq -\beta^2 I$ and therefore $Q = 0$. □

A slight refinement of the preceding theorem is given in

THEOREM 5. *Suppose $B \in E_c(\mu)$ has p real eigenvalues $\lambda_1, \dots, \lambda_p$ with corresponding eigenvectors $v_1, \dots, v_p$. If $\{Av_i\colon A \in S(\mu), 1 \le i \le p\}$ spans V, then B is diagonalizable with spectrum $\{\lambda_1, \dots, \lambda_p\}$. Thus if $\lambda_1 = \cdots = \lambda_p = \lambda$, $B = \lambda I$.*

PROOF. For $A \in S(\mu)$, $BAv_i = ABv_i = \lambda_i Av_i$, so Av_i is an eigenvector of B with eigenvalue λ_i. Hence there is a basis of V consisting of eigenvectors of B and so B is diagonalizable. □

COROLLARY. *In R^2 if $B \in E_c(\mu)$ and if there is a reflection $A \in S(\mu)$, then B is self-adjoint.*

PROOF. Select orthornormal vectors v_1 and v_2 so that $Av_1 = v_1$ and $Av_2 = -v_2$. Then $ABv_1 = Bv_1$ and $ABv_2 = -Bv_2$, so $Bv_1 = \lambda_1 v_1$ and $Bv_2 = \lambda_2 v_2$ where λ_1 and λ_2 are real. □

3. The Lévy measure. In this section we discuss the relationship between commuting exponents and the representation of the Lévy measure of μ. Since μ is infinitely divisible, one can write the characteristic function of μ in the canonical form

$$\hat{\mu}(y) = \exp\left\{i\langle y, a\rangle - \tfrac{1}{2}\langle y, \Sigma y\rangle + \int \psi(x, y) M(dx)\right\},$$

where $a \in V$, Σ is a nonnegative definite self-adjoint operator, M is a σ-finite measure satisfying

$$\int_V \|x\|^2 \wedge 1 M(dx) < \infty,$$

and

$$\psi(x, y) = \exp\{i\langle x, y\rangle\} - 1 - \frac{i\langle x, y\rangle}{1 + \langle x, x\rangle}.$$

For OS measures it has been shown that one can further decompose the Lévy measure M as follows. For an exponent B of μ set $L_B = \{\|x\| = 1 \text{ and } \|t^B x\| > 1$ for all $t > 1\}$ and define the *mixing measure* K_B on the Borel subsets A of L_B by

$$K_B(A) = M\big(\{t^B x\colon x \in A, t \ge 1\}\big).$$

Thus K_B assigns mass to the particular orbits $\{t^B x\colon t > 0\}$. Note that both L_B and K_B depend on the choice of exponent B. In terms of K_B the Lévy measure M is given by

$$M(S) = \int_{L_B} \int_0^\infty I_S(t^B x) t^{-2}\, dt\, dK_B(x). \tag{3}$$

(See [2] and [3].) It was necessary to introduce the subset L_B of U since for some exponents, orbits may intersect the unit sphere more than once.

We now introduce a new norm $|||\cdot|||$ which depends on the particular OS law but not on the choice of exponent. The unit sphere $U' = \{v\colon |||v||| = 1\}$ induced by this norm will intersect each orbit once and so may play the role of L_B. As above we define a mixing measure K on the Borel subsets A of U' by $K(A) = M\{t^B x\colon x \in A, t \geq 1\}$. This measure K also does not depend on the choice of exponent and the representation (3) of the Lévy measure M in terms of K is still valid. The new norm leads to a system of "polar" coordinates with nice properties (cf. Jurek [4]).

For $x \in V$, and $B \in E(\mu)$ define $|||x||| = \int_0^1 \int_{S(\mu)} \|gt^B x\| H(dg) t^{-1}\, dt$ where H again denotes Haar measure on $S(\mu)$ and $\|\cdot\|$ is the original norm on V.

PROPOSITION 2. *If μ is full and OS on V, then*

(i) $|||\cdot|||$ *does not depend on the choice of* $B \in E(\mu)$,
(ii) $|||\cdot|||$ *is a norm on* V,
(iii) *for* $A \in S(\mu)$, $|||Ax||| = |||x|||$,
(iv) $t \to |||t^B x|||$ *is strictly increasing on* $(0, \infty)$ *for each* $x \neq 0$, *and*
(v) *the map* $\Phi_B\colon U' \times (0, \infty) \to V \setminus \{0\}$ *defined by* $\Phi_B(x, t) = t^B x$ *is a homeomorphism when* $U' \times (0, \infty)$ *has the product topology.*

PROOF. (i) Let $B \in E(\mu)$ and let $B_0 \in E_c(\mu)$. By Theorem 1, $B - B_0 \in TS(\mu)$ so for all $t > 0$, $B_0 t^{B - B_0} = t^{B - B_0} B_0$. Differentiate to see that B_0 commutes with $B - B_0$ and consequently that $t^B = t^{B - B_0} t^{B_0}$. For $x \in V$, use the invariance property of Haar measure to obtain the equalities

$$
\begin{aligned}
|||x|||_B &= \int_0^1 \int_{S(\mu)} \|gt^B x\| t^{-1} H(dg)\, dt \\
&= \int_0^1 \int_{S(\mu)} \|gt^{B - B_0} t^{B_0} x\| t^{-1} H(dg)\, dt = |||x|||_{B_0}.
\end{aligned}
$$

This proves (i) and allows us to omit the subscript B.

(ii) This is obvious.

(iii) Let $A \in S(\mu)$. By (i) we may assume that $B \in E_c(\mu)$. Then

$$
\begin{aligned}
|||Ax||| &= \int_0^1 \int_{S(\mu)} \|gt^B Ax\| t^{-1} H(dg)\, dt \\
&= \int_0^1 \int_{S(\mu)} \|gAt^B x\| t^{-1} H(dg)\, dt = |||x|||.
\end{aligned}
$$

(iv) Suppose that $0 < r < s$. Then

$$
\begin{aligned}
|||r^B x||| &= \int_0^1 \int_{S(\mu)} \|g(tr)^B x\| t^{-1} H(dg)\, dt \\
&= \int_0^r \int_{S(\mu)} \|gu^B x\| u^{-1} H(dg)\, du \\
&< \int_0^s \int_{S(\mu)} \|gu^B x\| u^{-1} H(dg)\, du = |||s^B x|||.
\end{aligned}
$$

(v) It follows from (iv) that Φ_B is one-to-one. Since every point in $V \setminus \{0\}$ lies on some orbit, Φ_B is "onto." The continuity of Φ_B is well known and easily checked. To show Φ_B^{-1} is continuous write $\Phi_B^{-1}(x) = (l(x), \zeta(x))$ so that $l(x) \in U'$, $\zeta(x) > 0$, and $x = \zeta(x)^B l(x)$. Suppose that $x_n \to x \neq 0$. Assume some subsequence $\zeta(x_{n'})$ tends to infinity. Then since the eigenvalues of B have positive real parts, $\|x_{n'}\| = \|\zeta(x_{n'})^B l(x_{n'})\| \to \infty$ contrary to the convergence of x_n. It follows that $(l(x_n), \zeta(x_n))$ is a bounded sequence in $U' \times (0, \infty)$. Let $(l(x_{n'}), \zeta(x_{n'}))$ be any convergent subsequence and let $(l_0, \zeta_0) = \lim(l(x_{n'}), \zeta(x_{n'}))$. Then

$$x = \lim x_{n'} = \lim \zeta(x_{n'})^B l(x_{n'}) = \zeta_0^B l_0.$$

Since Φ_B is one-to-one, $\zeta(x) = \zeta_0$ and $l(x) = l_0$. Thus every convergent subsequence of $(l(x_n), \zeta(x_n))$ has the same limit, namely $(l(x), \zeta(x))$. This proves that Φ_B^{-1} is continuous. □

The proof that Φ_B^{-1} is continuous was given above for the sake of completeness, cf. [4].

Part (iv) of Proposition 2 implies that each orbit intersects U' exactly once. The fact that U' is closed and that Φ_B is a homeomorphism is useful in proving weak convergence results.

THEOREM 6. *Let μ be full OS with Lévy measure M and let $B \in E(\mu)$. Let F and E be any Borel subsets of $V \setminus \{0\}$ and U', respectively. Then*

$$M(F) = \int_{U'} \int_0^\infty I_F(s^B x) s^{-2}\, ds\, K(dx), \tag{4}$$

where K is a finite Borel measure on U' and

$$K(E) = M\{t^B x \colon x \in E, t \geq 1\}. \tag{5}$$

The measure K does not depend on the choice of $B \in E(\mu)$.

PROOF. The proof of (4) and (5) is similar to that of (3) in [2] or [3] and is therefore omitted.

The proof that K does not depend on the choice of exponent will involve an easy lemma.

LEMMA 1. *Let $g \in S(\mu)$ and $B \in E(\mu)$. If $gB = Bg$, then $gK_B = K_B$.*

PROOF. Let D be any Borel subset of U'. Then

$$\begin{aligned} gK_B(D) &= K_B(g^{-1}(D)) \\ &= M\{t^B x \colon x \in g^{-1}(D), t \geq 1\} \\ &= M\{t^B g^{-1} x \colon x \in D, t \geq 1\} \\ &= M(g^{-1}\{t^B x \colon x \in D, t \geq 1\}) \\ &= (gM)(\{t^B x \colon x \in D, t \geq 1\}). \end{aligned}$$

But $g \in S(\mu)$ and hence $gM = M$. Thus

$$gK_B(D) = M\{t^B x \colon x \in D, t \geq 1\} = K_B(D). \quad \square$$

Now let A be any Borel subset of $V \setminus \{0\}$. Then if $B \in E(\mu)$

$$M(A) = \int_0^\infty \int_{U'} I_A(t^B x) t^{-2} K_B(dx)\, dt$$
$$= \int_0^\infty K_B((t^{-B}A) \cap U') t^{-2}\, dt.$$

Let $B_0 \in E_c(\mu)$. It suffices to prove that $K_B = K_{B_0}$. So let D be any Borel subset of U', and put $A = \{s^B x\colon x \in D, s \geq 1\}$. Then

$$(t^{-B}A) \cap U' = \begin{cases} \phi & \text{if } t < 1, \\ D & \text{if } t \geq 1. \end{cases}$$

Hence

$$K_B(D) = M(A) = \int_0^\infty K_{B_0}((t^{-B_0}A) \cap U') t^{-2}\, dt.$$

But $B_0 \in E_c(\mu)$, so B_0 commutes with $B - B_0$. Furthermore $t^{B-B_0} \in S(\mu)$ and $t^{B-B_0}U' = U'$. It follows from Lemma 1 that

$$K_{B_0}((t^{-B}A) \cap U') = K_{B_0}((t^{B-B_0}(t^{-B}A)) \cap U')$$
$$= K_{B_0}((t^{-B_0}A) \cap U').$$

Therefore,

$$K_B(D) = \int_0^\infty K_{B_0}((t^{-B}A) \cap U') t^{-2}\, dt.$$
$$= \int_1^\infty K_{B_0}(D) t^{-2}\, dt = K_{B_0}(D). \quad \square$$

In the above, we have depended on the following "polar" representation. Namely, for each $x \neq 0$ and for each $B \in E(\mu)$, there is a unique $s > 0$ and $u \in U'$ such that $x = s^B u$. It can be shown that s does not depend on the choice of $B \in E(\mu)$. To see this let $B_0 \in E_c(\mu)$. Then for some $t \in 0$ and $v \in U'$

$$x = s^B u = t^{B_0} v.$$

Since B_0 commutes with B,

$$u = \left(\frac{t}{s}\right)^{B_0} s^{B_0 - B} v.$$

Now, $B_0 - B \in TS(\mu)$, so $s^{B_0 - B} \in U'$. But $u \in U'$ so by Proposition 2(iv) $t = s$.

Remark. There is a converse to Theorem 6. If B is an OS exponent, and if K is a finite Borel measure on $U' \cap V_p$, then the measure M defined by

$$M(F) = \int_{U'} \int_0^\infty I_F(s^B x) s^{-2}\, ds\, K(dx)$$

is the Lévy measure of some OS law with exponent B. Again see [2] or [3]. In [7] Sharpe characterized the set of OS exponents, i.e., those operators which are the exponent of some OS law.

We now consider the relationship between $S(K)$, the symmetry group of the measure K in Theorem 6, and $S(\mu)$.

THEOREM 7. *Let μ be a full* OS *measure on V. Then $S(\mu) \subset S(K)$.*

PROOF. Let $A \in S(\mu)$. Since by Proposition 2, $|||Ax||| = |||x|||$, $AU' = U'$. Since K does not depend on the choice of an exponent, we may assume $B \in E_c(\mu)$. Then $S(\mu) \subset S(K)$ follows from Lemma 1. □

The following example shows that even if an OS measure μ has no Gaussian component, if the original norm on V is used and if M is defined as in (3), then $S(K)$ may be much larger than $S(\mu)$ even though K is full. (To see that in this example μ has no Gaussian component, note that no eigenvalue of B has real part equal to $\frac{1}{2}$.)

EXAMPLE. Take $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Then L_B is the unit circle in R^2. Let K be the Lebesgue measure on the circle. Then K is full and $S(K)$ is the orthogonal group. Define M (and hence μ) in terms of K and B using equation (3). Then μ is a full OS measure with $B \in E(\mu)$ (see [2]). We now find $S(\mu)$. First note that $S(\mu)$ is closed and $V = R^2$ so if $S(\mu)$ were not discrete, $S(\mu)$ would be conjugate to the orthogonal group. Then by Theorem 4, B would have conjugate complex eigenvalues. Hence $S(\mu)$ is discrete, and $B \in E_c(\mu)$ by Proposition 1. Now suppose $D = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in S(\mu)$. Then since $B \in E_c(\mu)$ $BD = DB$ and so $c = b = 0$. Since $S(\mu)$ is a *compact* group, the fact that $D^n \in S(\mu)$ for all n shows $|a| = |d| = 1$. A direct computation now shows that $S(\mu) = S(M) = \left\{\pm I, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right\}$.

Added in proof. Instead of the norm $|||\cdot|||$, introduced in Section 3, one can also use the norm induced by the following inner product, $\langle x, y\rangle_1 := \int_0^1 \int_{S(\mu)} \langle gt^B x, gt^B y\rangle H(dg) t^{-1}\, dt$ where $\langle \cdot, \cdot \rangle$ is the original inner product on V.

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