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DECOUPLING INEQUALITIES FOR MULTILINEAR FORMS IN INDEPENDENT SYMMETRIC RANDOM VARIABLES

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Let $X^1, X^2, \ldots$ be independent copies of a sequence $X = (X_1, X_2, \ldots)$ of independent symmetric random variables. Let M be a symmetric multilinear form of rank s on $\mathbb{R}^{\mathbb{N}}$ whose components $a_{i_1, \ldots, i_s}$ relative to the standard basis of $\mathbb{R}^{\mathbb{N}}$ satisfy $a_{i_1, \ldots, i_s} = 0$ for all but finitely many multi-indices and whenever two indices agree. If ϕ is nondecreasing, convex, $\phi(0) = 0$ and ϕ satisfies a Δ_2 growth condition then

$$E\phi(|M(X, \ldots, X)|) \le cE\phi(|M(X^1, \ldots, X^s)|),$$

where c depends only on ϕ and s.

1. Introduction. Let $X = (X_1, X_2, \ldots)$ be an independent sequence of real-valued symmetric random variables. Let $\mathscr{M}_2$ denote the space of symmetric bilinear forms B on $\mathbb{R}^{\mathbb{N}}$ whose matrix $a = (a_{ij})$ with respect to the standard basis of $\mathbb{R}^{\mathbb{N}}$ satisfies $a_{ii} = 0$ for all i and $a_{ij} = 0$ for all but finitely many pairs (i, j). The quantity $B(X, X) = \sum_{i,j} a_{ij} X_i X_j$ is then a well-defined random variable.

Let $\tilde{X} = (\tilde{X}_1, \tilde{X}_2, \ldots)$ be an independent copy of X (assume both X and $\tilde{X}$ are defined on the same probability space). The goal of this paper is to establish the inequalities

$$cE|B(X, X)|^p \le E|B(X, \tilde{X})|^p \tag{1.1}$$

for $1 \le p < \infty$ and $B \in \mathscr{M}_2$, and more general results. The precise statements are given below. We call relations such as (1.1) *decoupling inequalities* because there is less dependence among the terms of $B(X, \tilde{X})$ than among the terms of $B(X, X)$.

Inequalities (1.1) continue to hold for bilinear forms B whose associated matrices contain infinitely many nonzero entries provided that we have almost sure convergence of the expressions

$$f_n = \sum_{i, j \le n} a_{ij} X_i \tilde{X}_j.$$

[This follows from (1.1), Fatou's lemma, and the observation that the f_n form a martingale.] It is not straightforward to formulate conditions on the a_{ij} for such convergence to take place, even if the X_i are i.i.d. See Varberg (1966) for the finite variance case and Cambanis et al. (1985) for the symmetric stable case.

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Inequalities (1.1) are a useful tool in the study of multiple stochastic integration since, if the X_i are i.i.d. symmetric stable random variables of index α, then $B(X, X)$ may be viewed as the double stochastic integral of an appropriate step function with respect to the Lévy symmetric stable process of index α. [See McConnell and Taqqu (1986) for this application.] In this special case Pisier has noted that (1.1) with $c = (2^{(2/\alpha)-1} - 1)^{-1}$ follows from the polarization identity,

$$2B(X, \tilde{X}) = B(X + \tilde{X}, X + \tilde{X}) - B(X, X) - B(\tilde{X}, \tilde{X}),$$

and the triangle inequality together with the observation that $X + \tilde{X}$ has the same distribution as $2^{1/\alpha}X$.

To state the results of the present paper in their full generality, let $\mathscr{M}_s$ denote the space of symmetric multilinear forms of rank s on $\mathbb{R}^{\mathbb{N}}$ whose components $a_{i_1}, \dots, {}_{i_s}$ relative to the standard basis of $\mathbb{R}^{\mathbb{N}}$ vanish for all but finitely many multi-indices, and also vanish whenever two multi-indices agree. Let $\mathscr{C}(\beta)$ denote the collection of nondecreasing convex functions ϕ which satisfy $\phi(0) = 0$ and the Δ_2 condition

$$\phi(2x) \le \beta\phi(x), \qquad x \ge 0. \tag{1.2}$$

(This class is empty unless $\beta \ge 2$.) Our main result is then the following.

THEOREM 1. *Let $X = (X_1, X_2, \dots)$ be a sequence of independent, symmetric random variables. For each $k = 1, 2, \dots,$ let $X^k = (X_1^k, X_2^k, \dots)$ be an independent copy of X. Then for each $\phi \in \mathscr{C}(\beta)$ and $M \in \mathscr{M}_s$ there is a constant $c = c(\beta, s)$ such that*

$$E\phi(|M(X, \dots, X)|) \le cE\phi(|M(X^1, \dots, X^s)|). \tag{1.3}$$

One source of technical difficulty in the proof of results like (1.3) is that the quantities on either side of the inequality sign do not define norms on $\mathscr{M}_s$. However, the theory of Orlicz spaces suggests several methods of constructing related norms. In particular, we follow Luxemburg (1955) and define seminorms $|\ |_1$ and $|\ |_2$ on $\mathscr{M}_s$ by

$$|M|_i = \inf\{\varepsilon > 0: \delta_i(M/\varepsilon) \le 1\}$$

for $i = 1$ and 2, where

$$\delta_1(M) = E\Phi(|M(X, \dots, X)|)$$

and

$$\delta_2(M) = E\Phi(|M(X^1, \dots, X^s)|).$$

The relations

$$|M|_i \le \max\{\beta\delta_i(M)^{1/\log_2\beta}, \delta_i(M)\} \tag{1.4}$$

and

$$\delta_i(M) \le \max\{\beta|M|_i^{\log_2\beta}, |M|_i\} \tag{1.5}$$

follow from the definition and the extension, $\phi(cx) \le \beta^{1+|\log_2 c|}\phi(x)$, of (1.2).

Inequality (1.3) then yields

COROLLARY 1. *There is a constant c_1 such that*

$$c_1|M|_1 \le |M|_2. \tag{1.6}$$

The constant c_1 depends only on β and s.

All of these results hold also for certain asymmetric multilinear forms. The following result contains an instance of this which is useful in the study of multiple stochastic integration. We shall use the notation

$$E\phi(|M(X,\dots,X)|) \approx E\phi(|M(X^1,\dots,X^s)|)$$

to mean that there are positive constants c_1 and c_2 independent of M such that

$$c_1E\phi(|M(X,\dots,X)|) \le E\phi(|M(X^1,\dots,X^s)|) \le c_2E\phi(|M(X,\dots,X)|).$$

THEOREM 2. *Let $M \in \mathcal{M}_s$ with components $a_{i_1,\dots,i_s}$. Let $\phi \in \mathscr{C}(\beta)$. Then*

$$\begin{aligned} E\phi(|M(X,\dots,X)|) &\approx E\phi\left(\left|\sum_{i_1<i_2<\cdots<i_s} a_{i_1,\dots,i_s}X_{i_1}\cdots X_{i_s}\right|\right) \\ &\le cE\phi\left(\left|\sum_{i_1<i_2<\cdots<i_s} a_{i_1,\dots,i_s}X^1_{i_1}\cdots X^s_{i_s}\right|\right) \\ &\approx E\phi(|M(X^1,\dots,X^s)|), \end{aligned}$$

where the constant c and the constants in $\approx$ depend only on β and s.

All the results described above continue to hold in a more general setting, for example, for random variables valued in ζ-convex Banach algebras. Details will appear elsewhere.

The paper is divided into three sections. In the second section we collect some preliminary inequalities for Rademacher functions which will be used in the proof of Theorem 1. This theorem, together with Corollary 1, is proved in Section 3.

For notational convenience we shall consider only the rank 2 case ($s = 2$) in the remainder. All proofs and preliminary results have natural extensions to the case of arbitrary finite rank.

Throughout the paper X will denote a symmetric sequence of random variables and $\tilde{X}$ an independent copy of X. The letter C will denote a constant, perhaps different from line to line.

2. Inequalities for Rademacher functions.

LEMMA 2.1. *Let $\{X_i\}$ be a symmetric sequence of random variables and let $\{\tilde{X}_i\}$ be an independent copy of $\{X_i\}$. Then for any $\phi \in \mathscr{C}(\beta)$, we have*

$$CE\phi\left(\left|\sum_{i<j} a_{ij}X_i\tilde{X}_j\right|\right) \le E\phi\left(\left|\sum_{i\ne j} a_{ij}X_i\tilde{X}_j\right|\right) \le \beta E\phi\left(\left|\sum_{i<j} a_{ij}X_i\tilde{X}_j\right|\right),$$

where the a_{ij} are symmetric and finitely many of them are nonzero.

PROOF. The inequality on the right-hand side follows from the Δ_2 inequality (1.2) and Jensen's inequality.

To prove the inequality on the left-hand side, it is enough to consider the case of the Rademacher functions (see below) since the X_i are symmetric. By the Burkholder–Davis–Gundy square function inequalities (see, e.g., Burkholder (1973), Theorem 15.1), and Lévy's inequality,

$$\begin{aligned} E\phi\left(\sum_{j=1}^{\infty}\left(\sum_{i\neq j} a_{ij} r_i\right)\tilde{r}_j\right) &\approx E\phi\left(\left(\sum_{j=1}^{\infty}\left(\sum_{i\neq j} a_{ij} r_i\right)^2\right)^{1/2}\right) \\ &= E\phi\left(\left\|\sum_{j=1}^{\infty}\left(\sum_{i\neq j} a_{ij} r_i\right) e_j\right\|_{l^2}\right) \qquad (2.1) \\ &= E\phi\left(\left\|\sum_{i=1}^{\infty}\left(\sum_{j\neq i} a_{ij} e_j\right) r_i\right\|_{l^2}\right), \end{aligned}$$

where e_j denotes the standard basis of l^2. Since the square function inequalities hold also for Hilbert space-valued martingales (with the same proof) we obtain

$$\begin{aligned} E\phi\left(\left\|\sum_{i=1}^{\infty}\left(\sum_{j\neq i} a_{ij} e_j\right) r_i\right\|_{l^2}\right) &\approx \phi\left(\left(\sum_{i=1}^{\infty}\left\|\sum_{j\neq i} a_{ij} e_j\right\|_{l^2}^2\right)^{1/2}\right) \\ &= \phi\left(\left(\sum_{i=1}^{\infty}\sum_{j\neq i} a_{ij}^2\right)^{1/2}\right) \\ &\geq \phi\left(\left(\sum_{i=1}^{\infty}\sum_{j> i} a_{ij}^2\right)^{1/2}\right). \end{aligned}$$

The desired left-hand inequality now follows by twice again applying the square function inequalities. □

Let $r_1, r_2, \dots$ be the Rademacher functions, or more generally i.i.d. random variables satisfying

$$P(r_i = 1) = P(r_i = -1) = \tfrac{1}{2}.$$

Let $r = (r_1, r_2, \dots)$ and let $\tilde{r}$ be an independent copy of r.

PROPOSITION 2.1. *Let $B \in \mathscr{M}_2$ and $\phi \in \mathscr{C}(\beta)$. Then*

$$E\phi(|B(r, r)|) \approx E\phi(|B(r, \tilde{r})|). \qquad (2.2)$$

PROOF. We prove first that $E\phi(|B(r, r)|)$ is majorized by $E\phi(|B(r, \tilde{r})|)$. Now, if $\hat{r}$ differs from r by having some of its components equal to zero, then $E(r_i r_j|\hat{r}) = \hat{r}_i\hat{r}_j$, and therefore, by Jensen's inequality,

$$\begin{aligned} E\phi(|B(\hat{r}, \hat{r})|) &= E\phi(|E(B(r, r)|\hat{r})|) \\ &\leq EE(\phi(|B(r, r)|)|\hat{r}) \qquad (2.3) \\ &= E\phi(|B(r, r)|). \end{aligned}$$

We may suppose that $r = (r_1, r_2, \dots, r_N)$ for large enough N since all but finitely many of the a_{ij} are zero, and we shall let $\tilde{r}$ be an independent copy of $r = (r_1, r_2, \dots, r_N)$.

Let J be a subset of $\{1, \dots, N\}$ and let $\hat{r}_J$ be r with $r_i = 0$ if and only if $i \in J$. Consider now $(r + \tilde{r})/2$ and note that its components take values $-1, 0, 1$. Let $A_J = \{(r_i + \tilde{r}_i)/2 = 0 \text{ iff } i \in J\}$. Then the conditional distribution of $(r + \tilde{r})/2$ given A_J is identical to the distribution of $\hat{r}_J$. Hence

$$\begin{aligned} E\phi\left(\left|B\left(\frac{r+\tilde{r}}{2}, \frac{r+\tilde{r}}{2}\right)\right|\right) &= \sum_{J \subseteq \{1,\dots,N\}} E\left\{\phi\left(\left|B\left(\frac{r+\tilde{r}}{2}, \frac{r+\tilde{r}}{2}\right)\right|\right)\middle|A_J\right\}P(A_J) \\ &= \sum_{J \subseteq \{1,\dots,N\}} E\phi(|B(\hat{r}_J, \hat{r}_J)|)P(A_J) \\ &\le E\phi(|B(r,r)|) \end{aligned}$$

by (2.3). By using the polarization identity,

$$\begin{aligned} B(r, \tilde{r}) &= \frac{1}{2}\left\{4B\left(\frac{r+\tilde{r}}{2}, \frac{r+\tilde{r}}{2}\right) - B(r,r) - B(\tilde{r}, \tilde{r})\right\} \\ &= \frac{1}{3}\left\{6B\left(\frac{r+\tilde{r}}{2}, \frac{r+\tilde{r}}{2}\right) - \frac{3}{2}B(r,r) - \frac{3}{2}B(\tilde{r}, \tilde{r})\right\}, \end{aligned}$$

the triangle inequality, the convexity of ϕ, and (1.2), we get

$$\begin{aligned} E\phi(|B(r,\tilde{r})|) &\le \frac{1}{3}\left\{E\phi\left(6\left|B\left(\frac{r+\tilde{r}}{2}, \frac{r+\tilde{r}}{2}\right)\right|\right) + 2E\phi\left(\frac{3}{2}|B(r,r)|\right)\right\} \\ &\le \frac{1}{3}\{\beta^{1+\log_2 6} + 2\beta^{1+\log_2 3/2}\}E\phi(|B(r,r)|). \end{aligned}$$

We now prove that $E\phi(|B(r,r)|)$ is majorized by $E\phi(|B(r,\tilde{r})|)$. Let $e_j = \sum_{i=1}^{j-1} a_{ij} r_i$. Then for some $N \ge 1$, $\sum_{i<j} a_{ij} r_i r_j = \sum_{j=1}^{N} e_j r_j$. Using the Δ_2 inequality (1.2), the Burkholder–Davis–Gundy inequalities [Burkholder (1973), Theorem 11.1], Levy's inequality, and Lemma 2.1 we get

$$\begin{aligned} E\phi(|B(r,r)|) &\le \beta E\phi\left(\left|\sum_{j=1}^{N} e_j r_j\right|\right) \le \beta E\phi\left(\sup_{n\le N}\left|\sum_{j=1}^{n} e_j r_j\right|\right) \\ &\le C_1 E\phi\left(\left|\sum_{j=1}^{N} e_j^2 r_j^2\right|^{1/2}\right) = C_1 E\phi\left(\left|\sum_{j=1}^{N} e_j^2 \tilde{r}_j^2\right|^{1/2}\right) \\ &\le C_2 E\phi\left(\sup_{n\le N}\left|\sum_{j=1}^{N} e_j \tilde{r}_j\right|\right) \le 2C_2 E\phi\left(\left|\sum_{j=1}^{N} e_j \tilde{r}_j\right|\right) \\ &\le 2C_2 E\phi(|B(r,\tilde{r})|), \end{aligned}$$

where the constants C_1 and C_2 depend only on β. □

EXAMPLE. If $\phi(x) = |x|^p$, $p \geq 1$, then

$$E|B(r, r)|^p \approx E|B(r, \tilde{r})|^p.$$

This follows also from the fact that both quantities are comparable in size to the expression $(\sum_{i,j} a_{ij}^2)^{p/2}$, a result of Bonami (1970). For completeness, we show next that this equivalence extends to $0 < p < 1$. While this is implicit in Bonami (1970), it seems worthwhile to provide the details.

PROPOSITION 2.2. *Let B be a bilinear form with matrix $a = (a_{ij})$ with $a_{ii} = 0$ for all i and finitely many nonzero entries a_{ij}. Assume either that $a_{ij} = a_{ji}$ or that $a_{ij} = 0$ for $i \geq j$. Then, for any $0 < p < \infty$,*

$$E|B(r, r)|^p \approx \left(\sum_{i,j} a_{ij}^2\right)^{p/2} \approx E|B(r, \tilde{r})|^p. \tag{2.4}$$

PROOF. The inequalities $E|B(r, \tilde{r})|^p \approx (\sum_{i,j} a_{ij}^2)^{p/2}$ for $p > 0$, follow from Appendix D of Stein (1970). By Lemma 2.1 and Proposition 2.1 we have $E|B(r, r)|^p \approx E|B(r, \tilde{r})|^p$ for $p \geq 1$. It remains to prove that for $0 < p < 1$, the inequalities

$$E|B(r, r)|^p \approx \left(\sum_{i,j} a_{ij}^2\right)^{p/2} \tag{2.5}$$

hold for any $a = (a_{ij})$ with $a_{ij} = 0$ for $i \geq j$. Hölder's inequality yields one side of (2.5) because

$$E|B(r, r)|^p \leq \left(E|B(r, r)|^2\right)^{p/2} = \left(\sum_{i<j} a_{ij}^2\right)^{p/2}.$$

It is therefore sufficient to prove that the reverse inequality

$$\left(\sum_{i<j} a_{ij}^2\right)^{p/2} \leq C'E|B(r, r)|^p$$

holds for $0 < p < 1$.

Let V be the subspace of $l^2(\mathbb{Z}_+^2)$ consisting of those $a = (a_{ij})$ for which $a_{ij} = 0$ for $i \geq j$. Consider the two following quasinorms on V: the l^2 norm given by $|a|_2 = (\sum^2 a_{ij})^{1/2}$ and the quasinorm ρ induced on V by $\rho(a) = E|\sum_{i<j} a_{ij} r_i r_j|^p$. We have noted that $\rho(a) \leq |a|_2^p$. By a well-known consequence of the closed graph theorem, the reverse inequality $|a|_2^p \leq C'\rho(a)$ will then follow if we prove that V is complete under ρ.

Choose then a sequence $a^{(n)} = (a_{ij}^{(n)})$ with $\rho(a^{(n)} - a^{(m)}) \to 0$ as $n, m \to \infty$. We first show that $a^{(n)}$ is Cauchy in l^2. Let $Z_n = \sum_{i<j} a_{ij}^{(n)} r_i r_j$. By Feller (1971, page 152), for any $0 < \lambda < 1$,

$$\begin{aligned} P(|Z_n - Z_m| > \lambda) &\geq \frac{\left(E|Z_n - Z_m|^2 - \lambda^2\right)^2}{E|Z_n - Z_m|^4} \\ &\geq \frac{\left(|a^{(n)} - a^{(m)}|_2^2 - \lambda^2\right)^2}{C_2|a^{(n)} - a^{(m)}|_2^4}, \end{aligned}$$

where we used the cases $p = 2$ and $p = 4$ of (2.5). Suppose that $a^{(n)}$ were not

Cauchy in l^2. Then, there would exist a subsequence n_k of n with $|a^{(n_k)} - a^{(n_l)}|_2^2$ bounded below by some ε as $k, l \to \infty$. Taking $\lambda = (\varepsilon/2)^{1/2}$, we get

$$\limsup_{n, m \to \infty} P\left(|Z_n - Z_m| > \left(\frac{\varepsilon}{2}\right)^{1/2} \right) \geq \inf_{x > \varepsilon} \frac{(x - \varepsilon/2)^2}{C_2 x^2} > 0.$$

But $E|Z_n - Z_m|^p = \rho(a^{(n)} - a^{(m)}) \to 0$ implies that Z_n converges in probability and this yields a contradiction. Therefore $a^{(n)}$ converges in l^2 to some $a = (a_{ij}) \in V$.

Now, let $Z = \sum_{i<j} a_{ij} r_i r_j$. To show that $\rho(a^{(n)} - a) = E|Z_n - Z|^p \to 0$, choose n so large that $E|Z_n - Z_m|^p < \varepsilon$ for all $m \geq n$. Then by Fatou's lemma,

$$E|Z_n - Z|^p \leq \lim_{m \to \infty} E|Z_n - Z_m|^p < \varepsilon.$$

This concludes the proof. □

3. Proof of the main theorems. The following lemma will be used in the proof of (1.3). It will be applied iteratively and therefore it is important that no extraneous constants appear in the inequality (3.1) below.

We need some notation. Let $b_1, b_2, \ldots, b_N$ be i.i.d. Bernoulli with $P(b_1 = 1) = P(b_1 = 0) = \frac{1}{2}$. Let $S_N = \{1, 2, \ldots, N\}$ and for each subset $J \subseteq S_N$, define the random variable

$$e_J = \left(\prod_{i \in J} b_i \right) \left(\prod_{j \in S_N \setminus J} (1 - b_j) \right),$$

so that $P(e_J = 1) = 2^{-N}$ and $P(e_J = 0) = 1 - 2^{-N}$ for any J in S_N. Finally, let $\{\tilde{e}_J, J \subseteq S_N\}$ denote an independent copy of $\{e_J, J \subseteq S_N\}$.

LEMMA 3.1. *Let ξ_1 and ξ_2 be random variables satisfying $E(\xi_2|\xi_1) = E(\xi_1|\xi_2) = 0$ and independent of both $\{e_J, J \subseteq S_N\}$ and $\{\tilde{e}_J, J \subseteq S_N\}$. Then, for every $\phi \in \mathscr{C}(\beta)$, and each real constant a and sequences $\{x_J\}_{J \subseteq S_N}$ and $\{y_J\}_{J \subseteq S_N}$ of real constants, we have*

$$E\phi\left(\left| a + \tfrac{1}{2}\xi_1 \sum_{J \subseteq S_N} x_J e_J + \tfrac{1}{2}\xi_2 \sum_{J \subseteq S_N} y_J e_J \right| \right) \leq E\phi\left(\left| a + \xi_1 \sum_{J \subseteq S_N} x_J e_J + \xi_2 \sum_{J \subseteq S_n} y_J \tilde{e}_J \right| \right). \tag{3.1}$$

PROOF. Let $p = P(e_J = 1) = 2^{-N}$, $q = 1 - p$ and note that $p + (q/2) \geq \frac{1}{2}$. Because the e_J, $J \subseteq S_N$ have disjoint support,

$$\begin{aligned} E\phi\left(\left| a + \xi_1 \sum_J x_J e_J + \xi_2 \sum_J y_J \tilde{e}_J \right| \right) &= \sum_J \sum_K p^2 E\phi(|a + \xi_1 x_J + \xi_2 y_K|) \\ &= \sum_J p^2 E\phi(|a + \xi_1 x_J + \xi_2 y_J|) \\ &\quad + \tfrac{1}{2} \sum\sum_{J \neq K} p^2 E\phi(|a + \xi_1 x_J + \xi_2 y_K|) \\ &\quad + \tfrac{1}{2} \sum\sum_{J \neq K} p^2 E\phi(|a + \xi_1 x_J + \xi_2 y_K|). \end{aligned}$$

Now

$$\sum_{J \neq K}\sum p^2 E\phi(|a + \xi_1 x_J + \xi_2 y_K|) \geq \sum_{J \neq K}\sum p^2 E\phi(|a + \xi_1 x_J|)$$

$$= \sum_J pqE\phi(|a + \xi_1 x_J|),$$

where we used Jensen's inequality, $E(\xi_2|\xi_1) = 0$ and the fact that $(2^N - 1)p = 1 - 2^{-N} = q$. Similarly,

$$\sum_{J \neq K}\sum p^2 E\phi(|a + \xi_1 x_J + \xi_2 y_K|) \geq \sum_K pqE\phi(|a + \xi_2 y_K|).$$

Therefore,

$$E\phi\left(\left|a + \xi_1 \sum_J x_J e_J + \xi_2 \sum_J y_J \tilde{e}_J\right|\right)$$

$$\geq \sum_J p\left\{pE\phi(|a + \xi_1 x_J + \xi_2 y_J|) + \frac{q}{2}E\phi(|a + \xi_1 x_J|) + \frac{q}{2}E\phi(|a + \xi_2 y_J|)\right\}$$

$$\geq \sum_J p\left\{E\phi\left(\left|a + \left(p + \frac{q}{2}\right)\xi_1 x_J + \left(p + \frac{q}{2}\right)\xi_2 y_J\right|\right\}\right.$$

$$\geq \sum_J p\left\{E\phi(\left|a + \tfrac{1}{2}\xi_1 x_J + \tfrac{1}{2}\xi_2 y_J\right|\right\}$$

$$= E\phi\left(\left|a + \tfrac{1}{2}\xi_1 \sum_J x_J e_J + \tfrac{1}{2}\xi_2 \sum_J y_J e_J\right|\right),$$

where we used Jensen's inequality and the observation that $p + (q/2) + (q/2) = 1$. We also used the fact that the function $g(\alpha) = E\phi(|a + \alpha Z|)$ is monotone increasing in α for $\alpha > 0$, where $Z = \xi_1 x_J + \xi_2 y_J$ is a mean zero random variable. This is obviously true if $a = 0$. Now suppose that $a \neq 0$. The monotonicity of g is easy to check when ϕ is differentiable because g is convex in α and satisfies $g'(0) = 0$. The same conclusion follows for general ϕ by approximating it by $\phi_t = \phi * p_t \leq \phi$ where p_t is a Gaussian kernel with variance t. □

EXTENSION. Let $\{e_J^i,\ J \subseteq S_N\}$, $i = 1, \dots, s$, be s independent copies of $\{e_J,\ J \subseteq S_N\}$ and let $\xi_1, \xi_2, \dots, \xi_s$ be mean zero random variables independent of the $\{e_J^i,\ J \subseteq S_N\}$. Let a be real and $\{x_J^i,\ J \subseteq S_N\}$, $i = 1, \dots, s$, be sequences of real numbers. Then, for every $\phi \in \mathscr{C}(\beta)$,

$$E\phi\left(\left|a + \sum_{i=1}^{s} \xi_i \sum_{J \subseteq S_N} x_J^i e_J\right|\right) \leq E\phi\left(\left|a + \sum_{i=1}^{s-1} 2^i \xi_i \sum_{J \subseteq S_N} x_J^i e_J^i + 2^{s-1}\xi_s \sum_{J \subseteq S_N} x_J^s e_J^s\right|\right)$$

$$\leq E\phi\left(\left|a + 2^{s-1}\sum_{i=1}^{s} \xi_i \sum_{J \subseteq S_N} x_J^i e_J^i\right|\right).$$

LEMMA 3.2. *Let X be a nonnegative random variable and ϕ be a nonconstant, nonnegative convex function such that $E\phi(X) < \infty$. Then there is a sequence of nonnegative random variables X_N of the form*

$$X_N = \sum_{J \subseteq S_n} x_J e_J \tag{3.2}$$

with $x_J \geq 0$ such that, as $N \to \infty$, $X_N \to X$ a.s. and $E\phi(X_N) \to E\phi(X)$.

PROOF. Note that $EX < \infty$ because, by convexity, $\phi(EX) \leq E\phi(X) < \infty$. Now realize X on $[0,1]$ and let $\mathscr{F}_N$ be the σ-algebra generated by the Nth-dyadic partition. Set $X_N = E(X|\mathscr{F}_N)$ and note that $X_N \to X$ a.s. by the martingale convergence theorem. Moreover, $\lim_{N\to\infty} E\phi(X_N) = E\phi(X)$ because, on one hand, $E\phi(X_N) = E\phi(E(X|\mathscr{F}_N)) \leq E\phi(X)$ and, on the other hand, $E\phi(X) \leq \lim_{N\to\infty} E\phi(X_N)$ by Fatou's lemma.

The dyadic random variable X_N can be expressed as

$$X_N = \sum_{k=0}^{2^N-1} x_k 1(I_{k,N}),$$

where $I_{k,N} = [k/2^N, (k+1)/2^N)$, $1(I_{k,N})$ is the indicator function of the interval $I_{k,N}$ and $x_k = (1/|I_{k,N}|)\int_{I_{k,N}} X(\omega)\, d\omega$. We shall now change the labeling and associate to each $k = 0, 1, \dots, 2^{N-1}$ a unique subset $J = \{i_1, i_2, \dots, i_l\}$ of $S_N = \{1, 2, \dots, N\}$ by letting $i_1 < i_2 < \cdots < i_l$ be the positions of the "1"'s in the binary expansion of $k/2^N = 2^{-i_1} + 2^{-i_2} + \cdots + 2^{-i_l}$ and by letting $k = 0$ be associated with $J = \phi$. Moreover,

$$1(I_{k,N}) = \left(\prod_{i \in J} b_i\right)\left(\sum_{j \in S_N \setminus J} (1 - b_j)\right) = e_J, \tag{3.3}$$

where $b_i(\omega)$ is the ith binary digit of ω. Because of the one-to-one correspondence between k and J we can write $X_N = \sum_{J \subseteq S_N} x_J e_J$. □

PROOF OF THEOREM 1. To simplify the notation we shall consider the case $s = 2$ only—it should be clear how the proof may be adapted to the general case. In view of Lemma 2.1 we are to prove the following: Let $\phi \in \mathscr{C}(\beta)$. Then for all independent symmetric sequences $\{X_i\}$ of random variables and for all matrices (a_{ij}) of real numbers with at most finitely many nonzero entries, we have

$$E\phi\left(\left|\sum_{i<j} a_{ij} X_i X_j\right|\right) \leq C E\phi\left(\left|\sum_{i<j} a_{ij} X_i \tilde{X}_j\right|\right), \tag{3.4}$$

where C is a constant that depends only on ϕ and where the sequence $\{\tilde{X}_i\}$ is an independent copy of the sequence $\{X_i\}$.

Since the X_i are independent and symmetric, the sequence $(X_1, X_2, \dots)$ has the same distribution as $(r_1|X_1|, r_2|X_2|, \dots)$ where the r_i are i.i.d., independent of $\{X_i\}$, and satisfy $P(r_i = +1) = P(r_i = -1) = \frac{1}{2}$. We may thus suppose without loss of generality that $X_i = r_i|X_i|$.

We consider first the case where these $|X_i|$'s are simple, that is equal to

$$Y_i = \sum_{J \subseteq S_N} x_{iJ} e_J$$

[see (3.2)]. Here the x_{iJ} are nonnegative and the e_J's are independent of r_i. To establish the theorem for such simple symmetric random variables $r_i Y_i$ we must prove

$$E\phi\left(\left|\sum_{i<j} a_{ij} Y_i r_i Y_j r_j\right|\right) \leq CE\phi\left(\left|\sum_{i<j} a_{ij} Y_i r_i \tilde{Y}_j \tilde{r}_j\right|\right),$$

where $\{\tilde{Y}_i\}$ and $\{\tilde{r}_i\}$ are, respectively, independent copies of $\{Y_i\}$ and $\{r_i\}$.

After conditioning on the $\{Y_i\}$, we can use Proposition 2.1 to conclude that

$$E\phi\left(\left|\sum_{i<j} a_{ij} Y_i r_i Y_j r_j\right|\right) \approx E\phi\left(\left|\sum_{i<j} a_{ij} (Y_i r_i)(Y_j \tilde{r}_j)\right|\right). \tag{3.5}$$

We shall now apply Lemma 3.1 to the right-hand side of (3.5). We focus successively on all $u \in \mathbb{Z}^1$ for which $|a_{uj}| + |a_{iu}| > 0$ for some i and j. If $|a_{uj}| > 0$, then the random variable Y_u appears in the first factor of the right-hand side of (3.5), and if $|a_{iu}| > 0$, then the random variable Y_u appears in the second factor. For each such u, we apply Lemma 3.1 as follows. We set $Y = Y_u$, we let ξ_1 (respectively, ξ_2) be the sum of the coefficients of Y_u when Y_u appears in the first (respectively, second) factor, and we let a denote the terms of the right-hand side of (3.5) that do not involve Y_u. (Note that ξ_1 or ξ_2 may be zero, but not both.) Thus

$$\xi_1 = \sum_{j>u} a_{uj} r_u (Y_j \tilde{r}_j) = r_u \left(\sum_{j>u} a_{uj} Y_j \tilde{r}_j\right),$$

$$\xi_2 = \sum_{i<u} a_{iu} (Y_i r_i) \tilde{r}_u = \tilde{r}_u \left(\sum_{i<u} a_{iu} Y_i r_i\right),$$

and

$$a = \sum_{\substack{i \neq u \\ j \neq u \\ i<j}} a_{ij} (Y_i r_i)(Y_j \tilde{r}_j).$$

Thus

$$E\phi\left(\left|\sum_{i<j} a_{ij} (Y_i r_i)(Y_j \tilde{r}_j)\right|\right) = E\phi(|a + \xi_1 Y_u + \xi_2 Y_u|).$$

Let $\{\tilde{Y}_i\}$ be an independent copy of $\{Y_i\}$ and let $\mathscr{G}$ denote the σ-fields generated by all random variables Y_i, r_i, $\tilde{Y}_i$, and $\tilde{r}_i$ with $i \neq u$. Note that Y_u is independent of $\mathscr{G}$, and we have $E(\xi_1 | \xi_2, \mathscr{G}) = E(\xi_2 | \xi_1, \mathscr{G}) = 0$. Then by Lemma 3.1, we have

$$\begin{aligned} E\phi(|a + \xi_1 Y_u + \xi_2 Y_u|) &= E\big(E\phi(|a + \xi_1 Y_u + \xi_2 Y_u|) | \mathscr{G}\big) \\ &\leq E\big(E\phi(|a + 2\xi_1 Y_u + 2\xi_2 \tilde{Y}_u|) | \mathscr{G}\big) \\ &= E\phi(|a + 2\xi_1 Y_u + 2\xi_2 \tilde{Y}_u|). \end{aligned}$$

We now apply the same reasoning to each u in turn. (The definitions of ξ_1, ξ_2, and a must be slightly modified by changing Y_j, if the previous argument had been already applied to the index j. In that case the variable Y_j should then be changed to $2Y_j$ if it previously appeared in ξ_1.) Since the right-hand side is a quadratic form in the Y's and since the previous reasoning is applied to each Y exactly once, we obtain

$$E\phi\left(\left|\sum_{i<j} a_{ij}Y_i r_i Y_j \tilde{r}_j\right|\right) \le E\phi\left(4\left|\sum_{i<j} a_{ij}Y_i r_i \tilde{Y}_j \tilde{r}_j\right|\right)$$
$$\le \beta^2 E\phi\left(\left|\sum_{i<j} a_{ij}Y_i r_i \tilde{Y}_j \tilde{r}_j\right|\right)$$

by the Δ_2 condition (1.2).

The inequality (3.4) has now been established for simple symmetric random variables X_i's. To show that it holds for arbitrary random variables we proceed by approximation. It is sufficient to suppose $E\phi(|\Sigma a_{ij}X_i\tilde{X}_j|) < \infty$, and therefore $E\phi(|X_i\tilde{X}_j|) < \infty$, for each $i \neq j$. Then, by the Δ_2 condition, there is a constant B such that

$$E\phi(|X_i|) \le BE\phi\big(|X_i|E|\tilde{X}_j|\big) \le BE\phi\big(|X_i\tilde{X}_j|\big) < \infty.$$

Lemma 3.2 applies and we can approximate $X_i = r_i|X_i|$ by $X_i^{(N)} = r_i|X_i^{(N)}|$ where $|X_i^{(N)}|$ is of the form (3.2). Let $\tilde{X}_i^{(N)}$ be an independent copy of $X_i^{(N)}$. By Fatou's lemma and (3.4) applied to simple symmetric random variables, we have

$$E\phi\left(\left|\sum a_{ij}X_iX_j\right|\right) \le \lim_{N\to\infty} E\phi\left(\left|\sum a_{ij}X_i^{(N)}X_j^{(N)}\right|\right)$$
$$\le C \lim_{N\to\infty} E\phi\left(\left|\sum a_{ij}X_i^{(N)}\tilde{X}_j^{(N)}\right|\right).$$

Using the Δ_2 condition and the fact that only finitely many of the a_{ij} are nonzero, we can find a constant A such that

$$\phi\left(\left|\sum a_{ij}X_i^{(N)}\tilde{X}_j^{(N)}\right|\right) \le A\sum\phi\left(\left|X_i^{(N)}\tilde{X}_j^{(N)}\right|\right).$$

There remains only to show that for each fixed pair (i, j) the sequence $\phi(|X_i^{(N)}\tilde{X}_j^{(N)}|)$ is uniformly integrable. By construction (see the proof of Lemma 3.2) we have that

$$\left|X_i^{(N)}\right| = E\big(|X_i|\,|\mathscr{F}_N\big)$$

for an appropriate sequence of σ-fields $\mathscr{F}_N$ over the sample space of X_i. Let $\tilde{\mathscr{F}}_N$ denote the analogous σ-fields over the sample space of $\tilde{X}_j$, and $\mathscr{F}_N \otimes \tilde{\mathscr{F}}_N$ the product σ-fields. Then

$$\left|X_i^{(N)}\tilde{X}_j^{(N)}\right| = E\big(|X_i\tilde{X}_j|\,|\mathscr{F}_N \otimes \tilde{\mathscr{F}}_N\big)$$

and, since ϕ is convex,

$$\phi\left(\left|X_i^{(N)}\tilde{X}_j^{(N)}\right|\right) \le E\big(\phi(|X_i|\,|\tilde{X}_j|)\,|\mathscr{F}_N \otimes \tilde{\mathscr{F}}_N\big).$$

The latter sequence is uniformly integrable because $E\phi(|X_i|\,|\tilde{X}_j|) < \infty$. The proof is complete. □

PROOF OF COROLLARY 1. We consider the case $s = 2$ only. Let then $X = (X_1, X_2, \dots)$ be a symmetric sequence, $\tilde{X}$ an independent copy of X, and $\mathscr{M}_2' \subset \mathscr{M}_2$ be the space of symmetric bilinear forms B whose matrix $a = (a_{ij})$ also satisfies $a_{ij} = 0$ if $X_i = 0$ or $X_j = 0$. Let

$$\delta_1(a) = E\phi\left(\left|\sum a_{ij}X_iX_j\right|\right), \qquad \delta_2(a) = E\phi\left(\left|\sum a_{ij}X_i\tilde{X}_j\right|\right),$$

and let $|\ |_1$ and $|\ |_2$ be the Luxemburg-type norms on $\mathscr{M}_2'$ defined by

$$|a|_i = \inf\{\varepsilon > 0: \delta_i(a/\varepsilon) \le 1\}, \qquad i = 1,2. \tag{3.6}$$

[For convenience, we write $\delta_i(a)$ and $|a|_i$ instead of $\delta_i(B)$ and $|B|_i$.] We are to prove

$$c_1|a|_1 \le |a|_2 \tag{3.7}$$

where c_1 depends only on β.

Let $c = c(\beta, 2)$ be the constant that appears in Theorem 1. By that theorem, $\delta_1(a/|a|_2) \le c\delta_2(a/|a|_2)$. Using (1.4) and since $\delta_2(a/|a|_2)_1 \le 1$, we get

$$\left|\frac{a}{|a|_2}\right|_1 \le \max\left\{\beta\delta_1\left(\frac{a}{|a|_2}\right)^{1/\log_2\beta}, \delta_1\left(\frac{a}{|a|_2}\right)\right\} \le \max\{\beta c^{1/\log_2\beta}, c\},$$

so that

$$|a|_1 \le \max\{\beta c^{1/\log_2\beta}, c\}|a|_2. \tag{3.8}$$

□

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