The Annals of Probability
1986, Vol. 14, No. 3, 793–801

BROWNIAN MOTION AND HARMONIC FUNCTIONS ON ROTATIONALLY SYMMETRIC MANIFOLDS[1]

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We consider Brownian motion X on a rotationally symmetric manifold $M_g = (\mathbb{R}^n, ds^2)$, $ds^2 = dr^2 + g(r)^2\, d\theta^2$. An integral test is presented which gives a necessary and sufficient condition for the nontriviality of the invariant σ-field of X, hence for the existence of nonconstant bounded harmonic functions on M_g. Conditions on the sectional curvatures are given which imply the convergence or the divergence of the test integral.

1. Introduction. When $\mathbb{R}^n$ is given a Riemannian metric which can be written in polar coordinates as

$$ds^2 = dr^2 + g(r)^2\, d\theta^2,$$

$$g(0) = 0, \qquad g'(0) = 1, \qquad g(r) > 0 \quad \forall r > 0,$$

it becomes a rotationally symmetric manifold M_g whose Laplace–Beltrami operator in these coordinates is

$$\Delta_g = \frac{\partial^2}{\partial r^2} + (n-1)g^{-1}(r)g'(r)\frac{\partial}{\partial r} + g^{-2}(r)\Delta_{\mathbb{S}^{n-1}}.$$

Of course when $g(r) = r$, M_g is the Euclidean $\mathbb{R}^n$. M_g is called a weak model by Greene and Wu (1979) and a Ricci model by Cheeger and Yau (1981). Choi (1984) gives a characterization of manifolds isometric to some M_g. These manifolds are used as comparison manifolds in geometry.

Recent interest has centered on the question of the existence of nonconstant harmonic functions on general Riemannian manifolds [e.g., Yau (1975), Greene and Wu (1979), Choi (1984), and Anderson (1983) using geometric methods; Prat (1971), (1975), Pinsky (1978), Kifer (1976), and Sullivan (1983) using probabilistic methods]. In this note we settle the question for weak models as follows:

The invariant σ-field of Brownian motion on M_g is nontrivial if and only if $J(g) = \int_1^\infty g^{n-3}(r)\, dr \int_r^\infty g^{1-n}(\rho)\, d\rho < \infty$.

In case $J(g) < \infty$ the existence of nonconstant bounded harmonic functions follows easily and if $J(g) = \infty$ there are none such. Let $c_2 = 1$, $c_n = \frac{1}{2}$, $n \geq 3$. Under the assumption that the radial curvature $k(r) = -(g''/g)(r)$ is nonpositive we find that

$$J(g) < \infty \quad \text{if } k(r) \leq \frac{-c}{r^2 \log r} \quad \text{for } c > c_n \text{ and large } r,$$

$$J(g) = \infty \quad \text{if } k(r) \geq \frac{-c}{r^2 \log r} \quad \text{for } c < c_n \text{ and large } r.$$

Received August 1984; revised November 1985.

[1] Research supported in part by NSF grant MCS-82-01599.

AMS 1980 *subject classifications*. Primary 60G65; secondary 58G32.

Key words and phrases. Skew product, invariant σ-field, sectional curvature.

This alternative has been observed before by Milnor (1977) in dimension 2 and in part by Choi (1984) in dimension $n \geq 3$.

2. An integral test. Brownian motion X_t on M_g can be represented as a skew product as follows. Introduce a one-dimensional standard Brownian motion β_t and consider the process r_t satisfying

$$dr_t = d\beta_t + \frac{n-1}{2} g^{-1}(r_t) g'(r_t)\, dt.$$

It has scale function

$$s(r) = \int_1^r g^{1-n}(\rho)\, d\rho$$

and speed measure

$$m(dr) = 2g^{n-1}(r)\, dr.$$

Because $g(0) = 0$ and $g'(0) = 1$, it is easy to check that $r = 0$ is an entrance, nonexit boundary [Itô and McKean (1965), pages 130–131].

Let $l(t, r)$ be the local time (with respect to m) of r_t. With

$$\lambda(dr) = 2g^{n-3}(r)\, dr,$$

$$\tau(t) = \int l(t, r)\lambda(r) = \int_0^t g^{-2}(r_s)\, ds,$$

and θ_t the Brownian motion on $\mathbb{S}^{n-1}$ we consider the skew-product diffusion [Itô and McKean (1965), Section 7.15]

$$X_t = \left[r_t, \theta_{\tau(t)} \right].$$

Its generator acting on smooth, compact functions of $\theta \in \mathbb{S}^{n-1}$ and $r > 0$ is

$$\begin{aligned}
&\left(\frac{d}{dm}\frac{d}{ds} + \frac{1}{2}\frac{d\lambda}{dm}(r)\Delta_{\mathbb{S}^{n-1}} \right) u(r, \theta) \\
&\quad = \frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + (n-1)g^{-1}(r)g'(r)\frac{\partial}{\partial r} + g^{-2}(r)\Delta_{\mathbb{S}^{n-1}} \right) u(r, \theta) \\
&\quad = \frac{1}{2}\Delta_g u(r, \theta).
\end{aligned}$$

Following the discussion in Itô and McKean (1965, Section 7.16) and modifying it slightly to account for the possibility that the lifetime of r_t may be finite, one finds that because $\int_0^1 s(r)\lambda(dr) = -\infty$, X_t can be considered as a diffusion on all of $\mathbb{R}^n$. Thus X_t is a representation of Brownian motion on M_g via the exponential map.

LEMMA 1. (i) *r_t is transient if and only if* $\int_1^\infty g^{1-n}(r)\, dr < \infty$.

(ii) $P_r[\tau(\zeta) < \infty] = 1$ *or* 0 *according as*

$$J(g) = \int_1^\infty g^{n-3}(r) \int_r^\infty g^{1-n}(\rho)\, d\rho \leq \infty,$$

where ζ is the lifetime of r_t.

PROOF. (i) Recall that r_t is recurrent if $P_r[T_\rho < \zeta] = 1$ for each $0 < r, \rho < \infty$, transient if $P_r[T_\rho < \zeta] = < 1$ for some $0 < r, \rho < \infty$, where T_ρ is the passage time to ρ, and that r_t is either recurrent or transient. If $r \le \rho$ then because $s(0) = -\infty$

$$P_r[T_\rho < \zeta] = P_r[T_\rho < T_{0+}],$$

$$T_{0+} = \lim_{\varepsilon \downarrow 0} T_\varepsilon = \lim_{\varepsilon \downarrow 0} \frac{s(r) - s(\varepsilon)}{s(\rho) - s(\varepsilon)} = 1.$$

On the other hand, if $r > \rho$ then

$$P_r[T_\rho \le \zeta] = P_r[T_\rho \le T_\infty],$$

$$T_\infty = \lim_{R \uparrow \infty} T_R = \lim_{R \to \infty} \frac{s(R) - s(r)}{s(R) - s(\rho)},$$

whose limit is 1 if $s(\infty) = \infty$ and less than one otherwise. Since $s(\infty) = \int_1^\infty g^{1-n}(r)\, dr$ this proves (i).

(ii) Since $s(0) = -\infty$, we have for each $R > 0$, $P_r[T_R < T_{0+}] = 1$, hence $P_r[\zeta = T_\infty] = 1$. Because $\tau(t)$ is increasing

$$\begin{aligned}\tau(T_R) &= \inf\{\tau(t)\colon r_t = R\}\\ &= \inf\{t\colon r_{\sigma(t)} = R\},\end{aligned}$$

where $\sigma(t)$ is the inverse of the additive functional τ. Thus $\tau(T_R)$ is the passage time to R of $r_{\sigma(t)}$, a process with speed $\lambda(dr)$ and scale $s(r)$. This means $P_r[\tau(T_\infty) < \infty] = 1$ or 0 according as $+\infty$ is an exit or nonexit boundary for $r_{\sigma(t)}$, that is according as

$$\int_1^\infty ds(\rho) \int_1^\rho \lambda(dr) = 2\int_1^\infty g^{-3}(r)\, dr \int_r^\infty g^{1-n}(\rho)\, d\rho = 2J(g) \le \infty. \qquad \square$$

To discuss the invariant σ-field let us regard X_t as a diffusion process $X = (\Omega, \mathscr{F}, \mathscr{F}_t, \theta_t, X_t, P_x)$ with state space $\mathbb{R}^n$. (In this paragraph only θ_t is the shift operator; in the sequel the letter θ denotes spherical Brownian motion). An event $\Lambda \in \mathscr{F}$ is an invariant event if its indicator function is invariant under the shift: $I_\Lambda \circ \theta_t = I_\Lambda$. The invariant σ-field is

$$\mathscr{I}(X) = \{\Lambda \in \mathscr{F}\colon \exists \Lambda^0 \in \mathscr{F} \text{ such that } P_x[\Lambda = \Lambda^0] = 1\ \forall x \text{ and } \Lambda^0 \text{ is invariant}\}.$$

Let us recall the relationship between bounded harmonic functions h and bounded invariant random variables H:

1. If h is bounded and harmonic then $h(X_t)$ is a bounded P_x-martingale and $H \equiv \lim_{t\to\infty} h(X_t)$ is an invariant random variable.
2. If H is bounded and invariant then $h(x) \equiv E_x[H]$ is a bounded harmonic function and $\lim_{t\to\infty} h(X_t) = H$, P_x a.s.
3. $h(x)$ is a harmonic function [i.e., $h(X_t)$ is a P_x, martingale for each $x \in \mathbb{R}^n$] if and only if $h \in C^2$ and $\frac{1}{2}\Delta_g h = 0$.

These facts are discussed in Dynkin (1965, Chapters XII and XIII), for example.

LEMMA 2. *The σ-field $\mathscr{I}(x)$ consists of exit sets Λ_A, $A \in R^n$, A open, where*

$$\Lambda_A = \{\omega\colon \text{there exists } T(\omega) \geq 0 \text{ such that } X_t(\omega) \in A\, \forall\, t \geq T(\omega)\}$$
$$= \{X_t \in A \text{ eventually}\}.$$

PROOF. Certainly the exit sets Λ_A are invariant. Let $\Lambda \in \mathscr{I}(X)$. Then $h(x) = P_x[\Lambda]$ is a harmonic function and $h(X_t)$ a bounded martingale. By 3 above $A = \{x \in \mathbb{R}^n\colon h(x) > \frac{1}{2}\}$ is open. Now $\Lambda_A = \{h(X_t) > \frac{1}{2} \text{ eventually}\}$ and by the martingale convergence theorem (2 above) $P_x[\lim_{t\to\infty} h(X_t) = I_\Lambda] = 1$ for every $x \in \mathbb{R}^n$. Thus $P_x[\Lambda = \Lambda_A] = 1$, which proves the lemma. □

THEOREM 1. *The invariant σ-field $\mathscr{I}(X)$ is nontrivial if and only if $J(g) = \int_1^\infty g^{n-3}(r)\,dr \int_r^\infty g^{1-n}(\rho)\,d\rho < \infty$.*

PROOF. Suppose r_t is recurrent. Then X_t can have no nontrivial exit sets. By Lemma 1, $\int_1^\infty g^{1-n}(\rho)\,d\rho = \infty$ which forces $J(g) = \infty$.

Let us suppose r_t is transient and $J(g) = \infty$. Then $\tau(\zeta) = \infty$, P_r a.s., which means that $P_r[\theta_{\tau(t)}$ is recurrent$] = 1$, since θ_τ is just a time change of spherical Brownian motion. Denote $P_{r,\theta} = P_r \times P_\theta$ where P_θ is spherical Wiener measure starting at θ. By Lemma 2 it is enough to check that $P_{r,\theta}[\Lambda_A] = 0, 1$ for sets $A = (a, b) \times B$, $B \subset \mathbb{S}^{n-1}$ an open set. Write $C = (a, b) \times \mathbb{S}^{n-1}$ and $D = (0, \infty) \times B$ so that $A = C \cap D$. Then

$$\begin{aligned} P_{r,\theta}[\Lambda_A] &= P_{r,\theta}[\Lambda_C \cap \Lambda_D] \\ &= E_r\big[I_{\Lambda_C} P_\theta[\Lambda_D]\big] \\ &= \delta P_r[\Lambda_C], \end{aligned}$$

where $\delta = P_\theta[\Lambda_D] = 0, 1$ by the recurrence of θ_τ. Because r_t is transient, $P_r[\Lambda_C] = 1$ if $(a, b) = (a, \infty)$ and zero if $b < \infty$, and this proves the 0,1 statement above.

Finally, assume $J(g) < \infty$. Then r_t is transient and $P_r[\tau(\zeta) < \infty] = 1$. Thus

$$\begin{aligned} P_{r,\theta}\big[\theta_{\tau(\zeta)} \in d\phi\big] &= \int_0^\infty P_\theta[\theta_t \in d\phi] P_r[\tau(\zeta) \in dt] \\ &\equiv d\phi \int_0^\infty p(t, \theta, \phi) q(r, dt) \end{aligned}$$

[$p(t, \theta, \phi)\,d\phi$ the transition density of spherical Brownian motion] is a probability density on $\mathbb{S}^{n-1}$. This shows that nonempty, disjoint open sets in $\mathbb{S}^{n-1}$ determine distinct, nontrivial exit sets. □

REMARKS. 1. In dimension 2 transience of X_t is equivalent to the nontriviality of $\mathscr{I}(X)$, i.e., to the existence of nonconstant, bounded harmonic functions.

2. The random variable $\theta_{\tau(\zeta)}$ generates the invariant σ-field so that every bounded harmonic function has the representation $h(r, \theta) = E_{r,\theta} f(\theta_{\tau(\zeta)})$ for $f \in L^\infty(\mathbb{S}^{n-1})$.

3. It is natural to guess and not hard to prove, following the argument of Kifer (1976), that $K(r, \theta, \phi) = \int_0^\infty p(t, \theta, \rho) q(r, dt)$, the density of $\theta_{\tau(\zeta)}$, is the Martin kernel of Δ_g.

3. Radial curvature. For each $p \in M_g$ and any 2-plane σ in $T_p M_g$ containing the radial vector dr, the sectional curvature $K(\sigma)$ is a function of $r = r(p)$ alone, call it $k(r)$, and one has Jacobi's equation

$$g'' + kg = 0,$$
$$g(0) = 0, \qquad g'(0) = 1.$$

Thus $k = -(g''/g)$[k is called the radial curvature, Greene and Wu (1979)].

LEMMA 3. *Suppose f and g are positive functions satisfying $f(0) = g(0) = 0$, $f'(0) = g'(0) = 1$, and $-(g''/g) \leq -(f''/f)$ for $r > 0$. Then $f'/f \leq g'/g$ and $f \leq g$ for $r > 0$.*

PROOF. The proof is based on the identities

$$\left(g^2\left(\frac{f}{g}\right)'\right)' = (gf' - fg')' = fg\left(\frac{f''}{f} - \frac{g''}{g}\right).$$

The details are omitted. □

LEMMA 4. *Suppose f and g are C^2 functions such that*

$$f(0) = g(0) = 0, \qquad f'(0) = g'(0) = 1, \quad \text{and} \quad -\frac{g''}{g} \leq -\frac{f''}{f} \leq 0$$

for all $r > 0$. Then $J(f) < \infty$ implies $J(g) < \infty$ and $J(g) = \infty$ implies $J(f) = \infty$.

PROOF. By Lemma 3, $1/r \leq f'/f \leq g'/g$ and $r \leq f(r) \leq g(r)$. Consider solutions of the equations

$$dr_t = d\beta_t + \frac{n-1}{2} g^{-1}(r_t) g'(r_t)\, dt,$$
$$d\rho_t = d\beta_t + \frac{n-1}{2} f^{-1}(\rho_t) f'(\rho_t)\, dt$$

defined up to their respective lifetimes ζ and η. Since zero is an entrance, nonexit boundary for both processes we have as in Lemma 1, $\zeta = \inf\{t\colon r_t = +\infty\}$ and $\eta = \inf\{t\colon \rho_t = +\infty\}$. With the understanding that $r_{\zeta+t} = \rho_{\eta+t} = +\infty$ for all $t \geq 0$, we claim $P_r[\rho_t \leq r_t\ \forall\, t \geq 0] = 1$. This follows directly from the comparison theorem for one dimensional diffusions [e.g., Ikeda and Watanabe (1981), Chapter VI, Theorem 1.1] since by Lemma 3 $((n-1)/2) f'/f \leq ((n-1)/2) g'/g$. However, our situation is simple enough that we recall the proof here.

Let ϕ_N be a sequence of smooth functions satisfying

$$\phi_N(x) = 0, \qquad x \leq 0; \qquad \phi_N(x) \uparrow x^+ = \max\{x, 0\} \quad \text{and} \quad 0 \leq \phi_N'(x) \leq 1.$$

Let us set $F = ((n-1)/2)f'/f$, $G = ((n-1)/2)g'/g$ and suppose that F and G are bounded and globally Lipschitz with constant K. By Itô's formula:

$$\begin{aligned}
&\phi_N(\tau_t - r_t)\\
&\quad = \int_0^t \phi_N'(\rho_s - r_s)[F(\rho_s) - G(r_s)]\,ds\\
&\quad = \int_0^t \phi_N'(\rho_s - r_s)[F(\rho_s) - F(r_s) + F(r_s) - G(r_s)]\,ds\\
&\quad \le \int_0^t \phi_N'(\rho_s - r_s)[F(\rho_s) - F(r_s)]\,ds\\
&\quad = \int_0^t \phi_N'(\rho_s - r_s) I_{\{\rho_s > r_s\}}[F(\rho_s) - F(r_s)]\,ds\\
&\quad \le K\int_0^t I_{\{\rho_s > r_s\}}|\rho_s - r_s|\,ds\\
&\quad \le K\int_0^t (\rho_s - r_s)^+\,ds.
\end{aligned}$$

Taking expectations and letting $N \to \infty$ one finds

$$E_r(\rho_t - r_t)^+ \le K\int_0^t E_r(\rho_s - r_s)^+\,ds,$$

hence $P_r[\rho_t \le r_t] = 1$ for each $t \ge 0$. Thus $P_r[\rho_t \le r_t\ \forall\, t \ge 1] = 1$ by continuity of paths. Now the bounded Lipschitz hypothesis can be removed by a simple localization argument.

Since f, g are increasing and $f \le g$ we have $f(\rho_t) \le g(r_t)$. Since $f(\infty) = g(\infty) = \infty$ and $\zeta \le \eta$ one finds

$$\tau_g(\zeta) = \int_0^\zeta g^{-2}(r_t)\,dt \le \int_0^\eta f^{-2}(\rho_t)\,dt = \tau_f(\eta) \quad P_r \text{ a.s.}$$

The lemma follows if one remembers (Lemma 1) the probabilistic meaning of $J(g)$ and $J(f)$. □

REMARK. The reader is referred to the appendix for an alternative proof of Lemma 4 which was kindly supplied by the referee.

LEMMA 5. *Suppose f and g are C^3 with the same initial conditions as before. Suppose that*

$$-\frac{g''}{g} \le 0, \qquad -\frac{f''}{g} \le 0 \quad \text{for all } r > 0$$

and

$$-\frac{g''}{g} \le -\frac{f''}{f} \quad \text{for all } r \ge R > 0.$$

Then $J(f) < \infty$ implies $J(g) < \infty$ and $J(g) = \infty$ implies $J(f) = \infty$.

PROOF. Set $K_+ = \max\{-(g''/g), -(f''/f)\}$, $k_- = \min\{-(g''/g), -(f''/f)\}$. These functions are Lipschitz. Let $u_\pm$ be the C^2 solutions of

$$w'' + k_\pm w = 0,$$
$$w(0) = 0, \qquad w'(1) = 1.$$

There are positive constants such that

$$\alpha_1 u_+(R) \le f(R) \le \alpha_2 u_+(R), \qquad \alpha_1 u'_+(R) \le f'(R) \le \alpha_2 u'_+(R).$$

On (R, ∞), u_+ and f satisfy the same equation, hence

$$\alpha_1 u_+(r) \le f(r) \le \alpha_2 u_+(r), \qquad r \ge R,$$

and so

$$\gamma_1 + \beta^{-1} J(u_+) \le J(f) \le \gamma_2 + \beta J(u_+)$$

for some constants γ_1, γ_2, and β. Similarly $\gamma_1 + \beta^{-1} J(u_-) \le J(g) \le \gamma_2 + \beta J(u_-)$. By Lemma 4, $J(u_+) < \infty$ implies $J(u_-) < \infty$ and $J(u_-) = \infty$ implies $J(u_+) = \infty$. By the inequalities above these implications hold with f, g replacing u_+, u_-, respectively.

THEOREM 2. *Let M_g be the weak model with metric $ds^2 = dr^2 + g(r)^2 d\theta^2$, $g \in C^3$ and $k(r) \le 0$ the radial curvature. Let $c_2 = 1$ and $c_n = \frac{1}{2}$, $n \ge 3$. If $k(r) \le -c/(r^2 \log r)$ for $c > c_n$ and all large r then M_g has nonconstant bounded harmonic functions. If $k(r) \ge -c/(r^2 \log r)$ for $c < c_n$ and all large r, then M_g has none.*

PROOF. Define $\phi(r) = r(\log r)^\alpha$, $\alpha > 0$. Choose $\beta > 1$ such that $\phi'(\beta) > 0$ and $\phi''(\beta) > 0$ and set $f(r) = (\phi(r + \beta) - \phi(\beta))/\phi'(\beta)$. Then $f(0) = 0$, $f'(0) = 1$, $-f''/f \le 0$ and

$$\frac{-f''}{f}(r) \sim \frac{-\phi''}{\phi}(r) = \frac{-\alpha}{r^2 \log r}\left(1 - \frac{\alpha - 1}{\log r}\right) \quad \text{as } r \to \infty,$$

Because $f(r) \sim \phi(r)$ as $r \to \infty$, $J(f)$ and $J^* = \int_2^\infty \phi(r)^{n-3}\, dr \int_r^\infty \phi(\rho)^{1-n}\, d\rho$ are finite or infinite together. When $n = 2$, $J^* < \infty$ if and only if $\alpha > 1$. When $n \ge 3$ one finds

$$\int_r^\infty \phi(\rho)^{1-n}\, d\rho \sim \left((n-2) r^{n-2} (\log r)^{(n-1)\alpha}\right)^{-1} \quad \text{as } r \to \infty.$$

Thus J^* behaves like $\int_2^\infty dr/(r(\log r)^{2\alpha})$. Hence $J^* < \infty$ if and only if $\alpha > \frac{1}{2}$.
Applying Lemma 5 to f and g finishes the proof. □

APPENDIX

Alternative proof of Lemma 4. Define

(1) $$J_s(f) = \int_1^s f^{n-3}(r)\, dr \int_r^\infty f^{1-n}(\rho)\, d\rho, \qquad s \ge 1.$$

We want to show under the assumptions in Lemma 4 that $J_s(f) \ge J_s(g)$, $s \ge 1$.

From Lemma 3, this inequality is obvious for $n = 2$. Therefore if suffices to show the inequality under additional assumptions; $n \geq 3$ and

$$\int_1^\infty f^{1-n}(\rho)\, d\rho < \infty. \tag{2}$$

Then

$$\frac{dJ_s(f)}{ds} = f^{n-3}(s) \int_s^\infty f^{1-n}(\rho)\, d\rho \geq 0.$$

So

$$\frac{1}{f^{n-3}(s)} \frac{dJ_s(f)}{ds} = \int_s^\infty f^{1-n}(\rho)\, d\rho \downarrow 0 \quad \text{as } s \uparrow \infty. \tag{3}$$

Differentiate (3):

$$\frac{d}{ds}\left(\frac{1}{f^{n-3}(s)} \frac{dJ_s(f)}{ds} \right) = -f^{1-n}(s).$$

Thus

$$\frac{d^2 J_s(f)}{ds^2} - (n-3) \frac{f'(s)}{f(s)} \frac{dJ_s(f)}{ds} + \frac{1}{f(s)^2} = 0.$$

From the assumptions in Lemma 4, we have

$$\frac{d^2 J_s(f)}{ds^2} - (n-3) \frac{g'(s)}{g(s)} \frac{dJ_s(f)}{ds} + \frac{1}{g^2(s)} \leq 0, \qquad s \leq 1.$$

Thus

$$\begin{aligned} -\frac{1}{g^{n-1}(s)} &\geq \frac{1}{g^{n-3}(s)} \frac{d^2 J_s(f)}{ds^2} - (n-3) \frac{g'(s)}{g^{n-2}(s)} \frac{dJ_s(f)}{ds} \\ &= \frac{d}{ds}\left(\frac{1}{g^{n-3}(s)} \frac{dJ_s(f)}{ds} \right). \end{aligned} \tag{4}$$

Integrate (4) from s to ∞:

$$-\int_s^\infty \frac{d\rho}{g^{n-1}(\rho)} \geq \lim_{\rho \to \infty} \left[\frac{1}{g^{n-3}(s)} \frac{dJ_s(f)}{ds} \right] - \frac{1}{g^{n-3}(s)} \frac{dJ_s(f)}{ds}.$$

Hence

$$g^{n-3}(s) \int_s^\infty g^{1-n}(\rho)\, d\rho \leq \frac{dJ_s(f)}{ds}. \tag{5}$$

Integrate (5) from 1 to s (≥ 1):

$$J_s(g) = \int_1^s g(r)^{n-3}\, dr \int_r^\infty g(\rho)^{1-n}\, d\rho \leq J_s(f) - J_s(1) = J_s(f). \qquad \square$$

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