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RECURRENCE OF RANDOM WALKS ON COMPLETELY SIMPLE SEMIGROUPS

BY P. B. CERRITO

University of South Florida

A completely simple semigroup has the form $S = X \times G \times Y$. This paper considers the relationship between S and G. Given a recurrent random walk on S we determine under what conditions G is also recurrent and conversely. In particular we generalize the results of Larisse.

1. Introduction. Let S be a locally compact, Hausdorff, second countable semigroup and let $\{X_i\}$ be a sequence of i.i.d. random elements of law μ defined on S. Let X_0 of law μ_0 be independent of X_i for all i where μ_0 is not necessarily equal to μ. Then $X_0 Z_n = X_0 X_1 \cdots X_n$ is a right random walk on S. Note that Z_n has law μ^n where μ^n denotes the n-fold convolution of μ. For properties of Z_n on S see Mukherjea and Tserpes [3].

We introduce some terminology concerning Z_n. Write $x \to y$ if there exists some $n > 0$ such that $P_x(Z_n \in N_y) = P(Z_n \in x^{-1}N_y) > 0$ for any neighborhood N_y of y and $x \to y$ i.o. if $P_x(Z_n \in N_y \text{ i.o.}) = 1$ for any N_y of y. Also $x \in S$ is recurrent if $x \to x$ i.o. If there exists some recurrent $x \in S$ then we say Z_n is recurrent. We also can say S is recurrent if there exists some recurrent random walk defined on S where the support of the measure generates S.

A nonempty subset I of S is a right ideal if $IS \subset I$, a left ideal if $SI \subset I$, and an ideal if $IS \subset I$ and $SI \subset I$. An element $e \in S$ is idempotent if $e^2 = e$. It is a primitive idempotent if $ef = fe = e$ for any other idempotent element f of S. S is completely simple if it contains no proper ideals and contains a primitive idempotent. If S is completely simple then we can write $S = X \times G \times Y$ where G is a group and X, Y are sets such that there exists a mapping Φ: $Y \times X \to G$. The multiplication in S is defined by

$$(x, g, y)(x', g', y') = (x, g\Phi(y, x')g', y').$$

We may assume $\Phi(y, x) = yx$. Also X and Y are left and right zero semigroups, respectively. Note that x is a left zero semigroup means that for any $x, y \in X$, $xy = x$. For properties of S see Paalman-deMiranda [4].

Random walks defined on completely simple semigroups have been extensively studied (see Mukherjea and Tserpes [3]). They represent the first major step beyond the compact case in the field of probability theory on semigroups. Also the results can be applied to the study of random walks defined on matrices.

In this paper we study the random walk Z_n on the completely simple semigroup S. In Section 2, we show that the recurrence of μ is equivalent to the recurrence of μ^n for some $n > 0$. In Section 3, we show some conditions for which

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the recurrence of S implies the recurrence of G and conversely. For recurrence properties of random walks on groups see Revuz [5].

2. The recurrence of μ^n. Let S be a completely simple semigroup: $S = X \times G \times Y$ and let $X_0 Z_n$ be defined as above. We will show that $X_0 Z_n$ is recurrent independent of the choice of the initial element X_0 if and only if $\mu_0(xD) = 1$ for some $x \in S$, where D is the semigroup of S generated by the support $S\mu$ of the measure μ, $D = \overline{\bigcup_n (S\mu)^n}$. If S is a group, the result will hold for any X_0 where the support of μ_0 is contained in D. From this it will follow that μ is recurrent if and only if μ^n is recurrent for some (any) $n > 0$.

To show these results we require some preliminary lemmas. The first two are from Mukherjea and Tserpes [3]. A simple compactness argument will prove the third.

LEMMA 1. *Z_n is recurrent on S if and only if there exist some elements x, y such that*

$$\sum_{n \geq 1} P_x(Z_n \in N_y) = \infty$$

for all neighborhoods N_y of y.

LEMMA 2. *Let $D = \overline{\bigcup_n (S\mu)^n}$ be the semigroup generated by the support $S\mu$ of μ. If Z_n is recurrent, $x \to y$ i.o. if and only if $y \in xD$.*

LEMMA 3. *Z_n is recurrent if and only if there exists a compact set K and an element $x \in S$ such that*

$$\sum_{n \geq 1} P_x(Z_n \in K) = \infty.$$

THEOREM 1. *Let X_0 have law μ_0. If there exists a compact set $K \subset S$ such that $P(X_0 Z_n \in K$ i.o.$) = 1$ (i.e., $X_0 Z_n$ is recurrent) then Z_n is recurrent on S. Conversely if Z_n is recurrent then $X_0 Z_n$ is recurrent if and only if there exists $x \in S$ such that $\mu_0(xD) = 1$.*

PROOF. Suppose $P(X_0 Z_n \in K \text{ i.o.}) = 1$ for K compact and Z_n is not recurrent. By Lemma 3, $P_y(Z_n \in K \text{ i.o.}) = 0$ for all $y \in S$. Therefore,

$$P(X_0 Z_n \in K \text{ i.o.}) = \int P_y(Z_n \in K \text{ i.o.}) \mu_0(dy)$$

$$= \int 0 \mu_0(dy) = 0.$$

This is a contradiction and Z_n is recurrent.

Conversely, assume Z_n is recurrent and let $x \in S$. Then for any open set N containing x,

$$P(X_0 Z_n \in N \text{ i.o}) = \int P_y(Z_n \in N \text{ i.o.})\mu_0(dy)$$

$$= \int_{xD} P_y(Z_n \in N \text{ i.o.})\mu_0(dy) + \int_{xD^c} P_y(Z_n \in N \text{ i.o.})\mu_0(dy)$$

$$= \mu_0(xD).$$

Therefore $x \in S$ is recurrent with respect to $X_0 Z_n$ if and only if $\mu_0(xD) = 1$. □

COROLLARY. *If μ^n is recurrent for some $n > 0$ then so is μ. If μ is recurrent then so is μ^n for any n.*

PROOF. If μ^n is recurrent then by Lemma 3,

$$\sum_{k \geq 1} P_x(Z_{nk} \in K) = \infty$$

for some compact K (Z_n has law μ^n). This implies that

$$\sum_{k \geq 1} P_x(Z_k \in K) = \infty$$

and μ is recurrent. Conversely, suppose

$$\infty = \sum_{k \geq 1} P_x(Z_k \in K) = \sum_{0}^{n-1} \sum_{k \geq 1} P_x(Z_{nk+j} \in K).$$

Then there exists some $j \in \{0, 1, 2, \ldots, n-1\}$ such that

$$\sum_{k \geq 1} P_x(Z_{nk+j} \in K) = \infty.$$

By applying Theorem 1 to $X_0 = xX_1 \cdots X_j$, μ^n is recurrent. □

If S is a group then for any x in the subgroup D of S generated by μ and any open set containing an element of D, Z_n is recurrent and X_0 has law μ_0,

$$P(X_0 Z_n \in N \text{ i.o.}) = \int P_x(Z_n \in N \text{ i.o.})\mu_0(dx)$$

$$= \mu_0(D)$$

so that $X_0 Z_n$ is recurrent for any X_0 such that the support of μ_0 is contained in D.

3. The relationship between S and G. For $S = X \times G \times Y$ it has been conjectured that S is recurrent if and only if G is recurrent. Mukherjea, Sun, and Tserpes [2] showed the result when G is a compact group. Larisse [1] considered discrete semigroups and showed the result when G is abelian or locally finite. G is locally finite provided every finite subset generates a finite group. We will expand the result of Larisse using a much shorter proof utilizing the properties of

Lemmas 1, 2, and 3. We will also prove the conjecture in the special case where $S = G \times Y$ ($S = X \times G$) is a right (left) group. By definition, S is a right group if it contains an idempotent and has no proper right ideals. It can easily be shown that any left group has the form $S = G \times Y$.

LEMMA 4. *Suppose $S = X \times G \times Y$ and $\mu = \nu \times \sigma$ is recurrent on S where ν, σ are defined on $X \times G$ and Y, respectively. Then there exists a recurrent measure on S.*

PROOF. Let $x = (x_1, x_2, x_3) \in S$ be recurrent with respect to μ. Let $X_i = (X_{i1}, X_{i2}, X_{i3})$ have law μ and let $Z_n = X_1 X_2 \cdots X_n$. Then for any neighborhood N_x of x, $N_x = N_1 \times N_2 \times N_3$, $P_x(Z_n \in N_x \text{ i.o.}) = 1$. Therefore,

$$\begin{aligned} 1 &= P_x(Z_n \in N_x \text{ i.o.}) \\ &= P(x_1 \in N_1, X_{n3} \in N_3, x_2 x_3 X_{11} X_{12} X_{13} \cdots X_{n-1,3} X_{n,1} X_{n,2} \in N_2 \text{ i.o.}) \\ &\le P(Y_0 Y_1 \cdots Y_n \in N_2 \text{ i.o.}) \end{aligned}$$

where

$$Y_0 = x_2 x_3 X_{11} X_{12}, \qquad Y_i = X_{i3} X_{i+1,1} X_{i+2,2} \quad \text{for } i \ge 1.$$

Since X_{i3} is independent of X_{i1} and X_{i2} for all i, by Theorem 1, $Y_0 Y_1 \cdots Y_n$ is a recurrent random walk on G. □

COROLLARY. *Let S be a right group, $S = G \times Y$. Then S is recurrent if and only if G is recurrent.*

PROOF. Any measure defined on S satisfies the conditions of the above lemma. Conversely, if μ_g is recurrent on G and μ_y is any measure on Y then $\mu_g \times \mu_y$ is recurrent on S. □

Consider the following example. Let S consist of the nonzero complex numbers with multiplication defined by $a \cdot b = |a|b$. Then S is a right group with $G = (\mathbb{R}^+, \cdot)$, the positive real numbers under multiplication and $Y = \{a \in \mathbb{C}: |a| = 1\}$. Since $(\mathbb{R}^+, \cdot)$ is isomorphic to $(\mathbb{R}, +)$ we define a recurrent measure on G as follows: Let σ be a normal distribution with mean zero. Define

$$\mu_g(B) = \sigma\{x: e^x \in B\}$$

for any Borel set in S and define μ_y to be normalized arclength on Y. Then $\mu_g \times \mu_y$ is recurrent on S.

We close this section with an extension of a result of Larisse [1] concerning discrete groups that are either abelian or locally finite.

THEOREM 2. *Let G be discrete and locally finite. Then any measure defined on G is recurrent.*

PROOF. Let μ be an arbitrary measure defined on $G = \langle a_1, a_2, \dots \rangle$ where $\{a_1, a_2, \dots\}$ are the elements of the support of μ. Then $\mu(a_i) > 0$ for all i and

$\mu(x) = 0$ for all $x \neq a_i$. Define $G_n = \langle a_1, \dots, a_n \rangle$ for all n. Then since G is locally finite, G_n is a finite group such that $G_n \uparrow G$. Define a measure μ_n on G_n by

$$\mu_n(a_i) = \mu(a_i), \qquad i = 1, \dots, n,$$

$$\mu_n(e) = \mu(e) + \sum_{i \geq n+1} \mu(a_i) \quad \text{where } e \text{ is the identity of } G.$$

Since G_n is compact, μ_n is recurrent on G. Also $\mu_n \to \mu$ setwise. Let $Z_{kn} = X_{1n}X_{2n} \cdots X_{kn}$ be a random walk on G_n of law μ_n. Then

$$\begin{aligned} P(Z_{kn} = e) &= \mu_n^k(e) \\ &= \sum_{\substack{x_1 x_2 \cdots x_k = e \\ x_j \in \{a_1, a_2, \dots, a_n\}}} \mu_n(x_1)\mu_n(x_2) \cdots \mu_n(x_k) \\ &= \sum_K \mu_n(x_1) \cdots \mu_n(x_j)\mu_n(e)^{k-j}, \end{aligned}$$

where

$$K = \{x_1 x_2 \cdots x_j = e, x_i \neq e, j \leq k, x_i \in G_n \text{ for all } i\}.$$

Therefore,

$$\begin{aligned} \sum_k P(Z_{kn} = e) &= \sum_{\substack{k \geq 1 \\ K}} \sum \mu_n(x_1) \cdots \mu_n(x_j)\mu_n(e)^{k-j} \\ &= \sum_{k \geq 0} \mu_n(e)^k \sum_{\substack{x_1 \cdots x_j = e \\ x_i \neq e}} \mu(x_1) \cdots \mu(x_j) \\ &= \sum_{k \geq 0} \mu_n(e)^k \cdot M. \end{aligned}$$

Suppose $M < \infty$. Then the above sum is also finite since $\mu_n(e) < 1$. However,

$$\sum_{k \geq 1} P(Z_{kn} = e) = \infty$$

since μ_n is recurrent. This is a contradiction and so $M = \infty$. If $Z_k = X_1 X_2 \cdots X_k$ is a random walk on G of law μ then $M \leq \sum P(X_k = e)$. Therefore μ is recurrent on G. □

COROLLARY. *Suppose $S = X \times G \times Y$ where G is discrete and locally finite. Then S is recurrent.*

PROOF. Let μ be any measure on S and let μ_x, μ_g, μ_y be the marginal distributions defined on X, G, Y, respectively. By Theorem 2, μ_g is recurrent on G. By Lemma 4, $\mu_x \times \mu_g \times \mu_y$ is recurrent on S. □

Note that although G defined above is discrete, there are no restrictions on X and Y. For example suppose $X = Y = \mathbb{R}$ and $G = \langle a_1, a_2, \dots \rangle$ such that $a_i^2 = a_i$. Define Φ: $Y \times X \to G$ such that $(y, x) \to a_i$ where i is the greatest integer less than or equal to $x + y$. Then any measure defined on S is recurrent.

We now consider the case where G is a discrete abelian group and μ is a recurrent measure defined on G.

LEMMA 5. *Let G be discrete and abelian. If G is recurrent then any subsemigroup of G containing the identity is recurrent.*

PROOF. Let μ be a recurrent measure on $G = \langle a_1, a_2, \dots \rangle$ where $A = \{a_1, a_2, \dots\}$ is the support of μ. Let H be any subsemigroup of G. Then H is generated as a semigroup by the set $B \subset A$ and H contains no element of the set $A \setminus B$ except possibly the identity. We define a measure σ on H as follows:

$$\sigma(a_i) = \mu(a_i) \quad \text{if } a_i \in B \quad \text{and}$$
$$\sigma(e) = \sum_{a_i \notin B} \mu(a_i).$$

Let $Z_n = X_1 X_2 \cdots X_n$ be a random walk of law μ defined on H. Then

$$P(Z_n = e) = \sum_{\substack{x_1 \cdots x_n = e \\ x_i \in B}} \sigma(x_1)\sigma(x_2) \cdots \sigma(x_n)$$
$$= \sum_K \sigma(x_1) \cdots \sigma(x_k)\sigma(e)^{n-k},$$

where

$$K = \{x_1 \cdots x_k = e, x_i \neq e, x_i \in B \text{ for all } i\}.$$

That is,

$$P(Z_n = e) = \sum_{k \geq 0} \sigma(x_1) \cdots \sigma(x_k) \sum_{\substack{x_i \notin B \\ x_{k+1} \cdots x_n = e}} \mu(x_{k+1}) \cdots \mu(x_n)$$
$$= \sum_{\substack{x_\iota \cdots x_n = e \\ x_i \in A}} \mu(x_1) \cdots \mu(x_n) \quad \text{since } G \text{ is abelian.}$$

Therefore $\sum_n P(Z_n = e) = \infty$ since μ is recurrent on G and σ is recurrent on H. □

THEOREM 3. *Let $S = X \times G \times Y$ where G is discrete and abelian. If G is recurrent then so is S.*

PROOF. Let μ be recurrent on G. Then YX is a subsemigroup of G ($Y \subset eS$, $X \subset Se$, $G = eSe$). Define a measure σ on YX using the method of Lemma 5. Then σ is recurrent on YX. By using the procedure in Lemma 5 to relate σ to μ and applying Theorem 1 to the result, $\sigma * \mu$ is recurrent on G.

Let $\{X_i\}$ and $\{Y_i\}$ be sequences of random variables such that $Y_j X_i$ has law σ for all i and j. The procedure for doing this is as follows: Let ν_1 and ν_2 be measures on X and Y, respectively. Then $\nu_1 * \nu_2$ is defined on $X \times Y$. Let $A_g = \{(x, y)\colon yx = g\}$. Define a measure ν on $X \times Y$ such that

$$\nu(B) = \sum_{g \in G} \nu_1 * \nu_2(B \cap A_g) \cdot \sigma(g)/\nu_1 * \nu_2(A_g).$$

Then for any g, $\nu(A_g) = \sigma(g)$. Let $\{G_i\}$ be a sequence of i.i.d. random variables of law μ. Then

$$P_e(X_1Y_1 \cdots Y_{n-1}X_nG_1 \cdots G_n \in N_e \text{ i.o.}) = 1.$$

Therefore

$$P(e \in N_e, Y_n \in N_e, X_1G_1Y_1 \cdots Y_{n-1}X_nG_n \in N_e \text{ i.o.}) = 1.$$

That is,

$$P_e((X_1, G_1, Y_1) \cdots (X_n, G_n, Y_n) \in N_e \times N_e \times N_e \text{ i.o.}) = 1.$$

This holds for any neighborhood of e in S. Therefore if $Z_i = (X_i, G_i, Y_i)$ then $Z_1Z_2 \cdots Z_n$ is a recurrent random walk on S. □

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTH FLORIDA
TAMPA, FLORIDA 33620