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ON THE RATE OF CONVERGENCE IN THE CENTRAL LIMIT THEOREM FOR MARTINGALES WITH DISCRETE AND CONTINUOUS TIME

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Heyde and Brown (1970) established a bound on the rate of convergence in the central limit theorem for discrete time martingales having finite moments of order $2 + 2\delta$ with $0 < \delta \leq 1$. In the present paper a modification of the methods developed by Bolthausen (1982) is applied to show the validity of this result for all $\delta > 0$. Moreover, an example is constructed demonstrating that this bound is asymptotically exact for all $\delta > 0$. The result for discrete time martingales is then used to derive the corresponding bound on the rate of convergence in the central limit theorem for locally square integrable martingales with continuous time.

1. Introduction and statements of results. For each integer $n \geq 1$, let the real-valued random variables $X_{n1}, \dots, X_{nk_n}$ form a square integrable martingale difference sequence (mds for short) w.r.t. the σ-fields $\mathscr{F}_{n0} \subset \mathscr{F}_{n1} \subset \cdots \subset \mathscr{F}_{nk_n}$, that is, suppose that X_{ni} is measurable w.r.t. $\mathscr{F}_{ni}$ with $E(X_{ni}^2) < \infty$ and $E(X_{ni}|\mathscr{F}_{n,i-1}) = 0$ a.s. for all n and i. According to one of the basic results of martingale central limit theory, the "conditional Lindeberg condition"

$$(1.1) \qquad \sum_{i=1}^{k_n} E\big(X_{ni}^2 I(|X_{ni}| > \varepsilon)|\mathscr{F}_{n,i-1}\big) \to_P 0, \quad \text{as } n \to \infty \text{ for each } \varepsilon > 0,$$

and the "conditional normalizing condition"

$$(1.2) \qquad \sum_{i=1}^{k_n} E\big(X_{ni}^2|\mathscr{F}_{n,i-1}\big) \to_P 1, \quad \text{as } n \to \infty,$$

together imply asymptotic normality of $S_{nk_n} \equiv \sum_{i=1}^{k_n} X_{ni}$, that is,

$$(1.3) \qquad S_{nk_n} \to_{\mathscr{D}} N(0,1),$$

or, equivalently,

$$D_{nk_n} \equiv \sup_{x \in R} \big|P\big(S_{nk_n} \leq x\big) - \Phi(x)\big| \to 0, \quad \text{as } n \to \infty.$$

Here $N(0,1)$ denotes a standard normal random variable and Φ its distribution function. Obviously, (1.1) is satisfied in particular if for some $\delta > 0$ the

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"conditional Liapounov condition of order $2 + 2\delta$" holds, i.e.,

$$\sum_{i=1}^{k_n} E\left(|X_{ni}|^{2+2\delta} | \mathscr{F}_{n,i-1}\right) \to_P 0, \quad \text{as } n \to \infty. \tag{1.4}$$

For proofs of (1.3) under conditions (1.1) and (1.2) the reader is referred to Brown (1971), Dvoretzky (1972), Scott (1973) and McLeish (1974), and for general information about martingale central limit theory to the monograph of Hall and Heyde (1980).

Under various sets of assumptions, many authors have derived bounds on the rate at which D_{nk_n} converges to zero. As is to be expected, to obtain rapid rates of convergence, one has to impose conditions which often are much more stringent than (1.1) and (1.2); cf., for example, Ibragimov (1963), Grams (1972), Nakata (1976), Kato (1979) and Bolthausen (1982). On the other hand, as explained by Hall and Heyde (1981), it is also desirable to have bounds on D_{nk_n} under minimally strengthened versions of conditions (1.2) and (1.4), that is, demanding that these conditions hold in an L_p-norm instead of in probability. Clearly then, under such moment assumptions, one can w.l.o.g. consider bounds on

$$D_n \equiv \sup_{x \in R} |P(S_n \le x) - \Phi(x)|,$$

for a fixed square integrable mds $X_1, \dots, X_n$ w.r.t. the σ-fields $\mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_n$, where $S_n \equiv \sum_{i=1}^n X_i$, and where the bound is a function of the moment terms

$$L_{n,2\delta} \equiv \sum_{i=1}^{n} E\left(|X_i|^{2+2\delta}\right)$$

and

$$N_{n,2\delta} \equiv E\left(\left|\sum_{i=1}^{n} E\left(X_i^2 | \mathscr{F}_{i-1}\right) - 1\right|^{1+\delta}\right).$$

This way of combining the L_1-version of the Liapounov condition of order $2 + 2\delta$ with the $L_{1+\delta}$-norm in the normalizing condition (1.2) is quite natural in view of the results of Hall (1978) on the convergence of moments in the martingale central limit theorem. For $0 < \delta \le 1$, Heyde and Brown (1970) showed that there exists a finite constant C_δ depending only on δ such that for each square integrable mds $X_1, \dots, X_n$,

$$D_n \le C_\delta \left(L_{n,2\delta} + N_{n,2\delta}\right)^{1/(3+2\delta)}. \tag{1.5}$$

Of course, this inequality is nontrivial only if $L_{n,2\delta}$ and $N_{n,2\delta}$ are finite. In their proof Heyde and Brown (1970) applied the martingale version of the Skorokhod embedding. Erickson, Quine and Weber (1979) obtained an explicit bound for the constant in (1.5) for $0 < \delta \le 1/2$ using the classical characteristic function technique. It seems unclear how to use either method to establish the validity of the bound in (1.5) for $\delta > 1$. In Haeusler (1984) it was shown that a version of an iterative method developed by Bolthausen (1982) can be used to deal with this case. The bound obtained was equal to the right-hand side of (1.5) times $|\log(L_{n,2\delta} + N_{n,2\delta})|$ whenever $L_{n,2\delta} + N_{n,2\delta} \le 1/2$. In the present paper this approach is refined, proving that the additional logarithmic factor is superfluous for all $\delta > 1$. Thus we have

THEOREM 1. *For any $\delta > 0$ there exists a finite constant C_δ depending only on δ such that* (1.5) *holds for any square integrable mds $X_1, \ldots, X_n$ w.r.t. the σ-fields $\mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_n$.*

The proof of Theorem 1 will be given in Section 2. In Section 3 we shall construct a sequence of mds $X_{n1}, \ldots, X_{nn}$, $n \ge 2$, w.r.t. σ-fields $\mathscr{F}_{n0} \subset \mathscr{F}_{n1} \subset \cdots \subset \mathscr{F}_{nn}$, $n \ge 2$, such that for all $\delta > 0$ and all $n \ge 2$,

$$\begin{aligned} C_{\delta,1}\big(L_{n,2\delta}^{(n)} + N_{n,2\delta}^{(n)}\big)^{1/(3+2\delta)} &\le D_n^{(n)} \\ &\le C_{\delta,2}\big(L_{n,2\delta}^{(n)} + N_{n,2\delta}^{(n)}\big)^{1/(3+2\delta)} \to 0 \quad \text{as } n \to \infty \end{aligned} \tag{1.6}$$

holds with two constants $0 < C_{\delta,1} < C_{\delta,2} < \infty$ depending only on δ, where $D_n^{(n)}$, $L_{n,2\delta}^{(n)}$ and $N_{n,2\delta}^{(n)}$ are defined as D_n, $L_{n,2\delta}$ and $N_{n,2\delta}$, respectively, but in terms of $X_{n1}, \ldots, X_{nn}$ and $\mathscr{F}_{n0}, \ldots, \mathscr{F}_{nn}$ instead of $X_1, \ldots, X_n$ and $\mathscr{F}_0, \ldots, \mathscr{F}_n$. This example demonstrates that under the assumptions of Theorem 1 inequality (1.5) is asymptotically the best possible bound on D_n for each $\delta > 0$.

Let us compare (1.5) briefly with the corresponding result for independent random variables $X_1, \ldots, X_n$ with mean 0 and, for simplicity, with $\sum_{i=1}^n E(X_i^2) = 1$. In this case, for $0 < \delta \le 1/2$,

$$D_n \le CL_{n,2\delta},$$

where $C < \infty$ now is a universal constant; cf. Petrov (1975), Chapter V, Theorem 6. Thus for $0 < \delta \le 1/2$ the rate at which the bound on D_n in terms of $L_{n,2\delta}$ gets small as $L_{n,2\delta}$ decreases is much better for independent random variables than it is for mds. Another difference between the two cases is the following one. In the independent case, bounding D_n in terms of $L_{n,2\delta}$ for $\delta > 1/2$ leads to no improvement over bounding D_n in terms of $L_{n,1}$, because for $\delta > 1/2$ the asymptotically best possible bound on D_n is $CL_{n,2\delta}^{1/(2\delta)}$ (consider $X_{ni} = n^{-1/2}Y_i$, $i = 1, \ldots, n$, for an i.i.d. sequence Y_i, $i \ge 1$, of Bernoulli variables), but we always have $L_{n,1} \le L_{n,2\delta}^{1/(2\delta)}$ from $\sum_{i=1}^n E(X_i^2) = 1$ and Hölder's inequality. For mds in the dependent case the situation is different. Usually, for $\delta > \delta'$ the term $L_{n,2\delta} + N_{n,2\delta}$ is asymptotically of a smaller order than $L_{n,2\delta'} + N_{n,2\delta'}$, so that inequality (1.5) becomes nontrivial for each $\delta > 0$.

The main features of the central limit theory for sequences of mds have been extended to sequences of continuous time local martingales and semimartingales through the work of Rebolledo (1979, 1980), Liptser and Shiryayev (1980), Helland (1982) and many others. An expository review of the basic results is given in Gaenssler and Haeusler (1986). We shall demonstrate here how Theorem 1 carries over to locally square integrable martingales. Both the formulation of the result and its proof require some concepts from the "general theory of stochastic processes" and in particular from the theory of continuous time local martingales. For the necessary terminology and for background the reader is referred to one of the textbooks on the topic, for instance, Dellacherie and Meyer (1978, 1982), Elliott (1982), Métivier (1982) or Kopp (1984).

To fix our notation and framework, let $M = (M(t))_{0 \le t < \infty}$ be a locally square integrable martingale w.r.t. a filtration $\mathbb{F} = (\mathscr{F}(t))_{0 \le t < \infty}$ of σ-fields. W.l.o.g. we assume M to be "cadlag," that is, to have right-continuous paths with left-hand limits. We also suppose that $\mathbb{F}$ satisfies the "usual conditions" in the sense of Dellacherie; cf. Dellacherie and Meyer (1978), IV.47, 48. Furthermore, we always assume $M(0) = 0$ a.s. Since M is locally square integrable, its predictable quadratic variation $\langle M \rangle$ exists, which by definition is the unique predictable increasing process $\langle M \rangle$ such that $M^2 - \langle M \rangle$ is a local martingale vanishing almost surely at 0. The jump process ΔX of a cadlag process $X = (X(t))_{0 \le t < \infty}$ is defined by $\Delta X(t) = X(t) - X(t-)$ with $X(t-) = \lim_{s \uparrow t} X(s)$.

Let us consider for a moment a sequence M_n, $n \ge 1$, of locally square integrable martingales. For each $n \ge 1$ and $\varepsilon > 0$ set

$$\sigma^{\varepsilon}[M_n](t) = \sum_{0 \le s \le t} (\Delta M_n(s))^2 I(|\Delta M_n(s)| > \varepsilon), \qquad 0 \le t < \infty.$$

Then $\sigma^{\varepsilon}[M_n]$ is a locally integrable increasing process so that its predictable compensator $\tilde{\sigma}^{\varepsilon}[M_n]$ exists. The natural continuous time versions of the conditions (1.1) and (1.2) become

$$\tilde{\sigma}^{\varepsilon}[M_n](1) \to_P 0, \quad \text{as } n \to \infty \text{ for each } \varepsilon > 0$$

and

$$\langle M_n \rangle(1) \to_P 1, \quad \text{as } n \to \infty.$$

These conditions in fact imply

$$M_n(1) \to_{\mathscr{D}} N(0,1), \quad \text{as } n \to \infty. \tag{1.7}$$

This result is contained in the aforementioned work of Rebolledo, Liptser and Shiryayev and Helland, as a consequence of more general weak convergence results for the whole processes $(M_n(t))_{0 \le t < \infty}$.

To establish a bound on the rate of convergence in (1.7) which corresponds to (1.5) we drop the index n again and consider a fixed locally square integrable martingale M. Setting

$$D_c = \sup_{x \in R} |P(M(1) \le x) - \Phi(x)|,$$

$$L_{c,2\delta} = E\left(\sum_{0 \le t \le 1} |\Delta M(t)|^{2+2\delta} \right)$$

and

$$N_{c,2\delta} = E(|\langle M \rangle(1) - 1|^{1+\delta}),$$

we shall deduce the following result from Theorem 1.

THEOREM 2. *For any $\delta > 0$ there exists a finite constant C_δ depending only on δ such that*

$$D_c \le C_\delta (L_{c,2\delta} + N_{c,2\delta})^{1/(3+2\delta)} \tag{1.8}$$

holds for any locally square integrable martingale M.

The proof of Theorem 2 will be given in Section 4. The usual interpretation of a discrete time square integrable martingale as a continuous time square integrable martingale and the example given in Section 3 show that asymptotically (1.8) is the best possible bound on the rate of convergence in (1.7) under the stated conditions.

The structure of inequality (1.8) is comparable to that of inequality (6) in Theorem 1 of Liptser and Shiryayev (1982). In this result all time points $0 < t < \infty$ and normal distributions with mean 0 and variance $0 < V_t < \infty$ are considered. If one sets $t = 1$ and $V_t = 1$, then one obtains an estimate of D_c in terms of $L_{c,1}$ and $N_{c,0}$. Observe also that extensions of (1.8) to arbitrary time points and centered normal distributions are readily obtained by obvious transformation arguments.

As a final remark we note that the methods used to establish Theorems 1 and 2 also enable one to derive explicit numerical bounds on the constants C_δ occurring in these results. This will become apparent from the proofs, but we do not undertake to determine such bounds here.

Throughout Sections 2 to 4, the following conventions will be used to simplify the notation. The symbol C always denotes a generic finite absolute constant, whereas C_δ is always a generic finite constant depending only on δ. Equations, inequalities, etc., between random variables and random processes are always assumed to hold almost surely without explicit mention, especially when conditional expectations are involved.

2. Proof of Theorem 1. The proof is based on a suitable version of the method developed by Bolthausen (1982) in the proof of his Theorem 2. Its main part is a refinement of the proof of the Theorem in Haeusler (1984). Since the arguments in the latter paper are only sketched, and many of the details must be modified, for the sake of readability a complete proof of Theorem 1 is presented here.

We begin by summarizing some basic facts about the Lévy-metric. For this, let $L(F,G)$ denote the Lévy-distance of the two distribution functions F and G, i.e.,

$$L(F,G) = \inf\{\varepsilon > 0\colon F(x-\varepsilon) - \varepsilon \le G(x) \le F(x+\varepsilon) + \varepsilon \text{ for all } x \in R\}.$$

The following elementary inequalities are well known and are a standard tool in deriving rates of convergence in martingale central limit theory; cf. Móri (1977), Erickson, Quine and Weber (1979) and Liptser and Shiryayev (1982), the last-mentioned paper containing the proofs. We have

$$L(F,G) \le \sup_{x\in R} |F(x) - G(x)|, \tag{2.1}$$

and, assuming G has a bounded density g,

$$\sup_{x\in R} |F(x) - G(x)| \le (1 + \|g\|_\infty) L(F,G). \tag{2.2}$$

Furthermore,

$$L(F,G) \le E(|X - Y|^s)^{1/(1+s)}, \quad \text{for all } s > 0, \tag{2.3}$$

if X and Y are random variables having distribution functions F and G. By a slight abuse of notation, we shall write $L(X, Y)$ for $L(F, G)$.

In the course of the proof we shall have to extend the given sequence $\mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_n$ of σ-fields to an infinite sequence $\mathscr{F}_i$, $i \geq 0$. For $i \geq 1$ and a square integrable random variable X we shall use the abbreviation $\sigma_i^2(X) = E(X^2|\mathscr{F}_{i-1})$.

First, we shall prove the assertion under the additional assumption that $\sum_{i=1}^n \sigma_i^2(X_i) = 1$ holds, and, of course, $L_{n,2\delta} < \infty$. For fixed $\beta > 0$ we set

$$Y_i = X_i I(|X_i| \leq \beta^{1/2}/2) - E(X_i I(|X_i| \leq \beta^{1/2}/2)|\mathscr{F}_{i-1}), \qquad i = 1, \dots, n,$$

$$S_n' = \sum_{i=1}^n Y_i \quad \text{and} \quad D_n' = \sup_{x \in R} |P(S_n' \leq x) - \Phi(x)|.$$

Then by construction, $Y_1, \dots, Y_n$ is an mds, and

$$E(|S_n - S_n'|^2) \leq \sum_{i=1}^n E(X_i^2 I(|X_i| > \beta^{1/2}/2)) \leq C_\delta \beta^{-\delta} L_{n,2\delta},$$

which together with (2.1), (2.2) and (2.3) implies

$$(2.4) \qquad D_n \leq C\{L(S_n, S_n') + L(S_n', N(0,1))\} \leq C_\delta\{\beta^{-\delta/3} L_{n,2\delta}^{1/3} + D_n'\}.$$

Let $Y_{n+1}, Y_{n+2}, \dots$ be independent random variables with $P(Y_i = \beta^{1/2}) = 1/2 = P(Y_i = -\beta^{1/2})$ for all i, which are independent of $\mathscr{F}_n$. For $i \geq n+1$ set $\mathscr{F}_i = \sigma(\mathscr{F}_n, Y_{n+1}, \dots, Y_i)$. Then Y_i, $i \geq 1$, is an mds w.r.t. $\mathscr{F}_i$, $i \geq 0$. Observe that the random variable

$$\tau = \max\left\{ l \geq 1 \colon \sum_{i=1}^l \sigma_i^2(Y_i) \leq 1 \right\}$$

is a stopping time w.r.t. $\mathscr{F}_i$, $i \geq 0$, for which in view of

$$\sum_{i=1}^n \sigma_i^2(Y_i) \leq \sum_{i=1}^n \sigma_i^2(X_i) = 1$$

and $\sigma_i^2(Y_i) = \beta$ for $i \geq n+1$ we have $n \leq \tau \leq n + [\gamma]$, where $[\cdot]$ denotes the integer part and $\gamma = \beta^{-1}$ for notational convenience. For $i = 1, \dots, k \equiv n + [\gamma] + 1$ we set

$$Z_i = Y_i I(i \leq \tau) + \left\{ \left(1 - \sum_{j=1}^{\tau} \sigma_j^2(Y_j) \right) \Big/ \beta \right\}^{1/2} Y_i I(i = \tau + 1),$$

thus obtaining an mds w.r.t. $\mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_k$. Writing from now on

$$\sigma_i^2 = \sigma_i^2(Z_i) \quad \text{for } i = 1, \dots, k, \quad \text{and} \quad S_k'' = \sum_{i=1}^k Z_i,$$

we have

$$(2.5) \qquad \sum_{i=1}^k \sigma_i^2 = 1,$$

$$(2.6) \qquad |Z_i| \leq \beta^{1/2}, \qquad i = 1, \dots, k,$$

and (observe that $Z_i = Y_i$ for $i = 1, \dots, n$)

$$(2.7) \qquad E\left(|S_n' - S_k''|^2\right) = \sum_{i=n+1}^{k} E\left(Z_i^2\right) \le C_\delta \beta^{-\delta} L_{n,2\delta}.$$

Setting $D_k'' = \sup_{x \in R} |P(S_k'' \le x) - \Phi(x)|$ and applying (2.1), (2.2), (2.3) and (2.7) we obtain

$$D_n' \le C\{L(S_n', S_k'') + L(S_k'', N(0,1))\} \le C_\delta\{\beta^{-\delta/3} L_{n,2\delta}^{1/3} + D_k''\},$$

which in combination with (2.4) yields

$$(2.8) \qquad D_n \le C_\delta\{\beta^{-\delta/3} L_{n,2\delta}^{1/3} + D_k''\}.$$

To derive a bound for D_k'' we now proceed as in the proofs of Theorems 1 and 2 in Bolthausen (1982). For this, let $N_1, \dots, N_k$ be standard normal random variables and let ξ be a normal random variable with mean 0 and variance 3β such that $\mathscr{F}_k, N_1, \dots, N_k$ and ξ are independent. Then $N_k'' = \sum_{i=1}^k \sigma_i N_i$ is a standard normal random variable because of (2.5). Two applications of Lemma 1 of Bolthausen (1982) and the triangle inequality yield

$$(2.9) \qquad D_k'' \le C\left\{\sup_{x \in R} |P(S_k'' + \xi \le x) - P(N_k'' + \xi \le x)| + \beta^{1/2}\right\}.$$

Let $x \in R$ be fixed and set $U_m = \sum_{i=1}^{m-1} Z_i$, $\lambda_m^2 = \sum_{i=m+1}^k \sigma_i^2 + 3\beta$ and $T_m = \lambda_m^{-1}(x - U_m)$. If $\mathscr{F}_0$ contains all P-null sets, which can be assumed w.l.o.g. since null sets do not affect distributional properties, then λ_m^2 is $\mathscr{F}_{m-1}$-measurable because of (2.5). Together with the mds-property of $Z_1, \dots, Z_k$ this fact enables one to obtain

$$(2.10) \qquad \begin{aligned} &|P(S_k'' + \xi \le x) - P(N_k'' + \xi \le x)| \\ &\quad \le \tfrac{1}{6} \sum_{m=1}^{k} E\left(|\varphi''(T_m - \theta_m \lambda_m^{-1} Z_m)| \lambda_m^{-3} |Z_m|^3\right) \\ &\qquad + \tfrac{1}{6} \sum_{m=1}^{k} E\left(|\varphi''(T_m - \tilde{\theta}_m \lambda_m^{-1} \sigma_m N_m)| \lambda_m^{-3} \sigma_m^3 |N_m|^3\right) \equiv \tfrac{1}{6} I + \tfrac{1}{6} II, \end{aligned}$$

where $0 \le \theta_m, \tilde{\theta}_m \le 1$ and φ denotes the standard normal density; cf. the arguments leading from (4.2) to (4.4) in Bolthausen (1982). Observe that $\lambda_m^2 \ge 3\beta$ and (2.6) imply

$$(2.11) \qquad |\theta_m \lambda_m^{-1} Z_m| \le 1, \qquad m = 1, \dots, k.$$

The main feature of Bolthausen's (1982) method for deriving bounds on the rate of convergence in martingale central limit theorems now is in the treatment of terms like I and II. Adapting his ideas to the present situation we introduce stopping times $0 = \tau_0 \le \tau_1 \le \cdots \le \tau_{[\gamma]} \le \tau_{[\gamma]+1} = k$ defined by

$$\tau_j = \inf\left\{l \ge 1 \colon \sum_{i=1}^{l} \sigma_i^2 \ge j\beta\right\}, \qquad j = 1, \dots, [\gamma].$$

In view of (2.5) and (2.6) we have for $j = 1, \dots, [\gamma] + 1$ and $m = 1, \dots, k$ on the event $\{\tau_{j-1} < m \le \tau_j\}$

$$\lambda_m^2 \le 1 - (j-1)\beta + 3\beta \equiv \bar{\lambda}_j^2, \tag{2.12}$$

$$\lambda_m^2 \ge 1 - (j+1)\beta + 3\beta \equiv \underline{\lambda}_j^2 \ge \beta > 0, \tag{2.13}$$

$$1 \le \left(\bar{\lambda}_j/\underline{\lambda}_j\right)^2 \le 3, \tag{2.14}$$

and

$$\sum_{m=\tau_{j-1}+1}^{\tau_j} \sigma_m^2 \le 2\beta. \tag{2.15}$$

Now we write

$$I = \sum_{j=1}^{[\gamma]+1} E\left(\sum_{m=\tau_{j-1}+1}^{\tau_j} |\varphi''(T_m - \theta_m \lambda_m^{-1} Z_m)| \lambda_m^{-3} |Z_m|^3 \right) \equiv \sum_{j=1}^{[\gamma]+1} I_j, \tag{2.16}$$

and establish a bound for each I_j. For this, we set $R_m = \sum_{i=\tau_{j-1}+1}^{m-1} Z_i$ and $A_m = \{|R_m| \le |x - U_{\tau_{j-1}+1}|/2\}$ and use (2.13) and $\|\varphi''\|_\infty \le 1$ to obtain

$$\begin{aligned} I_j \le \underline{\lambda}_j^{-3} \Bigg\{ & E\left(\sum_{m=\tau_{j-1}+1}^{\tau_j} |\varphi''(T_m - \theta_m \lambda_m^{-1} Z_m)|\, |Z_m|^3 I(A_m) \right) \\ & + E\left(\sum_{m=\tau_{j-1}+1}^{\tau_j} |Z_m|^3 I(A_m^c) \right) \Bigg\} \\ \equiv \underline{\lambda}_j^{-3} & \{ I_{j,1} + I_{j,2} \}. \end{aligned} \tag{2.17}$$

We consider $I_{j,1}$ first. Let the function ψ: $R \to [0, \infty)$ be defined by $\psi(x) = \sup\{|\varphi''(y)|: |y| \ge (|x|/2) - 1\}$. By an application of the triangle inequality, combined with (2.11) and (2.12), we conclude that

$$\left|\varphi''(T_m - \theta_m \lambda_m^{-1} Z_m)\right| \le \psi\left(\bar{\lambda}_j^{-1}(x - U_{\tau_{j-1}+1})\right)$$

holds on $\{\tau_{j-1} < m \le \tau_j\} \cap A_m$. Consequently, since $U_{\tau_{j-1}+1}$ is measurable w.r.t. the σ-field $\mathscr{F}(\tau_{j-1})$ of all events known at time τ_{j-1},

$$I_{j,1} \le E\left\{ \psi\left(\bar{\lambda}_j^{-1}(x - U_{\tau_{j-1}+1})\right) E\left(\max_{\tau_{j-1} < m \le \tau_j} |Z_m| \sum_{m=\tau_{j-1}+1}^{\tau_j} Z_m^2 | \mathscr{F}(\tau_{j-1}) \right) \right\}. \tag{2.18}$$

According to the conditional Hölder inequality the conditional expectation on the right-hand side is less than or equal to

$$E\left(\max_{\tau_{j-1} < m \le \tau_j} |Z_m|^{2+2\delta} | \mathscr{F}(\tau_{j-1}) \right)^{1/(2+2\delta)}$$

$$\times E\left(\left| \sum_{m=\tau_{j-1}+1}^{\tau_j} Z_m^2 \right|^{(2+2\delta)/(1+2\delta)} \middle| \mathscr{F}(\tau_{j-1}) \right)^{(1+2\delta)/(2+2\delta)}.$$

Using the c_r-inequality and (2.15), the second conditional expectation can be bounded by

$$C_\delta\left\{E\left(\left|\sum_{m=\tau_{j-1}+1}^{\tau_j}\left(Z_m^2-\sigma_m^2\right)\right|^{(2+2\delta)/(1+2\delta)}\middle|\mathscr{F}(\tau_{j-1})\right)+\beta^{(2+2\delta)/(1+2\delta)}\right\}.$$

Since $Z_m^2-\sigma_m^2$, $m=1,\dots,k$, is an mds, we conclude from Burkholder's square function inequality, cf. Theorem 2.10 in Hall and Heyde (1980), that the last conditional expectation is less than or equal to

$$C_\delta E\left(\left|\sum_{m=\tau_{j-1}+1}^{\tau_j}\left(Z_m^2-\sigma_m^2\right)^2\right|^{(1+\delta)/(1+2\delta)}\middle|\mathscr{F}(\tau_{j-1})\right),$$

which because of $(1+\delta)/(1+2\delta)<1$ is less than or equal to

$$\begin{aligned}
&C_\delta E\left(\sum_{m=\tau_{j-1}+1}^{\tau_j}|Z_m^2-\sigma_m^2|^{(2+2\delta)/(1+2\delta)}\middle|\mathscr{F}(\tau_{j-1})\right)\\
&\quad\le C_\delta\left\{E\left(\sum_{m=\tau_{j-1}+1}^{\tau_j}|Z_m|^{2+2/(1+2\delta)}\middle|\mathscr{F}(\tau_{j-1})\right)\right.\\
&\qquad\left.+E\left(\sum_{m=\tau_{j-1}+1}^{\tau_j}\sigma_m^{2+2/(1+2\delta)}\middle|\mathscr{F}(\tau_{j-1})\right)\right\}\\
&\quad\le C_\delta\beta^{(2+2\delta)/(1+2\delta)},
\end{aligned}$$

where for the last inequality we have used (2.6) and (2.15) to obtain

$$\begin{aligned}
&E\left(\sum_{m=\tau_{j-1}+1}^{\tau_j}|Z_m|^{2+2/(1+2\delta)}\middle|\mathscr{F}(\tau_{j-1})\right)\\
&\quad\le\beta^{1/(1+2\delta)}E\left(\sum_{m=\tau_{j-1}+1}^{\tau_j}Z_m^2\middle|\mathscr{F}(\tau_{j-1})\right)\qquad(2.19)\\
&\quad=\beta^{1/(1+2\delta)}E\left(\sum_{m=\tau_{j-1}+1}^{\tau_j}\sigma_m^2\middle|\mathscr{F}(\tau_{j-1})\right)\le 2\beta^{(2+2\delta)/(1+2\delta)}.
\end{aligned}$$

In summary, we have shown

$$E\left(\left|\sum_{m=\tau_{j-1}+1}^{\tau_j}Z_m^2\right|^{(2+2\delta)/(1+2\delta)}\middle|\mathscr{F}(\tau_{j-1})\right)^{(1+2\delta)/(2+2\delta)}\le C_\delta\beta$$

and therefore obtain

$$I_{j,1}\le C_\delta\beta E\left\{\psi\left(\bar{\lambda}_j^{-1}\left(x-U_{\tau_{j-1}+1}\right)\right)E\left(\max_{\tau_{j-1}<m\le\tau_j}|Z_m|^{2+2\delta}\middle|\mathscr{F}(\tau_{j-1})\right)^{1/(2+2\delta)}\right\},$$

whence by Hölder's inequality and $\psi_\delta \equiv \psi^{(2+2\delta)/(1+2\delta)}$

$$I_{j,1} \le C_\delta \beta E\Big\{\psi_\delta\big(\bar{\lambda}_j^{-1}\big(x - U_{\tau_{j-1}+1}\big)\big)\Big\}^{(1+2\delta)/(2+2\delta)}$$

$$\times E\Big(\max_{\tau_{j-1} < m \le \tau_j} |Z_m|^{2+2\delta}\Big)^{1/(2+2\delta)}.$$

Now we need the following fact: For each integrable function g: $R \to [0, \infty)$ which is of bounded variation on R one has

$$E\Big\{g\big(\bar{\lambda}_j^{-1}\big(x - U_{\tau_{j-1}+1}\big)\big)\Big\} \le 3\max(\|g\|_V, \|g\|_1)\big(D_k'' + \bar{\lambda}_j\big), \tag{2.20}$$

where $\|g\|_V$ denotes the total variation of g and $\|g\|_1$ the L_1-norm w.r.t. Lebesgue measure. Inequality (2.20) follows from Lemmas 1 and 2 in Bolthausen (1982) combined with

$$E\left(\left(\sum_{m=\tau_{j-1}+1}^{k} Z_m\right)^2 \Big| \mathscr{F}(\tau_{j-1})\right) = E\left(\sum_{m=\tau_{j-1}+1}^{k} \sigma_m^2 | \mathscr{F}(\tau_{j-1})\right) \le \bar{\lambda}_j^2, \tag{2.21}$$

where (2.21) follows from (2.5) and the definition of τ_{j-1}. The function ψ_δ obviously satisfies the assumptions of (2.20), hence

$$I_{j,1} \le C_\delta \beta\big(D_k'' + \bar{\lambda}_j\big)^{(1+2\delta)/(2+2\delta)} E\Big(\max_{\tau_{j-1} < m \le \tau_j} |Z_m|^{2+2\delta}\Big)^{1/(2+2\delta)}. \tag{2.22}$$

Next, we consider $I_{j,2}$. By definition of A_m

$$A_m^c \cap \{\tau_{j-1} < m \le \tau_j\} \subset B_j \cap \{\tau_{j-1} < m \le \tau_j\}, \tag{2.23}$$

for

$$B_j \equiv \left\{\max_{\tau_{j-1} < l \le \tau_j} \left|\sum_{m=\tau_{j-1}+1}^{l} Z_m\right| > |x - U_{\tau_{j-1}+1}|/4\right\}.$$

[To obtain (2.23) it is enough to define B_j with $|x - U_{\tau_{j-1}+1}|/2$ instead of $|x - U_{\tau_{j-1}+1}|/4$; the B_j as defined above will be needed later.]

Since $A_m^c \cap \{\tau_{j-1} < m \le \tau_j\} \in \mathscr{F}_{m-1}$, we obtain

$$\begin{aligned} I_{j,2} &= E\left(\sum_{m=\tau_{j-1}+1}^{\tau_j} E\big(|Z_m|^3 | \mathscr{F}_{m-1}\big) I(A_m^c)\right) \\ &\le E\left(I(B_j) \sum_{m=\tau_{j-1}+1}^{\tau_j} E\big(|Z_m|^3 | \mathscr{F}_{m-1}\big)\right), \end{aligned} \tag{2.24}$$

by (2.23). By a repeated application of Hölder's inequality and (2.19) the right-hand side of (2.24) can be bounded by

$$E\left\{I(B_j)\left(\sum_{m=\tau_{j-1}+1}^{\tau_j} E\left(|Z_m|^{2+2\delta}|\mathscr{F}_{m-1}\right)\right)^{1/(2+2\delta)}\right.$$
$$\left.\times\left(\sum_{m=\tau_{j-1}+1}^{\tau_j} E\left(|Z_m|^{2+2/(1+2\delta)}|\mathscr{F}_{m-1}\right)\right)^{(1+2\delta)/(2+2\delta)}\right\}$$
$$\leq C_\delta \beta P(B_j)^{(1+2\delta)/(2+2\delta)} E\left(\sum_{m=\tau_{j-1}+1}^{\tau_j} |Z_m|^{2+2\delta}\right)^{1/(2+2\delta)}.$$

Now, since $U_{\tau_{j-1}+1}$ is $\mathscr{F}(\tau_{j-1})$-measurable,

$$P(B_j) \leq CE\left\{\min\left(1, |x - U_{\tau_{j-1}+1}|^{-2} \max_{\tau_{j-1}<l\leq\tau_j}\left|\sum_{m=\tau_{j-1}+1}^{l} Z_m\right|^2\right)\right\}$$
$$\leq CE\left\{\min\left(1, |x - U_{\tau_{j-1}+1}|^{-2} E\left(\max_{\tau_{j-1}<l\leq\tau_j}\left|\sum_{m=\tau_{j-1}+1}^{l} Z_m\right|^2\middle|\mathscr{F}(\tau_{j-1})\right)\right)\right\}.$$

By the conditional form of Doob's maximal inequality the conditional expectation on the right-hand side is less than or equal to

$$4E\left(\left(\sum_{m=\tau_{j-1}+1}^{k} Z_m\right)^2\middle|\mathscr{F}(\tau_{j-1})\right) \leq 4\bar{\lambda}_j^2,$$

cf. (2.21); hence $P(B_j) \leq CE\{f(\bar{\lambda}_j^{-1}(x - U_{\tau_{j-1}+1}))\}$, where f is the function defined by $f(x) = \min(1, |x|^{-2})$. Obviously, f satisfies the assumptions of (2.20) so that we arrive at

$$I_{j,2} \leq C_\delta \beta\left(D_k'' + \bar{\lambda}_j\right)^{(1+2\delta)/(2+2\delta)} E\left(\sum_{m=\tau_{j-1}+1}^{\tau_j} |Z_m|^{2+2\delta}\right)^{1/(2+2\delta)}. \tag{2.25}$$

Substituting (2.22) and (2.25) into the right-hand side of (2.17) and utilizing the result in (2.16) leads to

$$\begin{aligned} I &\leq C_\delta \beta \sum_{j=1}^{[\gamma]+1} \underline{\lambda}_j^{-3}\left(D_k'' + \bar{\lambda}_j\right)^{(1+2\delta)/(2+2\delta)} E\left(\sum_{m=\tau_{j-1}+1}^{\tau_j} |Z_m|^{2+2\delta}\right)^{1/(2+2\delta)} \\ &\leq C_\delta \beta L_{n,2\delta}^{1/(2+2\delta)}\left\{\sum_{j=1}^{[\gamma]+1} \underline{\lambda}_j^{(-6-6\delta)/(1+2\delta)}\left(D_k'' + \bar{\lambda}_j\right)\right\}^{(1+2\delta)/(2+2\delta)}, \end{aligned} \tag{2.26}$$

where the last inequality is a consequence of Hölder's inequality combined with

$$\sum_{j=1}^{[\gamma]+1} E\left(\sum_{m=\tau_{j-1}+1}^{\tau_j} |Z_m|^{2+2\delta}\right) = \sum_{m=1}^{k} E\big(|Z_m|^{2+2\delta}\big)$$

$$\leq \sum_{m=1}^{n} E\big(|Y_m|^{2+2\delta}\big) + \beta^{\delta} \sum_{m=n+1}^{k} E\big(Z_m^2\big) \leq C_\delta L_{n,2\delta},$$

holding by definition of $Y_1, \ldots, Y_n$, (2.6) and (2.7). The sum within $\{\cdots\}$ on the right-hand side of (2.26) is equal to

$$D_k'' \sum_{j=1}^{[\gamma]+1} \underline{\lambda}_j^{(-6-6\delta)/(1+2\delta)} + \sum_{j=1}^{[\gamma]+1} \bar{\lambda}_j \underline{\lambda}_j^{(-6-6\delta)/(1+2\delta)},$$

where

$$\sum_{j=1}^{[\gamma]+1} \underline{\lambda}_j^{(-6-6\delta)/(1+2\delta)} = \sum_{j=1}^{[\gamma]+1} \big(1-(j+1)\beta+3\beta\big)^{(-3-3\delta)/(1+2\delta)}$$

$$\leq \beta^{(-3-3\delta)/(1+2\delta)} \sum_{j=1}^{[\gamma]+1} j^{(-3-3\delta)/(1+2\delta)} \leq C_\delta \beta^{(-3-3\delta)/(1+2\delta)}$$

and, by (2.14),

$$\sum_{j=1}^{[\gamma]+1} \bar{\lambda}_j \underline{\lambda}_j^{(-6-6\delta)/(1+2\delta)} \leq 3^{1/2} \sum_{j=1}^{[\gamma]+1} \underline{\lambda}_j^{(-5-4\delta)/(1+2\delta)}$$

$$= 3^{1/2} \sum_{j=1}^{[\gamma]+1} \big(1-(j+1)\beta+3\beta\big)^{(-5-4\delta)/(2+4\delta)}$$

$$\leq C_\delta \beta^{(-5-4\delta)/(2+4\delta)}.$$

Thus we obtain from (2.26)

$$I \leq C_\delta L_{n,2\delta}^{1/(2+2\delta)} \Big\{ \big(D_k''\big)^{(1+2\delta)/(2+2\delta)} \beta^{-1/2} + \beta^{-1/(4+4\delta)} \Big\}. \tag{2.27}$$

Next we need to derive a bound for II on the right-hand side of (2.10). For this, we write

$$II = \sum_{j=1}^{[\gamma]+1} E\left(\sum_{m=\tau_{j-1}+1}^{\tau_j} |\varphi''(T_m - \tilde{\theta}_m \lambda_m^{-1} \sigma_m N_m)| \lambda_m^{-3} \sigma_m^3 |N_m|^3\right) \equiv \sum_{j=1}^{[\gamma]+1} II_j,$$

and consider each II_j separately. This time we set $\tilde{A}_m = \{|R_m| \leq |x - U_{\tau_{j-1}+1}|/4\}$

and $\tilde{B}_m = \{\sigma_m |N_m| \le |x - U_{\tau_{j-1}+1}|/8\}$ and use (2.13) and $\|\varphi''\|_\infty \le 1$ to obtain

$$II_j \le \underline{\lambda}_j^{-3}\left\{E\left(\sum_{m=\tau_{j-1}+1}^{\tau_j} |\varphi''(T_m - \tilde{\theta}_m \lambda_m^{-1}\sigma_m N_m)|\sigma_m^3 |N_m|^3 I(\tilde{A}_m \cap \tilde{B}_m)\right)\right.$$

$$+E\left(\sum_{m=\tau_{j-1}+1}^{\tau_j} \sigma_m^3 |N_m|^3 I(\tilde{A}_m^c)\right)$$

$$\left.+E\left(\sum_{m=\tau_{j-1}+1}^{\tau_j} \sigma_m^3 |N_m|^3 I(\tilde{B}_m^c)\right)\right\}$$

$$\equiv \underline{\lambda}_j^{-3}\{II_{j,1} + II_{j,2} + II_{j,3}\}.$$

Now we have $|\varphi''(T_m - \tilde{\theta}_m \lambda_m^{-1}\sigma_m N_m)| \le \psi(\bar{\lambda}_j^{-1}(x - U_{\tau_{j-1}+1}))$ on $\{\tau_{j-1} < m \le \tau_j\} \cap \tilde{A}_m \cap \tilde{B}_m$ leading to

$$II_{j,1} \le E\left\{\psi\left(\bar{\lambda}_j^{-1}(x - U_{\tau_{j-1}+1})\right) \sum_{m=\tau_{j-1}+1}^{\tau_j} \sigma_m^3 E\left(|N_m|^3 | \mathscr{F}_k\right)\right\}$$

$$\le CE\left\{\psi\left(\bar{\lambda}_j^{-1}(x - U_{\tau_{j-1}+1})\right) \sum_{m=\tau_{j-1}+1}^{\tau_j} E\left(|Z_m|^3 | \mathscr{F}_{m-1}\right)\right\},$$

where we have used independence of $\mathscr{F}_k$ and N_m, and $\sigma_m^3 \le E(|Z_m|^3|\mathscr{F}_{m-1})$. The expectation on the right-hand side of this inequality is less than or equal to the expectation on the right-hand side of (2.18) so that we get the same bound for $II_{j,1}$ as we previously obtained for $I_{j,1}$. By definition of $\tilde{A}_m$ we have

$$\tilde{A}_m^c \cap \{\tau_{j-1} < m \le \tau_j\} \subset B_j \cap \{\tau_{j-1} < m \le \tau_j\}$$

and

$$\tilde{A}_m^c \cap \{\tau_{j-1} < m \le \tau_j\} \in \mathscr{F}_{m-1},$$

hence

$$II_{j,2} = E\left(\sum_{m=\tau_{j-1}+1}^{\tau_j} \sigma_m^3 E\left(|N_m|^3 | \mathscr{F}_k\right) I(\tilde{A}_m^c)\right)$$

$$\le CE\left(I(B_j) \sum_{m=\tau_{j-1}+1}^{\tau_j} E\left(|Z_m|^3 | \mathscr{F}_{m-1}\right)\right).$$

Up to a constant factor the right-hand side of this inequality is equal to the right-hand side of (2.24), so that we get the same bound for $II_{j,2}$ as we got for $I_{j,2}$. Finally, taking (2.6), (2.13) and (2.14) into account, we have

$\bar{\lambda}_j^{-1}|x - U_{\tau_{j-1}+1}| < 8|N_m|$ on the event $\tilde{B}_m^c \cap \{\tau_{j-1} < m \le \tau_j\}$, thus

$$II_{j,3} \le E\left\{ \sum_{m=\tau_{j-1}+1}^{\tau_j} \sigma_m^3 E\Big(|N_m|^3 I\big(8|N_m| > \bar{\lambda}_j^{-1}|x - U_{\tau_{j-1}+1}|\big)|\mathscr{F}_k\Big)\right\}$$

$$\le E\left\{ h\Big(\bar{\lambda}_j^{-1}\big(x - U_{\tau_{j-1}+1}\big)\Big) \sum_{m=\tau_{j-1}+1}^{\tau_j} E\big(|Z_m|^3|\mathscr{F}_{m-1}\big)\right\},$$

where the function h is defined by $h(x) = E(|N_1|^3 I(8|N_1| > |x|))$. Here independence of $\mathscr{F}_k$ and N_m is crucial again. The right-hand side of this inequality can be bounded by the right-hand side of (2.18) except that the function h occurs instead of ψ. But since $h^{(2+2\delta)/(1+2\delta)}$ satisfies the assumptions of (2.20), the arguments leading from (2.18) to (2.22) may be copied to obtain the same bound for $II_{j,3}$ as previously obtained for $I_{j,1}$. Thus we see that we get the same bound for II as we obtained for I in (2.27). Since the right-hand side of (2.27) is independent of x, we conclude from (2.9) and (2.10) that

$$(2.28)\quad D_k'' \le C_\delta\Big\{ (D_k'')^{(1+2\delta)/(2+2\delta)} \beta^{-1/2} L_{n,2\delta}^{1/(2+2\delta)} + \beta^{-1/(4+4\delta)} L_{n,2\delta}^{1/(2+2\delta)} + \beta^{1/2}\Big\}.$$

Arguments similar to those that established (2.4) and (2.8) can be used to prove

$$D_k'' \le C_\delta\big\{\beta^{-\delta/3} L_{n,2\delta}^{1/3} + D_n\big\}.$$

Applying this inequality to the right-hand side of (2.28) and substituting the resulting bound for D_k'' into (2.8) we arrive at

$$(2.29)\qquad D_n \le C_\delta^*\big\{ D_n^{(1+2\delta)/(2+2\delta)} \beta^{-1/2} L_{n,2\delta}^{1/(2+2\delta)} + H_\delta(\beta, L_{n,2\delta})\big\},$$

where

$$H_\delta(\beta, L_{n,2\delta}) = \big(\beta^{-\delta/3} L_{n,2\delta}^{1/3}\big)^{(1+2\delta)/(2+2\delta)} \beta^{-1/2} L_{n,2\delta}^{1/(2+2\delta)}$$
$$+ \beta^{-1/(4+4\delta)} L_{n,2\delta}^{1/(2+2\delta)} + \beta^{1/2} + \beta^{-\delta/3} L_{n,2\delta}^{1/3},$$

C_δ^* is a certain finite constant depending only on δ and $\beta > 0$ is arbitrary. Now we consider two cases. If β satisfies

$$\beta^{-1/2} L_{n,2\delta}^{1/(2+2\delta)} \le D_n^{1/(2+2\delta)}/(2C_\delta^*),$$

then (2.29) implies $D_n \le 2C_\delta^* H_\delta(\beta, L_{n,2\delta})$. If not, then we have $D_n \le (2C_\delta^*)^{2+2\delta}\beta^{-1-\delta} L_{n,2\delta}$, which means that with a certain finite constant C_δ' depending only on δ we have for all $\beta > 0$,

$$(2.30)\qquad D_n \le C_\delta'\big\{ H_\delta(\beta, L_{n,2\delta}) + \beta^{-1-\delta} L_{n,2\delta}\big\}.$$

Easy computations now verify that for $\beta = L_{n,2\delta}^{2/(3+2\delta)}$ all terms on the right-hand side of (2.30) are equal to $L_{n,2\delta}^{1/(3+2\delta)}$. This finishes the proof of Theorem 1 under the assumption $\sum_{i=1}^n \sigma_i^2(X_i) = 1$. To remove this assumption one can apply the same reasoning as in the proof of the Main Result in Haeusler (1984), the only difference being the fact that now the logarithmic factor on the right-hand side of inequality (2.5) of that note does not appear. □

3. An example. Our example will demonstrate that (1.6) is satisfied for a martingale array which differs from an array of row-wise independent normal variables only by a single summand in each row.

Let $n \geq 2$ be a fixed integer. For brevity we write $\alpha_n = (\log n)^{-1}$. Let the function f_n: $R \to [0, \infty)$ be defined by

$$f_n(x) = \begin{cases} x^{-1}, & \text{if } \alpha_n^{1/2}/2 \leq x < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, let $X_{n1}, \dots, X_{n,n-1}$ be independent and normally distributed random variables with mean 0 and variance $(n-1)^{-1}$. We write $\mathbf{X}_{n-1} = (X_{n1}, \dots, X_{n,n-1})$, $\mathbf{x}_{n-1} = (x_1, \dots, x_{n-1})$ and $\Sigma \mathbf{x}_{n-1} = \Sigma_{i=1}^{n-1} x_i$ for $x_1, \dots, x_{n-1} \in R$. The one-point mass concentrated at x is denoted by ε_x. Let the random variable X_{nn} be defined such that its conditional distribution given that $\mathbf{X}_{n-1}$ is known equals

$$P(X_{nn} \in \cdot | \mathbf{X}_{n-1} = \mathbf{x}_{n-1}) = \tfrac{1}{2}\varepsilon_{-\alpha_n f_n(\Sigma \mathbf{x}_{n-1})}(\cdot) + \tfrac{1}{2}\varepsilon_{\alpha_n f_n(\Sigma \mathbf{x}_{n-1})}(\cdot),$$

whence for the standard normal random variable $N_{n-1} = \Sigma_{i=1}^{n-1} X_{ni}$ the conditional distribution of X_{nn} given N_{n-1} is

$$P(X_{nn} \in \cdot | N_{n-1} = x) = \tfrac{1}{2}\varepsilon_{-\alpha_n f_n(x)}(\cdot) + \tfrac{1}{2}\varepsilon_{\alpha_n f_n(x)}(\cdot).$$

Clearly, $E(X_{nn}|\mathbf{X}_{n-1}) = 0$, so that $X_{n1}, \dots, X_{nn}$ is an mds w.r.t. $\mathscr{F}_{n0} \subset \mathscr{F}_{n1} \subset \cdots \subset \mathscr{F}_{nn}$ for $\mathscr{F}_{ni} = \sigma(X_{n1}, \dots, X_{ni})$, $\mathscr{F}_{n0}$ being the trivial σ-field. Now we determine the behavior of $L_{n,2\delta}^{(n)}$, $N_{n,2\delta}^{(n)}$ and $D_n^{(n)}$ of (1.6) for an arbitrary $0 < \delta < \infty$ as n goes to infinity. We have

$$\sum_{i=1}^{n-1} E\big(|X_{ni}|^{2+2\delta}\big) \sim E\big(|N(0,1)|^{2+2\delta}\big) n^{-\delta}$$

and

$$\begin{aligned} E\big(|X_{nn}|^{2+2\delta}\big) &= \int_R \int_R |y|^{2+2\delta} P(X_{nn} \in dy | N_{n-1} = x) P(N_{n-1} \in dx) \\ &= (2\pi)^{-1/2} \alpha_n^{2+2\delta} \int_{\alpha_n^{1/2}/2}^{\infty} x^{-2-2\delta} e^{-x^2/2}\, dx \sim C_\delta \alpha_n^{(3+2\delta)/2}, \end{aligned}$$

for some $0 < C_\delta < \infty$, hence $L_{n,2\delta}^{(n)} \sim C_\delta \alpha_n^{(3+2\delta)/2}$. Furthermore, taking into account that $E(X_{ni}^2|\mathscr{F}_{n,i-1}) = (n-1)^{-1}$ for $i = 1, \dots, n-1$ and

$$E\big(X_{nn}^2|\mathscr{F}_{n,n-1}\big) = E\big(X_{nn}^2|\mathbf{X}_{n-1}\big) = E\big(X_{nn}^2|N_{n-1}\big),$$

we see that

$$\begin{aligned} N_{n,2\delta}^{(n)} &= E\Big(E\big(X_{nn}^2|N_{n-1}\big)^{1+\delta}\Big) \\ &= \int_R \bigg(\int_R y^2 P(X_{nn} \in dy | N_{n-1} = x)\bigg)^{1+\delta} P(N_{n-1} \in dx) \\ &= (2\pi)^{-1/2} \alpha_n^{2+2\delta} \int_{\alpha_n^{1/2}/2}^{\infty} x^{-2-2\delta} e^{-x^2/2}\, dx \sim C_\delta \alpha_n^{(3+2\delta)/2}. \end{aligned}$$

Summarizing, we obtain

$$(3.1) \qquad \left(L_{n,2\delta}^{(n)} + N_{n,2\delta}^{(n)}\right)^{1/(3+2\delta)} \sim C_\delta \alpha_n^{1/2},$$

for some $0 < C_\delta < \infty$. On the other hand, we get

$$\begin{aligned}
P\left(\sum_{i=1}^{n} X_{ni} \le 0\right) &= P(X_{nn} \le -N_{n-1}) = \int_R P(X_{nn} \le -x | N_{n-1} = x) P(N_{n-1} \in dx) \\
&= (2\pi)^{-1/2} \int_{-\infty}^{\alpha_n^{1/2}/2} \varepsilon_0((-\infty, -x]) e^{-x^2/2}\, dx \\
&\quad + (2\pi)^{-1/2} \int_{\alpha_n^{1/2}/2}^{\infty} \left(\tfrac{1}{2}\varepsilon_{-\alpha_n/x} + \tfrac{1}{2}\varepsilon_{\alpha_n/x}\right)((-\infty, -x]) e^{-x^2/2}\, dx \\
&= (2\pi)^{-1/2} \int_{-\infty}^{0} e^{-x^2/2}\, dx + \tfrac{1}{2}(2\pi)^{-1/2} \int_{\alpha_n^{1/2}/2}^{\alpha_n^{1/2}} e^{-x^2/2}\, dx \\
&= \tfrac{1}{2} + \tfrac{1}{4}(2\pi)^{-1/2} \alpha_n^{1/2}(1 + o(1));
\end{aligned}$$

hence in view of (3.1)

$$\begin{aligned}
D_n^{(n)} = \sup_{x \in R} \left| P\left(\sum_{i=1}^{n} X_{ni} \le x\right) - \Phi(x) \right| &\ge \tfrac{1}{4}(2\pi)^{-1/2} \alpha_n^{1/2}(1 + o(1)) \\
\sim C_\delta \left(L_{n,2\delta}^{(n)} + N_{n,2\delta}^{(n)}\right)^{1/(3+2\delta)},
\end{aligned}$$

for some $0 < C_\delta < \infty$. This establishes the left inequality in (1.6), whereas the right one follows from Theorem 1 and (3.1).

4. Proof of Theorem 2. The quantities $L_{c,2\delta}$ and $N_{c,2\delta}$ are, of course, assumed to be finite throughout the proof.

First, we shall verify the assertion for a square integrable martingale $M = (M(t))_{0 \le t < \infty}$ w.r.t. the filtration $\mathbb{F} = (\mathscr{F}(t))_{0 \le t < \infty}$, i.e., a martingale with $\sup_{0 \le t < \infty} E(M^2(t)) < \infty$. We always set $\mathscr{F}(\infty) = \sigma(\mathscr{F}(t)\colon 0 \le t < \infty)$. Since we are interested only in the time interval $[0, 1]$, we assume w.l.o.g. that $M(t) = M(1)$ holds for all $1 \le t < \infty$, which can be attained by stopping the given martingale at time $t = 1$. As a consequence, we have $M(\infty) = M(1)$ for the random variable $M(\infty)$ closing the square integrable martingale M on $[0, \infty]$, and $\langle M \rangle(t) = \langle M \rangle(1)$ for all $1 \le t \le \infty$. Doob's inequality implies that $M^*(1) \equiv \sup_{0 \le t \le 1} |M(t)|$ is a square integrable random variable.

Fix $\varepsilon > 0$ arbitrarily. Let the random variables T_j, $j \ge 0$, be inductively defined by $T_0 = 0$ and

$$T_{j+1} = \inf\{t \in [0, \infty)\colon t \ge T_j, \langle M \rangle(t) \ge \langle M \rangle(T_j) + \varepsilon\}, \qquad j \ge 0,$$

where $\inf \varnothing = \infty$. Then by construction of T_j and right continuity of $\langle M \rangle$

$$(4.1) \qquad 0 = T_0 < T_1 \le T_2 \le \cdots \le T_j \le \cdots,$$

$$(4.2) \qquad T_j < T_{j+1}, \quad \text{on } \{T_j < \infty\} \text{ for each } j \ge 0,$$

and

$$\sup_{j\geq 0} T_j = \infty. \tag{4.3}$$

Since the process $\langle M\rangle$ is predictable, an induction argument based on Remark IV.87(d) in Dellacherie and Meyer (1978) shows that the variables T_j are predictable stopping times w.r.t. $\mathbb{F}$.

For any stopping time R, let $\mathscr{F}(R)$ denote the σ-field of all events occurring up to time R, whereas $\mathscr{F}(R-)$ is the σ-field of all events strictly prior to time R. According to elementary results about stopping times we have

$$T_j \text{ is } \mathscr{F}(T_j -)\text{-measurable for } j \geq 1, \tag{4.4}$$

$$\mathscr{F}(T_j -) \subset \mathscr{F}(T_j), \quad \text{for } j \geq 1, \tag{4.5}$$

and

$$\mathscr{F}(T_j) \subset \mathscr{F}(T_{j+1} -), \quad \text{for } j \geq 0, \tag{4.6}$$

where (4.6) holds in view of (4.1) and (4.2). Furthermore, we need the following facts about measurability:

(4.7) $M(R)$ and $\langle M\rangle(R)$ are $\mathscr{F}(R)$-measurable for each stopping time R,

and

(4.8) $M(R-)$ and $\langle M\rangle(R-)$ are $\mathscr{F}(R-)$-measurable for each predictable stopping time $R > 0$.

Statement (4.7) follows from Theorem IV.64(b) in Dellacherie and Meyer (1978), and (4.8) by an application of (4.7) to a sequence R_k, $k \geq 1$, of stopping times announcing the predictable stopping time R.

For any integer $j \geq 0$ we set

$$X_{2j+1} = \big(M(T_{j+1} -) - M(T_j)\big)I(T_j < 1),$$
$$X_{2j+2} = \Delta M(T_{j+1})I(T_{j+1} \leq 1),$$
$$\mathscr{F}_{2j+1} = \mathscr{F}(T_{j+1} -),$$

and

$$\mathscr{F}_{2j+2} = \mathscr{F}(T_{j+1}).$$

Statements (4.5) and (4.6) yield

$$\mathscr{F}_0 \equiv \mathscr{F}(0) \subset \mathscr{F}_1 \subset \mathscr{F}_2 \subset \cdots \subset \mathscr{F}_i \subset \cdots, \tag{4.9}$$

and from predictability of T_j and (4.4) to (4.8) we conclude that

$$X_i \text{ is } \mathscr{F}_i\text{-measurable for each } i \geq 1. \tag{4.10}$$

Square integrability of $M^*(1)$ implies $E(X_i^2) < \infty$ for each $i \geq 1$. Moreover, we have for each $i \geq 1$

$$E(X_i | \mathscr{F}_{i-1}) = 0. \tag{4.11}$$

For $i = 2j + 2$ with $j \geq 0$, (4.11) follows from

$$E(X_i|\mathscr{F}_{i-1}) = \{E(M(T_{j+1})|\mathscr{F}(T_{j+1}-)) - M(T_{j+1}-)\}I(T_{j+1} \leq 1)$$

and the stopping theorem for predictable stopping times, cf. Dellacherie and Meyer (1982), Theorem VI.14, which gives $E(M(T_{j+1})|\mathscr{F}(T_{j+1}-)) = M(T_{j+1}-)$. To verify (4.11) for $i = 2j+1$ with $j \geq 0$, we fix a sequence R_k, $k \geq 1$, of stopping times announcing the predictable stopping time T_{j+1}. Taking $R_k \vee T_j$ instead of R_k, if necessary, the R_k then satisfy

$$T_j \leq R_k, \quad \text{for each } k \geq 1, \tag{4.12}$$

$$R_k < T_{j+1}, \quad \text{on } \{T_j < \infty\} \text{ for each } k \geq 1, \tag{4.13}$$

and

$$R_k \uparrow T_{j+1}, \quad \text{as } k \to \infty. \tag{4.14}$$

From (4.13) and (4.14) we obtain

$$M(T_{j+1}-)I(T_j < 1) = \lim_{k\to\infty} M(R_k)I(T_j < 1). \tag{4.15}$$

Since $|M(R_k)I(T_j < 1)| \leq M^*(1)$ for all $k \geq 1$, (4.15) and the dominated convergence theorem for conditional expectations imply

$$E(M(T_{j+1}-)I(T_j < 1)|\mathscr{F}(T_j)) = \lim_{k\to\infty} E(M(R_k)I(T_j < 1)|\mathscr{F}(T_j)).$$

Observing (4.12), the stopping theorem yields

$$E(M(R_k)I(T_j < 1)|\mathscr{F}(T_j)) = M(T_j)I(T_j < 1),$$

for each $k \geq 1$, cf. Dellacherie and Meyer (1982), Theorem VI.10, so that (4.11) follows from

$$E(X_i|\mathscr{F}_{i-1}) = E(M(T_{j+1}-)I(T_j < 1)|\mathscr{F}(T_j)) - M(T_j)I(T_j < 1).$$

By construction, we have $M(1) = \Sigma_{i=1}^{\infty} X_i$, hence $M(1) = \lim_{n\to\infty}\tilde{S}_n$ for $\tilde{S}_n \equiv \Sigma_{i=1}^{2n} X_i$. In view of (4.1), (4.2) and (4.3), this convergence is stationary in the sense that for each ω there exists an n_0 such that $M(1)(\omega) = \tilde{S}_n(\omega)$ for all $n \geq n_0$. Consequently, we have for all $x \in R$,

$$|P(M(1) \leq x) - \Phi(x)| = \lim_{n\to\infty} |P(\tilde{S}_n \leq x) - \Phi(x)|. \tag{4.16}$$

According to (4.9), (4.10) and (4.11), for each $n \geq 1$ the variables $X_1, \ldots, X_{2n}$ form a square integrable mds w.r.t. the σ-fields $\mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_{2n}$. Therefore, Theorem 1 implies for all $n \geq 1$,

$$|P(\tilde{S}_n \leq x) - \Phi(x)| \leq C_\delta(L_{2n,2\delta} + N_{2n,2\delta})^{1/(3+2\delta)}, \tag{4.17}$$

where

$$L_{2n,2\delta} = \sum_{i=1}^{2n} E(|X_i|^{2+2\delta})$$

and

$$N_{2n,2\delta} = E\left(\left|\sum_{i=1}^{2n} E(X_i^2|\mathscr{F}_{i-1}) - 1\right|^{1+\delta}\right).$$

First, we shall consider $L_{2n,2\delta}$. For this, we write

$$L_{2n,2\delta} = \sum_{j=0}^{n-1} E\left(|X_{2j+1}|^{2+2\delta}\right) + \sum_{j=0}^{n-1} E\left(|X_{2j+2}|^{2+2\delta}\right) \equiv L^{(1)}_{2n,2\delta} + L^{(2)}_{2n,2\delta}.$$

Obviously, we have

$$L^{(2)}_{2n,2\delta} \le E\left(\sum_{j=0}^{\infty} |\Delta M(T_{j+1})|^{2+2\delta} I(T_{j+1} < 1) \right). \tag{4.18}$$

Next, we deal with $L^{(1)}_{2n,2\delta}$. For fixed j we apply (4.15) and Fatou's lemma to obtain

$$E\left(|X_{2j+1}|^{2+2\delta}\right) \le \liminf_{k\to\infty} E\left(|M(R_k) - M(T_j)|^{2+2\delta} I(T_j < 1)\right). \tag{4.19}$$

Fix now $k \ge 1$ and let $\gamma > 0$ be arbitrary. We define the random variables S_m, $m \ge 0$, inductively by $S_0 = T_j$ and

$$S_{m+1} = \inf\{t \in [0,\infty): t \ge S_m, |M(t) - M(S_m)| \ge \gamma\}, \qquad m \ge 0,$$

where $\inf \varnothing = \infty$. By construction of S_m and the fact that M is a cadlag process, we have

$$T_j = S_0 \le S_1 \le S_2 \le \cdots \le S_m \le \cdots, \tag{4.20}$$

$$S_m < S_{m+1}, \quad \text{on } \{S_m < \infty\} \text{ for each } m \ge 0, \tag{4.21}$$

and

$$\sup_{m \ge 0} S_m = \infty. \tag{4.22}$$

An induction argument based on Theorem IV.50 in Dellacherie and Meyer (1978) shows that the variables S_m are stopping times. Therefore, the variables $S'_m = S_m \wedge R_k$, $m \ge 0$, are also stopping times. They satisfy

$$T_j = S'_0 \le S'_1 \le \cdots \le S'_m \le \cdots, \tag{4.23}$$

because of (4.20). From (4.20) and (4.22) we see that $S'_m \uparrow R_k$ as $m \to \infty$, and the convergence is stationary on $\{T_j < 1\}$ since $R_k < \infty$ on this event by (4.13). Consequently, Fatou's lemma yields

$$\begin{aligned} &E\left(|M(R_k) - M(T_j)|^{2+2\delta} I(T_j < 1)\right) \\ &\quad \le \liminf_{m\to\infty} E\left(|M(S'_m) - M(T_j)|^{2+2\delta} I(T_j < 1)\right). \end{aligned} \tag{4.24}$$

For fixed $m \ge 1$ we introduce the random variables

$$Y_l = (M(S'_l) - M(S'_{l-1})) I(T_j < 1), \qquad l = 1, \dots, m,$$

and the σ-fields

$$\mathscr{F}'_l = \mathscr{F}(S'_l), \qquad l = 0, \dots, m,$$

which are increasing because of (4.23). Clearly, by square integrability of $M^*(1)$ and the stopping theorem, the variables $Y_1, \dots, Y_m$ form an mds w.r.t. $\mathscr{F}'_0 \subset \mathscr{F}'_1 \subset \cdots \subset \mathscr{F}'_m$. Applying a convex function inequality for martingales, cf.

Theorem 2.11 in Hall and Heyde (1980), we get

$$
\begin{aligned}
E\big(|M(S'_m) - M(T_j)|^{2+2\delta} I(T_j < 1)\big) &= E\left(\left|\sum_{l=1}^{m} Y_l\right|^{2+2\delta}\right) \\
&\le C_\delta \left\{ E\left(\left|\sum_{l=1}^{m} E\big(Y_l^2 | \mathscr{F}'_{l-1}\big)\right|^{1+\delta}\right) \right. \qquad (4.25)\\
&\qquad \left. + E\Big(\max_{1 \le l \le m} |Y_l|^{2+2\delta}\Big)\right\} \\
&\equiv C_\delta\{I + II\}.
\end{aligned}
$$

First, we shall consider I. Applying the stopping theorem to the uniformly integrable martingale $M^2 - \langle M \rangle$ one obtains, cf. Dellacherie and Meyer (1982), VII. (41.1),

$$
\begin{aligned}
E\big(Y_l^2 | \mathscr{F}'_{l-1}\big) &= E\big\{\big(M(S'_l) - M(S'_{l-1})\big)^2 | \mathscr{F}(S'_{l-1})\big\} I(T_j < 1) \\
&= E(\langle M \rangle(S'_l) - \langle M \rangle(S'_{l-1}) | \mathscr{F}(S'_{l-1})) I(T_j < 1) \qquad (4.26)\\
&= E\big\{(\langle M \rangle(S'_l) - \langle M \rangle(S'_{l-1})) I(T_j < 1) | \mathscr{F}(S'_{l-1})\big\}.
\end{aligned}
$$

Observe that the variables $(\langle M \rangle(S'_l) - \langle M \rangle(S'_{l-1}))I(T_j < 1)$ are nonnegative since $\langle M \rangle$ is increasing. Therefore, Garsia's inequality, cf. Hall and Heyde (1980), Theorem A.8, implies

$$
\begin{aligned}
I &\le C_\delta E\left\{\left|\sum_{l=1}^{m} (\langle M \rangle(S'_l) - \langle M \rangle(S'_{l-1})) I(T_j < 1)\right|^{1+\delta}\right\} \\
&= C_\delta E\big(|\langle M \rangle(S'_m) - \langle M \rangle(T_j)|^{1+\delta} I(T_j < 1)\big).
\end{aligned}
$$

Hence, setting

$$
Z_{j+1} = \big(\langle M \rangle(T_{j+1}-) - \langle M \rangle(T_j)\big) I(T_j < 1), \qquad j \ge 0,
$$

observing that $S'_m \le R_k < T_{j+1}$ on $\{T_j < 1\}$ by (4.13), and taking the definition of T_{j+1} into account, we conclude that

$$
I \le C_\delta \varepsilon^\delta E\big(Z_{j+1}\big).
$$

Next, we shall derive a bound for II in (4.25). For this, we set

$$
m_0 = \max\{m \ge 0 \colon S_m \le R_k\}.
$$

Observe that m_0 is well defined on $\{T_j < 1\}$ because of $R_k < \infty$ on this event by (4.13), and since (4.22) holds. By construction, $S'_l = S'_{l-1} = R_k$ for $l \ge m_0 + 1$,

$$
\big|M\big(S'_{m_0+1}\big) - M\big(S'_{m_0}\big)\big| = \big|M(R_k) - M\big(S_{m_0}\big)\big| \le \gamma,
$$

by definition of S_{m_0+1} and $S_{m_0+1} > R_k$, and $S'_l = S_l$ for $l \le m_0$. Hence

$$
\max_{1 \le l \le m} |Y_l|^{2+2\delta} \le \max_{1 \le l \le m \wedge m_0} \big|M(S_l) - M(S_{l-1})\big|^{2+2\delta} I(T_j < 1) + \gamma^{2+2\delta},
$$

which by definition of S_l is less than or equal to

$$C_\delta\left\{\max_{1\le l\le m\wedge m_0}|\Delta M(S_l)|^{2+2\delta}I(T_j<1)+\gamma^{2+2\delta}\right\},$$

which in view of $T_j<S_1<S_2<\cdots<S_{m_0}\le R_k<T_{j+1}$ on $\{T_j<1\}$ by (4.13) and (4.21) is less than or equal to

$$C_\delta\left\{\max_{T_j<t<T_{j+1}}|\Delta M(t)|^{2+2\delta}I(T_j<1)+\gamma^{2+2\delta}\right\}.$$

Consequently,

$$II\le C_\delta\left\{E\left(\max_{T_j<t<T_{j+1}}|\Delta M(t)|^{2+2\delta}I(T_j<1)\right)+\gamma^{2+2\delta}\right\}.$$

Substituting the above bounds for I and II on the right-hand side of (4.25), combining the result with (4.24) and (4.19) and observing that $\gamma>0$ is arbitrary leads to

$$E\left(|X_{2j+1}|^{2+2\delta}\right)\le C_\delta\left\{\varepsilon^\delta E(Z_{j+1})+E\left(\max_{T_j<t<T_{j+1}}|\Delta M(t)|^{2+2\delta}I(T_j<1)\right)\right\},$$

from which it is easy to deduce that

$$L^{(1)}_{2n,2\delta}\le C_\delta\left\{\varepsilon^\delta E(\langle M\rangle(1))+E\left(\sum_{j=0}^{\infty}\left(\sum_{T_j<t<T_{j+1}}|\Delta M(t)|^{2+2\delta}\right)I(T_j<1)\right)\right\}.$$

Combining this bound with (4.18) and observing that $\{T_j<1\}$ is the disjoint union of $\{T_{j+1}\le 1\}$ and $\{T_j<1<T_{j+1}\}$, and $\Delta M(t)=0$ for $1<t\le\infty$, we obtain

$$L_{2n,2\delta}\le C_\delta\{\varepsilon^\delta E(\langle M\rangle(1))+L_{c,2\delta}\}. \tag{4.27}$$

It remains to establish a bound for

$$N_{2n,2\delta}=E\left(\left|\sum_{j=0}^{n-1}E\left(X^2_{2j+1}|\mathscr{F}_{2j}\right)+\sum_{j=0}^{n-1}E\left(X^2_{2j+2}|\mathscr{F}_{2j+1}\right)-1\right|^{1+\delta}\right). \tag{4.28}$$

Applying (4.15) and the dominated convergence theorem for conditional expectations, we see that

$$E\left(X^2_{2j+1}|\mathscr{F}_{2j}\right)=\lim_{k\to\infty}E\left\{\left(M(R_k)-M(T_j)\right)^2I(T_j<1)|\mathscr{F}(T_j)\right\}.$$

For each fixed $k\ge 1$ the stopping theorem implies, cf. the proof of (4.26),

$$\begin{aligned}E&\left\{\left(M(R_k)-M(T_j)\right)^2I(T_j<1)|\mathscr{F}(T_j)\right\}\\&=E\left\{\left(\langle M\rangle(R_k)-\langle M\rangle(T_j)\right)I(T_j<1)|\mathscr{F}(T_j)\right\}.\end{aligned}$$

As $k\to\infty$, these random variables converge to $E(Z_{j+1}|\mathscr{F}(T_j))$ by the dominated convergence theorem for conditional expectations so that

$$\sum_{j=0}^{n-1} E\left(X_{2j+1}^2 | \mathscr{F}_{2j}\right) = \sum_{j=0}^{n-1} E\left(Z_{j+1} | \mathscr{F}_{2j}\right). \tag{4.29}$$

Since T_{j+1} is a predictable stopping time, we can apply a corollary to the stopping theorem, cf. formula VII.(41.2) in Dellacherie and Meyer (1982), to obtain

$$\sum_{j=0}^{n-1} E\left(X_{2j+2}^2 | \mathscr{F}_{2j+1}\right) = \sum_{j=0}^{n-1} \Delta\langle M\rangle(T_{j+1}) I(T_{j+1} \le 1).$$

Easy computations show that the last sum equals

$$\begin{aligned} &\sum_{j=0}^{n-1} \left\{\left(\langle M\rangle(T_{j+1}) - \langle M\rangle(T_j)\right) I(T_{j+1} \le 1)\right. \\ &\qquad \left. + \left(\langle M\rangle(T_{j+1}-) - \langle M\rangle(T_j)\right) I(T_j < 1 < T_{j+1})\right\} - \sum_{j=0}^{n-1} Z_{j+1} \\ &\quad = \langle M\rangle(T_n) I(T_n \le 1) + \langle M\rangle(1) I(T_n > 1) - \sum_{j=0}^{n-1} Z_{j+1}. \end{aligned}$$

Combining this result with (4.28) and (4.29), we have

$$\begin{aligned} N_{2n,2\delta} = E&\left(\left|\sum_{j=0}^{n-1} \left\{E\left(Z_{j+1} | \mathscr{F}_{2j}\right) - Z_{j+1}\right\}\right.\right. \\ &\left.\left. + \left(\langle M\rangle(T_n) - \langle M\rangle(1)\right) I(T_n \le 1) + \langle M\rangle(1) - 1\right|^{1+\delta}\right) \\ &\le C_\delta \left\{E\left(\left|\sum_{j=0}^{n-1} V_{j+1}\right|^{1+\delta}\right) + l(n) + N_{c,2\delta}\right\}, \end{aligned} \tag{4.30}$$

where

$$V_{j+1} = Z_{j+1} - E\left(Z_{j+1} | \mathscr{F}_{2j}\right), \qquad j \ge 0,$$

and

$$l(n) = E\left(|\langle M\rangle(T_n) - \langle M\rangle(1)|^{1+\delta} I(T_n \le 1)\right) \to 0, \quad \text{as } n \to \infty, \tag{4.31}$$

by the dominated convergence theorem, which applies because of (4.3) and $E(\langle M\rangle(1)^{1+\delta}) < \infty$, following from $N_{c,2\delta} < \infty$. Furthermore, in view of (4.7) and (4.8), for each $j \ge 0$ the variable V_{j+1} is $\mathscr{F}_{2j+2}$-measurable. Consequently, V_{j+1}, $j \ge 0$, is an mds w.r.t. the increasing σ-fields $\mathscr{F}_{2j+2}$, $j \ge 0$. By definition of T_{j+1}

we have $|V_{j+1}| \le 2\varepsilon$ for each j. Hence, by Burkholder's square function inequality,

$$E\left(\left|\sum_{j=0}^{n-1} V_{j+1}\right|^{1+\delta}\right) \le C_\delta E\left(\left(\sum_{j=0}^{n-1} V_{j+1}^2\right)^{(1+\delta)/2}\right)$$
$$\le C_\delta \varepsilon^{(1+\delta)/2} E\left(\left(\sum_{j=0}^{n-1} |V_{j+1}|\right)^{1+\delta}\right)^{1/2},$$

which by an application of Garsia's inequality is less than or equal to

$$C_\delta \varepsilon^{(1+\delta)/2} E\left(\left(\sum_{j=0}^{n-1} Z_{j+1}\right)^{1+\delta}\right)^{1/2} \le C_\delta \varepsilon^{(1+\delta)/2} E\big(\langle M\rangle(1)^{1+\delta}\big)^{1/2}.$$

Thus we obtain from (4.30)

$$N_{2n,2\delta} \le C_\delta\left\{\varepsilon^{(1+\delta)/2} E\big(\langle M\rangle(1)^{1+\delta}\big)^{1/2} + l(n) + N_{c,2\delta}\right\}. \tag{4.32}$$

Inequality (1.8) is now an obvious consequence of (4.16), (4.17), (4.27), (4.31), (4.32) and the fact that $\varepsilon > 0$ in (4.27) and (4.32) is arbitrary.

Now we consider a locally square integrable martingale M. By definition, there exists an increasing sequence T_n, $n \ge 1$, of stopping times such that $T_n \to \infty$ as $n \to \infty$ and the stopped processes M^{T_n} defined by $M^{T_n}(t) = M(T_n \wedge t)$ for $0 \le t < \infty$ are square integrable martingales. Then $M(1) = \lim_{n\to\infty} M^{T_n}(1)$, and the convergence is stationary. Hence for each $x \in R$,

$$|P(M(1) \le x) - \Phi(x)| = \lim_{n\to\infty} \left|P\big(M^{T_n}(1) \le x\big) - \Phi(x)\right|. \tag{4.33}$$

By the previous part of the proof we have for each x and n,

$$\begin{aligned} &\left|P\big(M^{T_n}(1) \le x\big) - \Phi(x)\right| \\ &\quad \le C_\delta\left\{E\Big(\sum_{0\le t\le 1} |\Delta M^{T_n}(t)|^{2+2\delta}\Big) + E\big(|\langle M^{T_n}\rangle(1) - 1|^{1+\delta}\big)\right\}^{1/(3+2\delta)}. \end{aligned} \tag{4.34}$$

Obviously, $|\Delta M^{T_n}| \le |\Delta M|$, so that for all $n \ge 1$,

$$E\Big(\sum_{0\le t\le 1} |\Delta M^{T_n}(t)|^{2+2\delta}\Big) \le L_{c,2\delta}. \tag{4.35}$$

Furthermore, by formula VII.(41.4) in Dellacherie and Meyer (1982), we have $\langle M^{T_n}\rangle = \langle M\rangle^{T_n}$ for each $n \ge 1$. Thus we see that $\langle M^{T_n}\rangle(1) \to \langle M\rangle(1)$ as $n \to \infty$ and $\langle M^{T_n}\rangle(1) = \langle M\rangle(T_n \wedge 1) \le \langle M\rangle(1)$ for all $n \ge 1$, whence by the dominated convergence theorem

$$E\big(|\langle M^{T_n}\rangle(1) - 1|^{1+\delta}\big) \to E\big(|\langle M\rangle(1) - 1|^{1+\delta}\big), \quad \text{as } n \to \infty. \tag{4.36}$$

Inequality (1.8) is now an obvious consequence of (4.33) to (4.36). □

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