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THE GENERATORS OF A GAUSSIAN WAVE ASSOCIATED WITH THE FREE MARKOV FIELD[1]

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Suppose $\Phi = \{\phi_a;\ a \in A\}$ is a Gaussian random field. Let ρ be a function on the parameter set A with values in an open interval I. To every t in I, there corresponds a subfield $\Phi_t = \{\phi_a;\ \rho(a) = t\}$ of the field Φ. The family Φ_t, $t \in I$, can be viewed as a Gaussian stochastic process. With a proper modification, this setup can be applied to generalized random fields for which the values at single points are not defined, in particular to the free field. In the case of a linear function ρ, the Gaussian process Φ_t plays a fundamental role in quantum field theory. It is a stationary Gaussian Markov process, where its Markov semigroup is given by the Feynman–Kac–Nelson formula. We prove that for a wide class of functions ρ, Φ_t is a nonhomogeneous Markov process and we evaluate the generators of this process.

1. Introduction.

1.1. Let X be the Brownian motion on a d-dimensional Euclidean space R^d, with the exponential killing rate $\frac{1}{2}$. The transition function of X is

$$P_t(x, y) = (2\pi t)^{-d/2}\exp\left\{-\frac{1}{2t}|x-y|^2 - \frac{1}{2}t\right\}.$$

The Green function is $g(x, y) = \int_0^\infty P_t(x, y)\,dt$. There are two Hilbert spaces associated with X, namely, the Green space $\mathscr{K}$ and the Dirichlet space $\mathscr{H}$. Let μ, ν be signed measures on R^d. We define $(\mu, \nu)_{\mathscr{K}} = \int\mu(dx)g(x, y)\nu(dy)$. Let $\mathscr{M} = \{\mu;\ (\mu, \mu)_{\mathscr{K}} < \infty\}$. The Green space $\mathscr{K}$ is the completion of $\mathscr{M}$ with norm $|\mu|_{\mathscr{K}} = (\mu, \mu)_{\mathscr{K}}^{1/2}$. For $\mu \in \mathscr{M}$, we define $G\mu = \int g(x, y)\mu(dy)$. Let $G(\mathscr{M}) = \{G\mu:\ \mu \in \mathscr{M}\}$. We define an inner product $(\ ,\ )_{\mathscr{H}}$ on $G(\mathscr{M})$ such that the mapping $\mu \to G\mu$ is an isometry. The Dirichlet space $\mathscr{H}$ is defined to be the completion of $G(\mathscr{M})$ with the norm $|h|_{\mathscr{H}} = (h, h)_{\mathscr{H}}^{1/2}$, for h in $G(\mathscr{M})$. The mapping $\mu \to G\mu$ is therefore extended to an isometry from $\mathscr{K}$ onto $\mathscr{H}$. The Hilbert space $\mathscr{H}$ is also called the Sobolev space $H^{+1}(R^d)$.

The free field Φ in R^d can be defined as the Gaussian field indexed by $\mathscr{K}$. Namely, $\Phi = \{\phi_f,\ f \in \mathscr{K}\}$, where ϕ_f is a Gaussian random variable on a probability space $(\Omega, \mathscr{F}, P)$ such that $\int\phi_f(\omega)P(d\omega) = 0$, and $\int\phi_f(\omega)\phi_h(\omega)P(d\omega) = (f, h)_{\mathscr{K}}$. Because of the isometry between $\mathscr{K}$ and $\mathscr{H}$, we may consider Φ to be indexed by $\mathscr{H}$. In the rest of this paper, we shall consider Φ to be indexed by the Dirichlet space $\mathscr{H}$.

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1.2. Let B be a Borel subset in R^d. Let $\mathscr{M}(B) = \{\mu \in \mathscr{M}: \operatorname{supp}\mu \subset B\}$. Then $G(\mathscr{M}(B)) = \{G\mu: \mu \in \mathscr{M}(B)\} \subset \mathscr{H}$. We denote by $\mathscr{H}(B)$ the minimal closed subspace of $\mathscr{H}$ which contains $G(\mathscr{M}(B))$. We call $\Phi(B) = \{\phi_f;\ f \in \mathscr{H}(B)\}$ the subfield of Φ in B.

Let ρ be a measurable function from R^d onto an interval I. Let $\rho^{-1}(t) = \{x \in R^d;\ \rho(x) = t\}$; then $\Phi_t = \Phi(\rho^{-1}(t))$, $t \in I$, is a Gaussian process. We denote this process by (Φ, ρ), and we call it a Gaussian wave. Suppose ρ is a continuous function; then the Markov property of Φ (see, e.g., [1]) implies that the Gaussian wave (Φ, ρ) is in general a nonhomogeneous Markov process. Our aim in this paper is to evaluate the generator of (Φ, ρ).

1.3. Let Π_t be the orthogonal projection of $\mathscr{H}$ onto $\mathscr{H}(\rho^{-1}(t))$. Let s be an interior point of I and ε be positive such that $(s-\varepsilon, s+\varepsilon) \subset I$. Suppose for each t in $(s-\varepsilon, s+\varepsilon)$, there is a f_t in $\mathscr{H}$. Let $F(t) = \Pi_t f_t$. We define a mapping Φ_F from $(s-\varepsilon, s+\varepsilon)$ to $\mathscr{H}$ by $\phi_F(t) = \phi_{F(t)}$. Under some smoothness conditions on ρ and f_t, we prove that ϕ_F is in the domain of the generator $\overline{H}_s$ of (Φ, ρ) at time s, and

$$\overline{H}_s \phi_F = \phi_{\Pi_s(\tilde{A} f_s + \partial f_s/\partial t)}, \quad P\text{-a.s.}, \tag{1.1}$$

where, for $x \in \rho^{-1}(s)$,

$$\tilde{A} f(x) = |\nabla\rho(x)|^{-1}\left\{\frac{df}{dn}(x) + \mathfrak{D} f(x)\right\}. \tag{1.2}$$

Here $(df/dn)(x)$ is the derivative of f at x in the direction of the exterior normal to the boundary, $\rho^{-1}(s)$, of $\rho^{-1}(\le s) = \{x;\ \rho(x) \le s\}$. $\mathfrak{D} f(x)$ is the interior normal derivative at x of the harmonic function (with respect to $\Delta - 1$) in $\rho^{-1}(\le s)$ which coincides with f on $\rho^{-1}(s)$. We note that $\Pi_s \tilde{A} f$ depends on the values of $\tilde{A} f(x)$ only for x in $\rho^{-1}(s)$.

In (1.1), $\overline{H}_s$ acts on linear functionals of Φ. $\overline{H}_s$ can be naturally extended to act on L^2-functionals of Φ by the following formulae. Let $(\ ,\ )$ be the inner product of $L^2(\Omega, \mathscr{F}, P)$, then the generating functional of Hermite polynomials of ϕ_f is

$$:\exp\phi_f: = \exp\{\phi_f - (\phi_f, \phi_f)/2\}. \tag{1.3}$$

If ϕ_F is in the domain of $\overline{H}_s$, then $:\exp\phi_F:$ is in the domain of $\overline{H}_s$ and

$$\overline{H}_s :\exp\phi_F: = \left[\overline{H}_s\phi_F - \left(\phi_{F(s)}, \overline{H}_s\phi_F\right)\right] :\exp\phi_{F(s)}:. \tag{1.4}$$

The operator $\tilde{A}$ in (1.2) also has appeared in [4] as the generator of a certain Markov process $\tilde{X}$—a stochastic wave which is the result of a random time change (related to ρ) of X. [In [4], stochastic waves associated with diffusions without killing have been considered, and (1.2) is also true for processes with exponential killing.]

1.4. The following particular case is of special importance in Euclidean quantum field theory. Let ρ be a function on R^d defined by $\rho(x, s) = s$,

$x \in R^{d-1}$, $s \in R^1$. Then we have the following integral expression for $\tilde{A}$:

$$(\tilde{A}f)(x,s) = \frac{\partial f}{\partial s}(x,s) - f(x,s) + \int_{R^{d-1}} \left[f(y,s) - f(x,s) - \sum_{i=1}^{d-1} (y^i - x^i) 1_{|y-x|<C} \frac{\partial f}{\partial x^i}(x,s) \right] h(y-x)\,dy, \tag{1.5}$$

$$h(x) = 2^2(2\pi)^{-d/2}\left\{K'_{-1+d/2}(|x|)|x|^{-d/2} + K_{-1+d/2}(|x|)|x|^{-1-d/2}\right\}. \tag{1.6}$$

Here $K_{-1+d/2}$ is the modified Bessel function of the third kind. In particular, for $d = 3$,

$$h(x) = 2^{-1/2}\pi^{-1}e^{-|x|}\left\{|x|^{-2} + |x|^{-3}\right\}. \tag{1.7}$$

In combination with (1.1) and (1.4), formula (1.5) gives an integral expression for the Hamiltonian of the dynamics of the free quantum field. This is another form of the Feynman–Kac–Nelson formula (see, e.g., [7]).

The results described in Section 1.3 are proved by Theorem 1 which is based on Lemma 2 and an additional assumption that the sets $\rho^{-1}(\leq t)$, $t \in I$, are bounded. In the case of linear ρ, Φ_t, $t \in R$, is a stationary Gaussian Markov process. We shall prove (1.5) in Section 4.

If we disregard the deterministic component $\partial/\partial t$ of $\tilde{A}$ in (1.5), the stochastic wave behaves locally at point x as an infinitely divisible process with Lévy–Khintchine measure V satisfying $V\{\infty\} = 1$ and $V(dy) = h(y)\,dy$.

2. The Gaussian wave associated with the free field.

2.1. Let $(\Omega, \mathscr{F}, P)$ be the underlying probability space of Φ. Since ρ is continuous in R^d, the range $\tilde{I}$ of ρ is an interval. Let I be the interior of $\tilde{I}$. For each t in $\tilde{I}$, we let $\Phi_t = \Phi(\rho^{-1}(t))$. To each subinterval $J \subset \tilde{I}$, there corresponds a sub-σ-algebra $\mathscr{F}_J$ of $\mathscr{F}$ generated by $\{\Phi_t,\ t \in J\}$. We shall write L for $L^2(\Omega, \mathscr{F}, P)$, and L_J for $L^2(\Omega, \mathscr{F}_J, P)$. The σ-algebras $\mathscr{F}_{\leq t}$, $\mathscr{F}_{\geq t}$, $\mathscr{F}_t$ and subspaces $L_{\leq t}$, $L_{\geq t}$, L_t correspond to the subintervals $\tilde{I} \cap (-\infty, t]$, $\tilde{I} \cap [t, \infty)$ and $\tilde{I} \cap \{t\}$, respectively. For $s < t$, $s, t \in \tilde{I}$, a linear transformation T_s^t: $L_t \to L_s$ is defined by

$$T_s^t \xi_t = E[\xi_t | \mathscr{F}_{\leq s}], \quad \text{for all } \xi_t \text{ in } L_t.$$

By the Markov property of Φ (see, e.g., [1]) and the continuity of ρ, (Φ, ρ) is a Markov process. Therefore, the orthogonal projection of $L_{\geq t}$ onto $L_{\leq t}$ is contained in L_t. Hence, T_s^t is a linear transformation from L_t into L_s and we have

$$T_s^t T_t^u = T_s^u, \tag{2.1}$$

for all $s < t < u$, s, t, u in $\tilde{I}$. Let $\varepsilon > 0$ such that $(s - \varepsilon, s + \varepsilon) \subset I$. Let ξ be a mapping from $(s-\varepsilon, s+\varepsilon)$ into L such that $\xi(t)$ is in L_t. We define

$$H_s\xi = \lim (t-s)^{-1}\big(T_s^t\xi(t) - \xi(s)\big), \tag{2.2}$$

if the limit exists in L as $t \downarrow s$. The domain $D(H_s)$ consists of all ξ where $H_s\xi$ exists. H_s is a linear transformation from $D(H_s)$ into L_s, which is called the generator of (Φ, ρ) at time s. We also consider the weak generator $\overline{H}_s$ which is defined by (2.2) with strong convergence replaced by weak convergence in L.

2.2. For a subinterval J of $\tilde{I}$, we let $\mathscr{H}_J$ be the minimal closed subspace of $\mathscr{H}$ which contains $\{\mathscr{H}(\rho^{-1}(t));\ t \in J\}$. Let Π_J be the orthogonal projection of $\mathscr{H}$ onto $\mathscr{H}_J$. The subspaces $\mathscr{H}_{\leq t}, \mathscr{H}_{\geq t}, \mathscr{H}_t$ and projections $\Pi_{\leq t}, \Pi_{\leq t}, \Pi_t$ correspond to subintervals $\tilde{I} \cap (-\infty, t]$, $\tilde{I} \cap [t, \infty)$, $\tilde{I} \cap \{t\}$, respectively. We denote by $\Pi_J^{J'}$ the restriction of Π_J to the subspace $\mathscr{H}_{J'}$. Let F be a mapping from $(s-\varepsilon, s+\varepsilon)$ into $\mathscr{H}$ such that $F(t)$ is in $\mathscr{H}_t$. We define

$$A_sF = \lim (t-s)^{-1}\big(\Pi_sF(t) - F(s)\big), \tag{2.3}$$

if the limit exists in $\mathscr{H}$ as $t \downarrow s$. In this case, we say that F is in the domain $D(A_s)$ of A_s. The weak operator $\overline{A}_s$ is defined in an analogous way. We note that Π_s can be replaced by Π_s^t in (2.3).

2.3. For an interval J in $\tilde{I}$, we let $\Gamma^1(\Phi_J)$ be the minimal subspace of L_J, which contains the subfield $\{\Phi_t,\ t \in J\}$. We shall write $\Gamma^1(\Phi)$ for $\Gamma^1(\Phi_{\tilde{I}})$. It follows from the definition of Φ that the mapping ϕ: $h \to \phi_h$ is an isometry from $\mathscr{H}$ onto $\Gamma^1(\Phi)$ and from $\mathscr{H}_J$ onto $\Gamma^1(\Phi_J)$. $E[\phi_h|\mathscr{F}_J]$ is the orthogonal projection of ϕ_h onto L_J. By the Gaussian property of Φ, $E[\phi_h|\mathscr{F}_J]$ is in $\Gamma^1(\Phi_J)$. Therefore, we have

$$E[\phi_h|\mathscr{F}_J] = \phi_{\Pi_J h}, \tag{2.4}$$

for all $h \in \mathscr{H}$. By (2.1) and (2.4), we obtain

$$\Pi_s^t\Pi_t^u = \Pi_s^u, \tag{2.5}$$

for all $s, t, u \in \tilde{I}$, $s < t < u$. If we restrict the operators to act on the space of linear functionals of Φ, then $T_s^t, H_s, \overline{H}_s$ go, under the isometry ϕ, into operators $\Pi_s^t, A_s, \overline{A}_s$. More precisely, let s be in I and $\varepsilon > 0$ such that $(s-\varepsilon, s+\varepsilon) \subset I$. Let F: $(s-\varepsilon, s+\varepsilon) \to \mathscr{H}$ such that $F(t) \in \mathscr{H}_t$, for all t in $(s-\varepsilon, s+\varepsilon)$. Then we have

$$T_s^t\phi_{F(t)} = \phi_{\Pi_s^tF(t)}. \tag{2.6}$$

Moreover, F is in $D(A_s)$ if and only if ϕ_F is in $D(H_s)$, and

$$H_s\phi_F = \phi_{A_sF}. \tag{2.7}$$

We shall let T_s^t and H_s act on the generating functionals of Hermite polynomials of Φ to obtain the actions of T_s^t and H_s on L^2-functionals of Φ. The Gaussian property of Φ implies that

$$T_s^t{:}\exp \phi_{F(t)}{:} = {:}\exp T_s^t\phi_{F(t)}{:}, \tag{2.8}$$

where $:\exp \phi_{F(t)}:$ is defined in (1.3). By (2.8), it is easy to obtain that if ϕ_F is in $D(H_s)$ [or, equivalently, F is in $D(A_s)$], then

$$H_s{:}\exp \phi_F{:} = \left[H_s\phi_F - \left(\phi_{F(s)}, H_s\phi_F\right)\right]{:}\exp \phi_{F(s)}{:}\,. \tag{2.9}$$

The set of generating functionals $\{:\exp \phi_f: ;\ f \in Q\}$ is linearly dense in L_t if Q is dense in $\mathscr{H}_t$. Hence, for $t \in (s - \varepsilon, s + \varepsilon)$, $\{:\exp \phi_{F(t)}: ;\ F \in D(A_s)\}$ is linearly dense in L_t if $\{F(t);\ F \in D(A_s)\}$ is dense in $\mathscr{H}_t$.

Relations between $\overline{H}_s$ and $\overline{A}_s$ are analogous to the relations between H_s and A_s. Therefore, to know the actions of T_s^t, H_s, $\overline{H}_s$, it is sufficient to know the actions of Π_s^t, A_s and $\overline{A}_s$.

2.4. Let ρ be a function defined on R^d satisfying the following conditions.

2.A. ρ is continuous.

2.B. For every x in R^d, $\inf\{t > 0;\ \rho(X_t) - \rho(x) > 0\} = 0$, P^x-a.s. [i.e., x is a regular point of the set $\{y;\ \rho(y) \le \rho(x)\}$].

By 2.A, the range of ρ is an interval $\tilde{I}$. Let I be the interior of $\tilde{I}$. For $t > 0$, let $\tau_t = \inf\{u > 0;\ \rho(X_u) - \rho(X_0) > t\}$, and $\tilde{X}_t = X_{\tau_t}$. If $\{u > 0;\ \rho(X_u) - \rho(X_0) > t\}$ is an empty set, we put $\tau_t = \infty$, and $\tilde{X}_t$ equals the cemetery ∂ of X. In [4], the stochastic process $\tilde{X}_t$ has been called a stochastic wave corresponding to (X, ρ). The following results follow from Theorem 1 of [4]. [Processes without killing have been considered in [4]; the results of Theorem 1 of [4] are also true, while Theorem 2 of [4] needs a slight modification for the processes with an exponential killing. For instance, a term of $-f(x, s)$ appears in (1.5) is due to the existence of our exponential killing rate $\frac{1}{2}$. This term does not appear in Theorem 2 of [4].]

(a) A stochastic wave $\tilde{X}$ is a strong Markov process.

(b) The infinitesimal operator $\tilde{A}$ of $\tilde{X}$ is defined by the following limit: For $f \in D(\tilde{A})$,

$$\tilde{A}f(x) = s\text{-}\lim_{t \downarrow 0} \frac{1}{t}\left[E^x f(\tilde{X}_t) - f(x)\right],$$

where s-lim means uniform convergence in R^d, and $D(\tilde{A})$ consists of those bounded measurable functions on R^d for which the limit exists.

Let x be in R^d. Suppose ρ is in $C^\infty(R^d)$, $\rho^{-1}(\le \rho(x))$ is bounded, and $\nabla\rho$ does not vanish in a neighborhood of $\rho^{-1}(\rho(x))$. Then every C^∞-function f with a compact support is in $D(\tilde{A})$ and $\tilde{A}f(x)$ is given by (1.2).

(c) Let $\mathfrak{S}$ be the topology in R^d generated by sets $\{x;\ r < \phi(x) < u\}$, then $\tilde{A}$ is a local operator in the following sense. Suppose $f_1, f_2 \in D(\tilde{A})$ and $f_1 = f_2$ on a $\mathfrak{S}$-neighborhood of x, then $\tilde{A}f_1(x) = \tilde{A}f_2(x)$. We put $f \in D_x(\tilde{A})$ if there exists $f^* \in D(\tilde{A})$ such that $f = f^*$ on a $\mathfrak{S}$-neighborhood of x. For every $f \in D_x(\tilde{A})$, we put $\tilde{A}f(x) = \tilde{A}f^*(x)$.

2.5. Let $\sigma_t = \inf\{u > 0;\ X_u \in \rho^{-1}(t)\}$ be the first hitting time of X at $\rho^{-1}(t)$. Let h be in $\mathscr{H}$. Then the orthogonal projection $\Pi_t h$ of h onto $\mathscr{H}_t$ is given by the formula (see, e.g., Theorem 7.4 in [2])

$$\Pi_t h(x) = E^x h(X_{\sigma_t}). \tag{2.10}$$

We shall also write $\Pi_t h(x)$ for $E^x h(X_{\sigma_t})$, for all measurable functions h on R^d, for which the expectation makes sense.

THEOREM 1. *Let s be in I. Suppose ρ is in $C^\infty(R^d)$ such that $\nabla\rho \neq 0$ on $\rho^{-1}(s)$ and $\rho^{-1}(\leq \delta)$ is a compact set, for some $\delta > s$. For a measurable function f defined on $I \times R^d$, we put $f_t(x) = f(t,x)$ and $F(t) = \Pi_t f_t$, for t in a neighborhood of s. If $f \in C^\infty(I \times R^d)$, then $F \in D(\bar{A}_s)$, and*

$$\bar{A}_s F = \Pi_s\left(\frac{\partial f_s}{\partial t} + \tilde{A} f_s\right). \tag{2.11}$$

REMARK 1. The set C_0^∞ of all C^∞-functions with compact supports in R^d is dense in $\mathscr{H}$, and hence the set of their projections on $\mathscr{H}_t$ is dense in $\mathscr{H}_t$. For any $f \in C_0^\infty$, we let $F(t) = \Pi_t f$, for all t in a neighborhood of s. By Theorem 1, F is in $D(\bar{A}_s)$. Therefore, $\{F(t);\ F \in D(\bar{A}_s)\}$ is dense in $\mathscr{H}_t$, for all t in a neighborhood of s.

REMARK 2. (1.1) follows from (2.11) and (2.7) for weak operators.

3. Proof of Theorem 1.

3.1. We shall write $\|f\|_p$ for the L^p-norm of a measurable function f on R^d with respect to the Lebesgue measure. For an subset V of R^d, we shall write $\|f\|_{H^p(\nu)}$ for the norm of f in the Sobolev space $H^p(\nu)$. The proof of Theorem 1 is based on the following lemma.

LEMMA 2. *Let ρ be a function satisfying* 2.A *and* 2.B. *Let $F(t) = \Pi_t f_t$, where $f_t \in \mathscr{H}$, for all $t \in I$. For $t > s$, we put*

$$U_{st} = (t-s)^{-1}(\Pi_t f_s - \Pi_s f_s),$$

$$V_{st} = (t-s)^{-1}(\Pi_t f_t - \Pi_t f_s).$$

Suppose

(i) *f_s is in $D_x(\tilde{A})$, for all $x \in \rho^{-1}(s)$,*
(ii) *$\limsup_{t\to s}|\Pi_s U_{st}|_{\mathscr{H}}$ is finite,*
(iii) *$\lim_{t\to s} V_{st}(x) = \partial f_s(x)/\partial t$, for all $x \in \rho^{-1}(s)$,*
(iv) *$\limsup_{t\to s}\sup\{|V_{st}(x)|;\ x \in \rho^{-1}(s)\}$ is finite,*
(v) *$\limsup_{t\to s}|\Pi_s V_{st}|_{\mathscr{H}}$ is finite.*

Then F is in $D(\bar{A}_s)$ and $\bar{A}_s F = \Pi_s[\tilde{A} f_s + \partial f_s/\partial t]$.

PROOF OF LEMMA 2. We shall evaluate the weak limit of (3.1) in $\mathscr{H}$ as $t \downarrow s$,

$$(t-s)^{-1}(\Pi_s \Pi_t f_t - \Pi_s f_s) = \Pi_s U_{st} + \Pi_s V_{st}. \tag{3.1}$$

By (ii), (v) and the 3ε-argument, to prove Lemma 2, it is sufficient to prove

$$\lim_{t \downarrow s} (\Pi_s U_{st}, h)_{\mathscr{H}} = (\Pi_s \tilde{A} f_s, h)_{\mathscr{H}}, \tag{3.2}$$

$$\lim_{t \downarrow s} (\Pi_s V_{st}, h)_{\mathscr{H}} = (\Pi_s\, \partial f_s/\partial t, h)_{\mathscr{H}}, \tag{3.3}$$

for all $h \in C_0^\infty(R^d)$ which is a dense subset of $\mathscr{H}$.

Let $h \in C_0^\infty(R^d)$ and $\tilde{h} = (-\Delta + 1)h$. Then

$$(\Pi_s U_{st}, h)_{\mathscr{H}} = \int dx\, \tilde{h}(x) E^x U_{st}(X_{\sigma_s}). \tag{3.4}$$

By (i) and the definition of $\tilde{A} f_s$, U_{st} converges to $\tilde{A} f_s$ uniformly on $\rho^{-1}(s)$ as $t \downarrow s$. $\int dx\, \tilde{h}(x) P^x(d\omega)$ is a finite measure, therefore the right-hand side of (3.4) converges to

$$\int dx\, \tilde{h}(x) E^x \tilde{A} f_s(X_{\sigma_s}),$$

which is $(\Pi_s \tilde{A} f_s, h)_{\mathscr{H}}$. This proves (3.2).

By (iii), (iv), the dominated convergence theorem and the analogous expression (3.4) for V_{st}, we obtain (3.3). □

3.2. PROOF OF THEOREM 1. By the assumption there exists $\delta > 0$ such that $\rho^{-1}([s, \delta])$ is a compact set, we may choose two $\mathfrak{S}$-neighborhoods V_1, V_2 and $\varepsilon > 0$ such that $\rho^{-1}([s, \varepsilon]) \subset V_1 \subset \bar{V}_1 \subset V_2$, where $\bar{V}_1$ and $\bar{V}_2$ are compact sets. Let w be a cutoff function which equals 1 in $\bar{V}_1$ and vanishes in V_2^c. To prove Theorem 1, it is sufficient to show the assumptions satisfy the conditions of Lemma 2.

STEP 1. We shall show that $\Pi_t f_t \in \mathscr{H}_t$, for all $s \le t \le \varepsilon$. For all $t \in I$, $wf_t \in C_0^\infty(R^d)$ and hence in $\mathscr{H}$. By (2.10), $\Pi_t w f_t = \Pi_t f_t$, for all t, $s \le t \le \varepsilon$. Therefore, $\Pi_t f_t \in \mathscr{H}_t$, for all t, $s \le t \le \varepsilon$.

STEP 2. We shall prove that $f_s \in D_x(\tilde{A})$, for all $x \in \rho^{-1}(s)$. Since $wf_s \in C_0^\infty(R^d)$, by (b) of Section 2.4, we have $wf_s \in D(\tilde{A})$. Since $wf_s = f_s$ on V_1, by (c) of Section 2.4, $f_s \in D_x(\tilde{A})$ for all $x \in V_1$ and hence for all $x \in \rho^{-1}(s)$.

STEP 3. We shall show that $\limsup \sup\{|V_{st}(x)|;\ x \in \rho^{-1}(s)\}$ is finite as $t \downarrow s$. By (2.10),

$$\begin{aligned} V_{st}(x) &= E^x (t-s)^{-1}\big(f_t(X_{\sigma_t}) - f_s(X_{\sigma_t})\big) \\ &= E^x (t-s)^{-1}\big(wf_t(X_{\sigma_t}) - wf_s(X_{\sigma_t})\big), \end{aligned} \tag{3.5}$$

for all t, $s < t \le \varepsilon$. By the mean value theorem, there exists θ, $s < \theta < t$, such that (3.5) equals

$$E^x \frac{\partial w f_\theta}{\partial t}(X_{\sigma_t}). \tag{3.6}$$

Since $M = \sup\{|\partial w f_t(x)/\partial t|;\ x \in R^d,\ s \le t \le \varepsilon\}$ is finite, $|V_{st}(x)|$ is bounded by M, for all $x \in \rho^{-1}(s)$, $s < t \le \varepsilon$.

STEP 4. Let $x \in \rho^{-1}(s)$. By 2.A, 2.B and the right continuity of X_t, we have $\lim X_{\sigma_t} = x$, P^x-a.s. as $t \downarrow s$. Since $\partial w f_t/\partial t$ is in $C^\infty(I \times R^d)$ with support in V_2, (3.5) goes to $\partial w f_s(X_{\sigma_t})/\partial t = \partial f_s(x)/\partial t$, P^x-a.s. as $t \downarrow s$. By the dominated convergence theorem, $\lim V_{st}(x) = \partial f_s(x)/\partial t$ as $t \downarrow s$, for all $x \in \rho^{-1}(s)$.

STEP 5. We shall show that $\limsup|\Pi_s V_{st}|_{\mathscr{H}}$ is finite as $t \downarrow s$. Let

$$\Psi_u = (1 - \Delta)(w f_u)/2, \tag{3.7}$$

and $l_t(x) = (\Psi_t - \Psi_s)(x)(t-s)^{-1}$, $t > s$. Then $\Psi \in C^\infty(I \times R^d)$ and $\operatorname{supp}\Psi_u \subseteq \operatorname{supp} w$. Therefore, $M = \sup\{|\partial \Psi_t(x)/\partial t|,\ x \in R^d,\ s \le t \le \varepsilon\}$ is finite. By the mean value theorem, $|l_t(x)| \le M$, for all $x \in R^d$, $s < t < \varepsilon$. Let

$$h_t(x) = E^x \int_0^\infty l_t(X_u)\, du. \tag{3.8}$$

Then $|h_t|_{\mathscr{H}}$ equals

$$\int\int l_t(x) g(x, y) l_t(y)\, dx\, dy,$$

which is bounded by

$$M^2 \int_{V_2}\int_{V_2} g(x, y)\, dx\, dy \tag{3.9}$$

for all $s < t < \varepsilon$.

By (2.10) and (3.5), $\Pi_s V_{st} = \Pi_s \Pi_t h_t$. Therefore, $\lim\sup|\Pi_s V_{st}|_{\mathscr{H}} = \limsup|\Pi_s \Pi_t h_t|_{\mathscr{H}}$ which is bounded by $\sup\{|h_t|_{\mathscr{H}},\ s < t < \varepsilon\}$, and hence it is bounded by (3.9).

STEP 6. We shall prove that $\limsup|\Pi_s U_{st}|_{\mathscr{H}}$ is finite as $t \downarrow s$. For notational simplicity, we put $\alpha(x) = \Pi_s U_{st}(x)$, for $s < t < \varepsilon$. By (2.10), and the strong Markov property of X,

$$\alpha(x) = E^x \int \Psi_s(X_u)(t-s)^{-1}\, du, \tag{3.10}$$

where the integration is over $[\sigma_s, \sigma_{st}]$ and σ_{st} is the first hitting time of X at $\rho^{-1}(t)$ after hitting at $\rho^{-1}(s)$. The Dirichlet norm of α can be computed as (see, e.g., Theorem 5.1 of [3])

$$|\alpha|^2_{\mathscr{H}} = \lim_{\nu \to 0} \int \alpha(x) E^x[\alpha(X_0) - \alpha(X_\nu)]\nu^{-1}\, dx. \tag{3.11}$$

By the Schwarz inequality, (3.11) is bounded by

$$\|\alpha\|_2 \lim_{\nu \to 0} \|E^x[\alpha(X_0) - \alpha(X_\nu)]\nu^{-1}\|_2. \tag{3.12}$$

Let $\beta(x) = E^x(t-s)^{-1}(\sigma_{st} - \sigma_s)$. By (3.10)

$$|\alpha(x)| \le \|\Psi_s\|_\infty \beta(x). \tag{3.13}$$

By (3.10) and the Markov property of X,

$$(3.14)\qquad \begin{aligned} &|E^x[\alpha(X_0) - \alpha(X_\nu)]\nu^{-1}| \\ &= \left|E^x\int_{\sigma_s}^{\sigma_{st}}(t-s)^{-1}\nu^{-1}E^{Xu}[\Psi_s(X_0) - \Psi_s(X_\nu)]\,du\right|. \end{aligned}$$

Ψ_s is C^∞ with a compact support; therefore,

$$(3.15)\qquad \lim_{\nu\to 0} E^x\nu^{-1}[\Psi_s(X_0) - \Psi_s(X_\nu)] = (1-\Delta)\Psi_s(x)/2,$$

uniformly in $\rho^{-1}(\leq t)$. X_u is in $\rho^{-1}(\leq t)$, for all $\sigma_s \leq u \leq \sigma_{st}$. Hence (3.14) is bounded by

$$(3.16)\qquad \beta(x)(\|(1-\Delta)\Psi_s/2\| + 1),$$

if ν is sufficiently small.

Combining (3.12), (3.13) and (3.16), we see that to prove that $\limsup|\Pi_s U_{st}|_{\mathscr{H}}$ is finite, it is sufficient to show $\limsup\|\beta\|_2$ is finite as $t \downarrow s$. To this end, we shall use some estimates for solutions of partial differential equations.

Let $\gamma(x) = E^x\sigma_t$ for $x \in \rho^{-1}(\leq t)$. By the strong Markov property of X,

$$(3.17)\qquad \beta(x) = E^x\gamma(X_{\sigma_s})(t-s)^{-1}.$$

Let y be in $\rho^{-1}(s)$. By the assumption $|\nabla\rho| > 0$ on $\rho^{-1}(s)$, there exists $y_t \in \rho^{-1}(t)$ such that y_t is in the normal direction of $\rho^{-1}(s)$ through y and the line segment, $[y_t, y]$, between y_t and y, is in $\rho^{-1}(\leq t)$.

Since $\gamma(y_t) = 0$, the mean value theorem implies that there exists $\theta \in (y_t, y)$ such that

$$(3.18)\qquad \begin{aligned} |\gamma(y)(t-s)^{-1}| &= |\nabla\gamma(\theta)(y-y_t)|(t-s)^{-1} \\ &= \left|\nabla\gamma(\theta)\frac{(y-y_t)}{|y-y_t|}\right||y-y_t|\,|\rho(y)-\rho(y_t)|^{-1}. \end{aligned}$$

The function γ satisfies the equations

$$(3.19)\qquad \begin{aligned} (1-\Delta)\gamma(x)/2 &= 1, \quad \text{for } x \in \rho^{-1}(<t), \\ \gamma(x) &= 0, \quad \text{for } x \in \rho^{-1}(t). \end{aligned}$$

Let j be the smallest integer which is greater than $1 + d/2$. By Sobolev's inequality (see, e.g., Theorem 11.1 of [5]), there exists a constant $M(t)$ such that

$$(3.20)\qquad \sup_{x\in\rho^{-1}(\leq t)}\left|\frac{\partial\gamma(x)}{\partial x_i}\right| \leq M(t)\|\gamma\|_{H^j(\rho^{-1}(<t))}.$$

By an a priori estimate (see, e.g., Theorem 18.1 of [5]), the right-hand side of (3.20) is bounded by

$$(3.21)\qquad M(t)\{\|2\|_{H^{j-2}(\rho^{-1}(<t))} = \|\gamma\|_2\}.$$

By the proofs of Theorem 11.1 and Theorem 18.1 of [5], $M(t)$ may be chosen to be bounded as $t \downarrow s$. Since $j - 2 \geq 0$, the H^{j-2}-norm of 2 equals the L^2-norm of 2 in $\rho^{-1}(<t)$ and it is bounded as $t \downarrow s$. Let

$$\eta(x) = E^x\int_0^\infty 1_{V_1}(X_s)\,ds.$$

Then $\eta(x) \geq \gamma(x)$, for all $x \in R^d$. Therefore $\|\gamma\|_2 \leq \|\eta\|_2$. $\|\eta\|_2$ is finite because of the boundedness of V_1 and the exponential decay of $g(x, y)$, as $|x - y|$ goes to ∞. Hence (3.21) is finite as $t \downarrow s$.

By the mean value theorem and the assumption that $|\nabla\rho(y)| \neq 0$ on $\rho^{-1}(s)$, $|y - y_t|\,|\rho(y) - \rho(y_t)|^{-1}$ is bounded by a constant for all y, and all t in a neighborhood of s.

By (3.18) and the above estimates, there exists a constant c such that

$$|\gamma(y)(t-s)^{-1}| \leq c, \tag{3.22}$$

for all $y \in \rho^{-1}(s)$ and all t in a neighborhood of s. By (3.17) and (3.22), for all t in a neighborhood of s,

$$|\beta(x)| \leq cE^x 1(X_{\sigma_s}). \tag{3.23}$$

The L^2-norm of the right-hand side of (3.23) is bounded and is independent of t, therefore $\limsup\|\beta\|_2$ is finite as $t \downarrow s$. □

4. An example.

4.1. Let ρ be a linear function on R^d defined by $\rho(x, t) = t$, for $x \in R^{d-1}$, $t \in R^1$. The Gaussian wave Φ_t, $t \in R$, corresponding to ρ is of special importance for quantum field theory (see, e.g., [7]).

Let θ_t: $(x, s) \to (x, s + t)$ be a translation of R^d. We define $\theta_t f(x, s) = f(x, s - t)$, for all functions on R^d, and $\theta_t\phi_f = \phi_{\theta_t f}$. Because of the translation invariance of g, θ_t can be extended to be an isometry from L_s, $\mathscr{H}_s$, $\Gamma^1(\Phi_s)$ onto L_{s+t}, $\mathscr{H}_{s+t}$, and $\Gamma^1(\Phi_{s+t})$, respectively. Again by the translation invariance of g, $\{\theta_t(\phi_{f_1}), \ldots, \theta_t(\phi_{f_n})\}$ and $\{\phi_{f_1}, \ldots, \phi_{f_n}\}$ have the same probability distribution, for all f_i in $\mathscr{H}$, $i = 1, \ldots, n$ and t in R. Therefore, Φ_t, $t \in R$, is a stationary Gaussian Markov process. In this case, we are interested in computing $H_s\phi_F$, where $F(t) = \theta_t F(0)$, for some $F(0)$ in $\mathscr{H}_0$. We shall choose $F(0)$ from a dense subspace $\mathscr{P}$ of $\mathscr{H}_0$. Let ξ be in the Schwartz space $\mathscr{S}(R^{d-1})$ and ξ_a be a measure concentrated on $\rho^{-1}(a)$ defined by

$$\xi_a(dx\,dt) = \xi(x)\,dx\,\delta_a(dt),$$

where δ_a is the Dirac δ-function at a. We let $\mathscr{P} = \{G\xi_0;\ \xi \in \mathscr{S}(R^{d-1})\}$. We define l by

$$l(x, t) = (G\xi_0)(x, 0), \qquad x \in R^{d-1},\ t \in R. \tag{4.1}$$

Let $F(0) = G\xi_0$, then $F(t) = \theta_t F(0) = \Pi_t l$, for all t. In this case, (1.1) is equivalent to

$$\bar{A}_s F = \Pi_s \tilde{A} l. \tag{4.2}$$

The proof of (4.2) is as follows.

Let Δ be the Laplacian operator on R^{d-1}. By a Fourier transform argument (see, e.g., page 191 of [6]),

$$\Pi_s G\xi_{t+s} = G(S_t\xi)_s, \tag{4.3}$$

where $S_t = \exp\{-t(1-\Delta)^{1/2}\}$.

Let $\beta = -G((1-\Delta)^{1/2}\xi)_s$. By (4.3), for $t > 0$,

$$\begin{aligned}
&|t^{-1}[\Pi_s\Pi_{s+t}F(s+t) - \Pi_s F(s)] - \beta|^2_{\mathscr{H}} \\
&\quad = |t^{-1}[G(S_t\xi)_s - G\xi_s] - \beta|^2_{\mathscr{H}} \\
&\quad = \|t^{-1}[S_t\xi - \xi] + (1-\Delta)^{1/2}\xi\|^2_{\mathscr{H}^{-1/2}(R^{d-1})} \qquad (4.4)\\
&\quad = \int t^{-2}\{e^{-t\mu(k)} - 1 + t\mu(k)\}^2 \mu(k)^{-1}|\hat{\xi}(k)|^2\,dk,
\end{aligned}$$

where $\mu(k) = (1+|k|^2)^{1/2}$, $\hat{\xi}$ is the Fourier transform of ξ in R^{d-1}, and the integration is taken over R^{d-1}. By the monotone convergence theorem, the last expression of (4.4) goes to zero as $t \downarrow s$. Therefore,

$$A_s F = \beta. \qquad (4.5)$$

For the other side of (4.2), we let $t > s$. Then

$$(t-s)^{-1}(\Pi_t l - \Pi_s l)(x,s) = (t-s)^{-1}[G\xi_t(x,s) - G\xi_s(x,s)]. \qquad (4.6)$$

By the symmetry of g, we have $G\xi_t(x,s) = G\xi_s(x,t)$. Therefore, (4.6) equals

$$(t-s)^{-1}[G\xi_s(x,t) - G\xi_s(x,s)]. \qquad (4.7)$$

Since ξ is in $\mathscr{S}(R^{d-1})$, $G\xi_s$ has bounded derivatives of all orders in $\rho^{-1}(>s)$, and their continuation to $\rho^{-1}(s)$ exist. Therefore, (4.7) goes to $\partial G\,\xi_s(x,s)/\partial n_x$ uniformly in x as $t \downarrow s$. Here $\partial G\,\xi_s(x,s)/\partial n_x$ is the limit of $\partial G\,\xi_s(x,t)/\partial t$ as $t \downarrow s$. By Green's theorem, $\partial G\,\xi_s(x,s)/\partial n_x = -\xi(x)$. By the translation invariance of l along the direction t, we obtain that (4.7) goes to $-\xi(x)$ uniformly in $(x,s) \in R^d$. By the definition of $\tilde{A}$, we then have

$$\tilde{A}l(x,s) = -\xi(x). \qquad (4.8)$$

Let $\eta \in \mathscr{S}(R^{d-1})$. By (4.8), we have $(\Pi_s\tilde{A}l, G\eta_s)_{\mathscr{H}} = -\int \xi(x)\eta(x)\,dx$, and hence it equals the inner product of $-(1-\Delta)^{1/2}\xi$ and η in $H^{-1/2}(R^{d-1})$. Since the mapping $\eta \to \eta_s$ is an isometry from $H^{-1/2}(R^{d-1})$ into $H^{-1}(R^d)$ which is $\mathscr{K}$, $(\Pi_s\tilde{A}l, G\eta_s)_{\mathscr{H}}$ equals the inner product of $-((1-\Delta)^{1/2}\xi)_s$ and η_s in $\mathscr{K}$, and hence equals the product of β and $G\eta_s$ in $\mathscr{H}$. Since $\{G\eta_s,\ \eta \in \mathscr{S}(R^{d-1})\}$ is dense in $\mathscr{H}_s$, we have $\Pi_s\tilde{A}l = \beta$. By (4.5), we obtain $\Pi_s\tilde{A}l = A_sF$. This proves (4.2).

4.2. We shall derive the integral expression for $\tilde{A}$ given by (1.5). We shall prove (1.5) for the case of R^3 only; for the case of other dimensions, the proof can be performed in the same way in terms of Bessel functions.

Let $x \in R^2$, $t \in R^1$ and $\rho(x,t) = t$. Let $f \in C_0^\infty(R^3)$ and $f(\partial) = 0$. We put $X_{\sigma_t} = (Y_{\sigma_t}, t)$. Then $\tilde{A}f(0,0)$ is the limit of

$$E^0 t^{-1}[f(X_{\sigma_t}) - f(0)],$$

which may be written as

$$\begin{aligned}
&E^0t^{-1}[f(Y_{\sigma_t},t) - f(y_{\sigma_t},0)] + E^0t^{-1}[f(0,0)\exp\{-\sigma_t/2\} - f(0,0)] \\
&\quad + E^0t^{-1}[f(Y_{\sigma_t},0) - f(0,0)\exp\{-\sigma_t/2\}].
\end{aligned} \qquad (4.9)$$

Since $f \in C_0^\infty(R^3)$, $t^{-1}[f(x,t) - f(x,0)]$ goes to $\partial f(x,0)/\partial t$ uniformly as $t \downarrow 0$. The dominated convergence theorem and $\lim Y_{\sigma_t} = 0$, P^0-a.s. as $t \downarrow 0$ imply that the first term of (4.9) goes to $\partial f(0,0)/\partial t$ as $t \downarrow s$. The second term of (4.9) goes to $-f(0,0)$ as $t \downarrow 0$.

The third term of (4.9) equals

$$\int [f(x,0) - f(0,0)] t^{-1} Q_t(x)\, dx, \tag{4.10}$$

where

$$Q_t(x) = 2^{-1/2}\pi^{-1} t e^{-r}(r^{-2} + r^{-3}) \tag{4.11}$$

is the hitting probability density of X at $\rho^{-1}(t)$ starting at 0, and $r^2 = t^2 + |x|^2$. Q_t is an even function; therefore (4.10) equals

$$\int t^{-1} Q_t(x) \left[f(x,0) - f(0,0) - 1_{|x| \le c} \sum_{i=1}^{2} x_i \frac{\partial f}{\partial x_i}(0,0) \right] dx, \tag{4.12}$$

for any constant $c > 0$. The above integrand is dominated by

$$e^{-|x|}(|x|^{-2} + |x|^{-3}), \quad \text{for } |x| \ge c,$$

$$e^{-|x|}|x|^2(|x|^2 + |x|^{-3}), \quad \text{for } |x| < c,$$

which is integrable. By the dominated convergence theorem, (4.12) goes to the third term on the right-hand side of (1.5), with h given by (1.7). Because of the translation invariance of X, we have thus proved (1.5).

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