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A MINI-MAX VARIATIONAL FORMULA GIVING NECESSARY AND SUFFICIENT CONDITIONS FOR RECURRENCE OR TRANSIENCE OF MULTIDIMENSIONAL DIFFUSION PROCESSES

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Let $L = \frac{1}{2}\nabla \cdot a\nabla + b \cdot \nabla$ generate a diffusion process on R^d. An expression involving a and b on $1 \le |x| \le n$ and two functions g and h, varied over suitable domains, attains its mini-max value at λ_n. It is shown that $\lim_{n\to\infty}\lambda_n = 0$ or $\lim_{n\to\infty}\lambda_n > 0$ according to whether the process is recurrent or transient.

1. Introduction. In this paper we develop an analytic criterion, specifically a mini-max variational formula, which gives necessary and sufficient conditions for transience or recurrence of multidimensional diffusion processes. Our result is a generalization of a result of Ichihara [4] for reversible diffusions, that is, diffusions with self-adjoint generators, which is based on the Dirichlet principle. Ichihara considered a diffusion $X(t)$ in R^d generated by $L = \frac{1}{2}\nabla \cdot a\nabla$, where $a(x)$ is a positive definite $d \times d$ matrix at each $x \in R^d$ with smooth entries a_{ij}. Let $\tau_n = \inf\{t \ge 0: |X(t)| = n\}$ and let $\phi_n(x) = P_x(\tau_1 < \tau_n)$ for $1 \le |x| \le n$. Then the process is recurrent if $P_x(\tau_1 < \infty) = \lim_{n\to\infty}\phi_n(x) = 1$ for all $x \in R^d$ satisfying $|x| \ge 1$ and is transient otherwise. We have $\phi_n(x) = 1$ for $|x| = 1$ and $\phi_n(x) = 0$ for $|x| = n$, and by Itô's formula, ϕ_n solves $L\phi_n = 0$ for $1 < |x| < n$. Now consider the energy integral $\frac{1}{2}\int_{\Sigma_n - \Sigma_1}(\nabla\phi a \nabla\phi)\,dx$, where $\Sigma_j = \{x \in R^d: |x| < j\}$. The classical Dirichlet principle states that

$$(1.1)\qquad \lambda_n = \inf_{\substack{\phi \in W^{1,2}(\Sigma_n - \Sigma_1)\\ \phi = 1 \text{ on } \partial\Sigma_1\\ \phi = 0 \text{ on } \partial\Sigma_n}} \frac{1}{2}\int_{\Sigma_n - \Sigma_1}(\nabla\phi a \nabla\phi)\,dx$$

is attained uniquely at $\phi = \phi_n$. From this and the preceding characterization of recurrence, Ichihara was able to show quite easily that the process is recurrent or transient according to whether $\lim_{n\to\infty}\lambda_n = 0$ or $\lim_{n\to\infty}\lambda_n > 0$. Ichihara then took up the more difficult task of analyzing the preceding energy integral and was able to obtain explicit sufficiency conditions for recurrence and for transience in terms of the matrix a, which compare favorably with the results of Friedman and Khasminskii [2, 5], when these latter results are restricted to the case of reversible diffusions. In fact, Ichihara's method covers the general self-adjoint case $L = \frac{1}{2}\nabla \cdot a\nabla + a\nabla Q \cdot \nabla$ for a smooth function Q, since in this case L can be written as $L = \frac{1}{2}e^{-2Q}\nabla \cdot ae^{2Q}\nabla$, and the factor e^{-2Q} does not

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affect transience or recurrence. In this general reversible case, we have

$$\lambda_n = \inf_{\substack{\phi \in W^{1,2}(\Sigma_n - \Sigma_1) \\ \phi = 1 \text{ on } \partial\Sigma_1 \\ \phi = 0 \text{ on } \partial\Sigma_n}} \frac{1}{2}\int_{\Sigma_n - \Sigma_1} (\nabla\phi a \nabla\phi) e^{2Q}\, dx = \frac{1}{2}\int_{\Sigma_n - \Sigma_1} (\nabla\phi_n a \nabla\phi_n) e^{2Q}\, dx, \tag{1.2}$$

where $L\phi_n = 0$ in $\Sigma_n - \Sigma_1$, $\phi_n = 1$ on $\partial\Sigma_1$ and $\phi_n = 0$ on $\partial\Sigma_n$, and

$$\lim_{n\to\infty} \lambda_n = 0 \quad \text{or} \quad \lim_{n\to\infty} \lambda_n > 0 \tag{1.3}$$

according to whether the process is recurrent or transient.

In this paper we will obtain a result analogous to (1.2) and (1.3) for the general non-self-adjoint diffusion generated by $L = \frac{1}{2}\nabla \cdot a\nabla + b \cdot \nabla$. Our result depends on a generalized Dirichlet principle for the non-self-adjoint operator L, which we developed in [7] and which we now describe. In order to apply our generalized Dirichlet principle, we require that the entries a_{ij} of a and the components b_i of b be in $C^{1,\alpha}_{\text{loc}}(R^d)$. Let $\tilde{L} = \frac{1}{2}\nabla \cdot a\nabla - b \cdot \nabla - \nabla \cdot b$ be the formal adjoint to the operator L. With coefficients as before, there exists a unique solution $\phi_n \in C^{2,\alpha}(\overline{\Sigma_n - \Sigma_1})$ to $L\phi_n = 0$ for $1 < |x| < n$ with $\phi_n = 1$ on $\partial\Sigma_1$ and $\phi_n = 0$ on $\partial\Sigma_n$. For $k \in C^{2,\alpha}(R^d)$, there also exists a unique solution $\tilde{\phi}_n \in C^{2,\alpha}(\overline{\Sigma_n - \Sigma_1})$ to $\tilde{L}\tilde{\phi}_n = 0$ in $\Sigma_n - \Sigma_1$ with $\tilde{\phi}_n = e^{2k}$ on $\partial\Sigma_1$ and $\tilde{\phi} = 0$ on $\partial\Sigma_n$. (See [3], Theorem 6.14.) By Itô's formula, $\phi_n(x) = P_x(\tau_1 < \tau_n)$. Our generalized Dirichlet principle states that the mini-max variation

$$\lambda_n^{(k)} = \inf_{\substack{g \in W^{1,2}(\Sigma_n - \Sigma_1) \\ g = e^k \text{ on } \partial\Sigma_1,\, g = 0 \text{ on } \partial\Sigma_n \\ (\text{dist}(x, \partial\Sigma_n))^{-1} g(x) \in L^\infty(\Sigma_n - \Sigma_1)}} \times \sup_{\substack{h \in W^{1,2}(\Sigma_n - \Sigma_1,\, g^2\, dx) \\ h = k \text{ on } \partial\Sigma_1}} \left[\frac{1}{2}\int_{\Sigma_n - \Sigma_1}\left(\frac{\nabla g}{g} - a^{-1}b\right) a \left(\frac{\nabla g}{g} - a^{-1}b\right) g^2\, dx - \frac{1}{2}\int_{\Sigma_n - \Sigma_1} (\nabla h - a^{-1}b) a (\nabla h - a^{-1}b) g^2\, dx\right] \tag{1.4}$$

is attained at the pair (g_n, h_n), where $g_n = (\phi_n\tilde{\phi}_n)^{1/2}$ and $h_n = \frac{1}{2}\log(\tilde{\phi}_n/\phi_n)$, and that

$$\lambda_n^{(k)} = \frac{1}{2}\int_{\Sigma_n - \Sigma_1} (\nabla\phi_n a \nabla\phi_n) \frac{\tilde{\phi}_n}{\phi_n}\, dx. \tag{1.5}$$

Note that an arbitrary function k appears in the variational formula (1.4). Thus, in fact, we obtain a family of generalized Dirichlet principles indexed by k. The function k plays the following role. If L is self-adjoint, that is $a^{-1}b = \nabla Q$, then the mini-max variational formula (1.4) reduces to the classical variational formula (1.2) if and only if $Q = k$ on $\partial\Sigma_1$. Thus, for fixed k, the generalized Dirichlet principle does not reduce to the classical one simultaneously for all self-adjoint L; however, given any Q, it is always possible to pick an appropriate k so that the generalized Dirichlet principle reduces to the classical one in the case $a^{-1}b = \nabla Q$. To be a little more specific about this and to see more clearly how (1.4) is related to (1.2), we can proceed as follows.

The supremum over all h in (1.4) is attained at h_g, which satisfies

$$\int_{\Sigma_n - \Sigma_1} \left(\nabla h_g - a^{-1}b\right) a \nabla q g^2 \, dx = 0$$

for all $q \in W^{1,2}(\Sigma_n - \Sigma_1)$ satisfying $q = 0$ on $\partial \Sigma_1$. Using this, it can be shown that (1.4) reduces to

$$\lambda_n^{(k)} = \inf_{\substack{g \in W^{1,2}(\Sigma_n - \Sigma_1) \\ g = e^k \text{ on } \partial\Sigma_1,\, g = 0 \text{ on } \partial\Sigma_n \\ (\operatorname{dist}(x, \partial\Sigma_n))^{-1} g(x) \in L^\infty(\Sigma_n - \Sigma_1)}} \frac{1}{2} \int_{\Sigma_n - \Sigma_1} \left(\frac{\nabla g}{g} - \nabla h_g\right) \times a \left(\frac{\nabla g}{g} - \nabla h_g\right) g^2 \, dx. \tag{1.6}$$

Now, the point is that h_g has the property that ∇h_g is the projection in $L^2(\Sigma_n - \Sigma_1, g^2\, dx)$ of $a^{-1}b$ onto the subspace of gradients ∇h which satisfy $h = k$ on $\partial\Sigma_1$. Thus, in particular, if $a^{-1}b = \nabla Q$, then in the case $k = Q$, one obtains $\nabla h_g = \nabla q$ independent of g and (1.6) becomes

$$\lambda_n^{(Q)} = \inf_{\substack{g \in W^{1,2}(\Sigma_n - \Sigma_1) \\ g = e^Q \text{ on } \partial\Sigma_1,\, g = 0 \text{ on } \partial\Sigma_n}} \int_{\Sigma_n - \Sigma_1} \left(\frac{\nabla g}{g} - \nabla Q\right) a \left(\frac{\nabla g}{g} - \nabla Q\right) g^2 \, dx. \tag{1.7}$$

Making the change of variables $\phi = ge^{-Q}$, reduces (1.7) to (1.2).

As noted in [7], if we wish, we may convert the mini-max principle (1.3) to the minimum principle

$$\lambda_n^{(k)} = \inf_{\substack{g \in W^{1,2}(\Sigma_n - \Sigma_1) \\ g = e^k \text{ on } \partial\Sigma_1,\, g = 0 \text{ on } \partial\Sigma_n \\ (\operatorname{dist}(x, \partial\Sigma_n))^{-1} g(x) \in L^\infty(\Sigma_n - \Sigma_1)}} \times \inf_{\substack{z = (z_1, \dots, z_n) \\ \nabla \cdot (g^2 z) = 0 \text{ in } D}} \left[\frac{1}{2} \int_{\Sigma_n - \Sigma_1} \left(\frac{\nabla g}{g} - a^{-1}b\right) a \left(\frac{\nabla g}{g} - a^{-1}b\right) g^2 \, dx + \int_D \left(\frac{1}{2}(zaz) + z(b - a\nabla k)\right) g^2 \, dx\right]. \tag{1.8}$$

We will replace the classical Dirichlet principle (1.2) by (1.4) and (1.5). However, the conclusion of (1.3) with λ_n replaced by $\lambda_n^{(k)}$ does not follow so readily from (1.4) and (1.5) as (1.3) did from (1.2) in the self-adjoint case. That this conclusion does in fact hold is the main result of this paper.

Theorem 1. *Let $L = \frac{1}{2}\nabla \cdot a\nabla + b \cdot \nabla$ be the generator of a diffusion process in R^d where $a(x)$ is a positive definite $d \times d$ matrix at each $x \in R^d$.*

Assume the entries a_{ij} of a and the components b_i of b are in $C^{1,\alpha}_{\mathrm{loc}}(R^d)$. For $k \in C^{2,\alpha}(R^d)$ an arbitrary fixed function, define $\lambda_n^{(k)}$ by (1.4) [*or* (1.8)]. *Then*

$$\lim_{n\to\infty} \lambda_n^{(k)} = 0 \quad \text{or} \quad \lim_{n\to\infty} \lambda_n^{(k)} > 0 \tag{1.9}$$

according to whether the process is recurrent or transient. Furthermore, in the reversible case, that is, the case $a^{-1}b = \nabla Q$, (1.4), (1.5) *and* (1.9) *reduce to* (1.2) *and* (1.3) *for all k satisfying $k = Q$ on $\partial \Sigma_1$.*

Before we prove Theorem 1, we give an application.

THEOREM 2. *Let $L = \frac{1}{2}\nabla \cdot a\nabla + a\nabla Q \cdot \nabla + b \cdot \nabla$, where $Q \in C^{2,\alpha}_{\mathrm{loc}}(R^d)$ and a and b are as in Theorem* 1. *Let $L_0 = \frac{1}{2}\nabla \cdot a\nabla + a\nabla Q \cdot \nabla$ and assume that $d \geq 2$.*

(i) *If $\nabla \cdot (e^{2Q}b) \equiv 0$ and the reversible process generated by L_0 is transient, then the process generated by L is also transient.*

(ii) *If $\nabla \cdot (e^{2Q}b) \geq 0$ for all x and $\nabla \cdot (e^{2Q}b) \not\equiv 0$, then the process generated by L is transient.*

In fact, if in the preceding conditions we replace $e^{2Q}b$ by $(a\nabla Q + b)$ and replace L_0 by $\frac{1}{2}\nabla \cdot a\nabla$, then the conclusion continues to hold.

From Theorem 2, we immediately obtain the following

COROLLARY. *Let $L = \frac{1}{2}\Delta + b \cdot \nabla$, where b is as in Theorem* 2.

(i) *If $d \geq 3$, then $\nabla \cdot b(x) \geq 0$ for all x is a sufficient condition for transience.*

(ii) *If $d = 2$, then $\nabla \cdot b(x) \geq 0$ for all x and $\nabla \cdot b \not\equiv 0$ is a sufficient condition for transience.*

PROOF OF THEOREM 2. First we note that the final statement of the theorem follows by writing $L = \frac{1}{2}\nabla \cdot a\nabla + \tilde{b}$, where $\tilde{b} = a\nabla Q + b$ and by setting $\tilde{Q} \equiv 0$. Now apply the first part of the theorem to L with $\tilde{b}$ and $\tilde{Q}$ in place of b and Q.

To prove (i) and (ii), let $k = Q$ and consider $\lambda_n^{(Q)}$ given by (1.4). Note that the b which appears there must be replaced by $\tilde{b} \equiv a\nabla Q + b$. Picking $h = Q$, we obtain

$$\begin{aligned}
\lambda_n^{(Q)} \geq \inf\Bigg[&\frac{1}{2}\int_{\Sigma_n - \Sigma_1}\left(\frac{\nabla g}{g} - \nabla Q\right)a\left(\frac{\nabla g}{g} - \nabla Q\right)g^2\,dx \\
&+ \int_{\Sigma_n - \Sigma_1}(g^2\nabla Qb - g\nabla gb)\,dx\Bigg] \\
= \inf\Bigg[&\frac{1}{2}\int_{\Sigma_n - \Sigma_1}\left(\frac{\nabla g}{g} - \nabla Q\right)a\left(\frac{\nabla g}{g} - \nabla Q\right)g^2\,dx \\
&+ \frac{1}{2}\int_{\Sigma_n - \Sigma_1}(\nabla \cdot b + 2\nabla Qb)g^2\,dx\Bigg] - \frac{1}{2}\int_{\partial\Sigma_1}(b \cdot \nu)\,e^{2Q}\,d\sigma,
\end{aligned} \tag{1.10}$$

where ν is the outward unit normal to $\Sigma_n - \Sigma_1$ at $\partial\Sigma_1$ and where the infimum is calculated over all $g \in W^{1,2}(\Sigma_n - \Sigma_1)$ which satisfy $g = e^Q$ on $\partial\Sigma_1$ and $g = 0$ on $\partial\Sigma_n$. [The condition $(\operatorname{dist}(x, \partial\Sigma_n))^{-1}g(x) \in L^\infty(\Sigma_n - \Sigma_1)$ can be dropped once we have fixed an h to use for all g.] Now the last term on the right-hand side of (1.10) satisfies $-\frac{1}{2}\int(b \cdot \nu)e^{2Q}\,d\sigma = \int_{\Sigma_1}\nabla \cdot (e^{2Q}b)\,dx$. Thus from (1.10) we have

$$\lambda_n^{(Q)} \ge \inf\left[\frac{1}{2}\int_{\Sigma_n - \Sigma_1}\left(\frac{\nabla g}{g} - \nabla Q\right)a\left(\frac{\nabla g}{g} - \nabla Q\right)g^2\,dx + \frac{1}{2}\int_{\Sigma_n - \Sigma_1}(\nabla \cdot b + 2\nabla Q b)g^2\,dx\right] + \int_{\Sigma_1}\nabla \cdot (e^{2Q}b)\,dx. \tag{1.11}$$

We first prove case (i). In this case, (1.11) reduces to

$$\lambda_n^{(Q)} \ge \inf\left[\frac{1}{2}\int_{\Sigma_n - \Sigma_1}\left(\frac{\nabla g}{g} - \nabla Q\right)a\left(\frac{\nabla g}{g} - \nabla Q\right)g^2\,dx\right]. \tag{1.12}$$

Now the right-hand side of (1.12) is exactly (1.7) which, as mentioned, reduces to (1.2), that is, to the λ_n corresponding to the generator $e^{2Q}L_0$. By assumption, the process corresponding to L_0 (or equally $e^{2Q}L_0$) is transient; thus $\lambda_\infty \equiv \lim_{n\to\infty}\lambda_n > 0$. From (1.12) it follows that $\lim_{n\to\infty}\lambda_n^{(Q)} \ge \lambda_\infty > 0$, and by Theorem 1, the process is transient.

We now consider case (ii). We note that of course there was no reason in particular to use $\partial\Sigma_1$ as the inner boundary in the preceding theory; everything goes through just as well if we use $\partial\Sigma_\rho$ for arbitrary $\rho > 0$. Thus consider (1.11) with Σ_1 replaced by Σ_ρ. By assumption, $\nabla \cdot (e^{2Q}b) \not\equiv 0$. Thus, for large enough ρ, the term $\int_{\Sigma_\rho}\nabla \cdot (e^{2Q}b)\,dx$ is strictly positive. Thus, for $n > \rho$, it follows that $\lambda_n^{(Q)} > \int_{\Sigma_\rho}\nabla \cdot (e^{2Q}b)\,d\sigma > 0$ and consequently $\lim_{n\to\infty}\lambda_n^{(Q)} > 0$. By Theorem 1, the process is transient. □

2. Proof of Theorem 1. As a first step in the proof, we need to analyze the function $\tilde{\phi}_n$ which solves $\tilde{L}\tilde{\phi}_n = 0$ in $\Sigma_n - \Sigma_1$ with $\tilde{\phi}_n = e^{2k}$ on $\partial\Sigma_1$ and $\tilde{\phi}_n = 0$ on $\partial\Sigma_n$. To do this, let $\hat{X}(t)$ be the diffusion generated by $\hat{L} = \frac{1}{2}\nabla \cdot a\nabla - b \cdot \nabla$ with corresponding probability measure $\hat{P}_x$ and expectation $\hat{E}_x$ on paths starting at $x \in R^d$. Let $\sigma_n = \inf\{t \ge 0: |\hat{X}(t)| = n\}$. By Itô's formula, $\tilde{\phi}_n(\hat{X}(t))\exp(-\int_0^t(\nabla \cdot b)(\hat{X}(s))\,ds)$ is a $\hat{P}_x$-local martingale for $x \in \Sigma_n - \Sigma_1$ and thus, for any $t \ge 0$ and $x \in \Sigma_n - \Sigma_1$,

$$\tilde{\phi}_n(x) = \hat{E}_x\tilde{\phi}_n\big(\hat{X}(t \wedge \sigma_1 \wedge \sigma_n)\big)\exp\left(-\int_0^{t\wedge\sigma_1\wedge\sigma_n}(\nabla \cdot b)\big(\hat{X}(s)\big)\,ds\right).$$

We now prove that

$$\lim_{t\to\infty}\hat{E}_x\tilde{\phi}_n\big(\hat{X}(t \wedge \sigma_1 \wedge \sigma_n)\big)\exp\left(-\int_0^{t\wedge\sigma_1\wedge\sigma_n}(\nabla \cdot b)\big(\hat{X}(s)\big)\,ds\right) = \hat{E}_x\tilde{\phi}_n\big(\hat{X}(\sigma_1 \wedge \sigma_n)\big)\exp\left(-\int_0^{\sigma_1\wedge\sigma_n}(\nabla \cdot b)\big(\hat{X}(s)\big)\,ds\right), \tag{2.1}$$

thus giving us the representation

$$(2.2)\qquad \tilde{\phi}_n(x) = \hat{E}_x\big(e^{2k(\hat{X}(\sigma_1))}\big)\exp\Big(-\int_0^{\sigma_1}(\nabla\cdot b)(\hat{X}(s))\,ds;\ \sigma_1<\sigma_n\Big).$$

To prove (2.1), we write

$$\begin{aligned}
&\lim_{t\to\infty}\hat{E}_x\tilde{\phi}_n(\hat{X}(t\wedge\sigma_1\wedge\sigma_n))\exp\Big(-\int_0^{t\wedge\sigma_1\wedge\sigma_n}(\nabla\cdot b)(\hat{X}(s))\,ds\Big)\\
&\quad=\lim_{t\to\infty}\hat{E}_x\Big(\tilde{\phi}_n(\hat{X}(\sigma_1\wedge\sigma_n))\exp\Big(-\int_0^{\sigma_1\wedge\sigma_n}(\nabla\cdot b)(\hat{X}(s))\,ds\Big);\ \sigma_1\wedge\sigma_n<t\Big)\\
&\qquad+\lim_{t\to\infty}\hat{E}_x\Big(\tilde{\phi}_n(\hat{X}(t))\exp\Big(-\int_0^{t}(\nabla\cdot b)(\hat{X}(s))\,ds\Big);\ t<\sigma_1\wedge\sigma_n\Big)\\
&\quad=\hat{E}_x\tilde{\phi}_n(\hat{X}(\sigma_1\wedge\sigma_n))\exp\Big(-\int_0^{\sigma_1\wedge\sigma_n}(\nabla\cdot b)(\hat{X}(s))\,ds\Big)\\
&\qquad+\lim_{t\to\infty}\hat{E}_x\Big(\tilde{\phi}_n(\hat{X}(t))\exp\Big(-\int_0^{t}(\nabla\cdot b)(\hat{X}(s))\,ds\Big);\ t<\sigma_1\wedge\sigma_n\Big),
\end{aligned}$$

by the monotone convergence theorem and the fact that $\tilde{\phi}_n \geq 0$ on $\partial\Sigma_1 \cup \partial\Sigma_n$. To complete the proof, we will show that

$$\lambda_0 \equiv \lim_{t\to\infty}\frac{1}{t}\log\hat{E}_x\Big(\tilde{\phi}_n(\hat{X}(t))\exp\Big(-\int_0^{t}(\nabla\cdot b)(\hat{X}(s))\,ds\Big);\ t<\sigma_1\wedge\sigma_n\Big)<0.$$

A result of Donsker and Varadhan [1] shows that λ_0 is related to the spectrum of the operator $\hat{L} - \nabla\cdot b$ on $\Sigma_n - \Sigma_1$ with the Dirichlet boundary condition on $\partial\Sigma_n$ and $\partial\Sigma_1$ by the formula $\lambda_0 = \sup(\mathrm{Re}(\mathrm{spec}(\hat{L} - \nabla\cdot b)))$. Now $\hat{L} - \nabla\cdot b = \tilde{L}$ and the operators $\tilde{L}$ and L on $\Sigma_n - \Sigma_1$ with the Dirichlet boundary condition on $\partial\Sigma_n$ and $\partial\Sigma_1$ have the same spectrum since they are adjoints of one another. It is well known that the spectrum of L is negative and bounded away from zero; thus $\lambda_0 < 0$. (Actually, what is happening is the following. By the Donsker–Varadhan theory,

$$\lambda_0 = \inf_{\substack{\mu\in P(R^d)\\ \mathrm{supp}(\mu)\subset\overline{\Sigma_n-\Sigma_1}}}\Big[\hat{I}(\mu) - \int_{\Sigma_n-\Sigma_1}(\nabla\cdot b)\,d\mu\Big],$$

where $\hat{I}(\mu)$ is the I-function for $\hat{X}(t)$. Using the explicit representation of the I-function [7], one can show that $\hat{I}(\mu) - \int_{\Sigma_n-\Sigma_1}(\nabla\cdot b)\,d\mu = I(\mu)$, where $I(\mu)$ is the I-function for $X(t)$. Thus we obtain $\lambda_0 = \inf_{\mu\in P(R^d),\,\mathrm{supp}\,\mu\subset\overline{\Sigma_n-\Sigma_1}} I(\mu)$ which, by [1], is equal to $\sup(\mathrm{Re}(\mathrm{spec}(L)))$.)

Now define $\phi_n(x) = \tilde{\phi}_n(x) \equiv 0$ for $|x| > n$. From (2.2) one sees that $\tilde{\phi}_n(x)$ is positive for $1 \leq |x| < n$ and that $\tilde{\phi}_n(x)$ is monotone increasing in n for each $x \in R^d$. With this fact, one direction of the proof may be proved easily. Assume that $\lim_{n\to\infty}\lambda_n^{(k)} = 0$; hence by (1.5)

$$(2.3)\qquad \lim_{n\to\infty}\int_{\Sigma_n-\Sigma_1}(\nabla\phi_n a\nabla\phi_n)\frac{\tilde{\phi}_n}{\phi_n}\,dx = 0.$$

We will demonstrate recurrence by proving that $\phi_\infty(x) \equiv \lim_{n\to\infty}\phi_n(x) = 1$ for $|x| \geq 1$. Fix $N > 1$ and note that $\sup_{n\geq N+1}\inf_{1\leq|x|\leq N}\tilde{\phi}_n(x) > 0$. This, together with (2.3) and the fact that $0 \leq \phi_n \leq 1$, gives us

$$\lim_{n\to\infty}\int_{\Sigma_N-\Sigma_1}|\nabla\phi_n|^2\,dx = 0. \tag{2.4}$$

We also have by the bounded convergence theorem

$$\lim_{n\to\infty}\int_{\Sigma_N-\Sigma_1}|\phi_\infty - \phi_n|^2\,dx = 0. \tag{2.5}$$

From (2.4) and (2.5), it follows easily that $\phi_n \to \phi_\infty$ strongly in $W^{1,2}(\Sigma_N - \Sigma_1)$ and that $\nabla\phi_\infty = 0$ a.e. in $\Sigma_N - \Sigma_1$. Consequently, by elliptic regularity, since $\phi_\infty(x) = 1$ for $|x| = 1$, we have $\phi_\infty(x) = 1$ for $1 \leq |x| \leq N$, where N is arbitrary.

The proof of the other direction is considerably more involved. Assume recurrence, that is, $\phi_\infty(x) = \lim_{n\to\infty}\phi_n(x) = 1$ for $|x| \geq 1$. We must show that $\lim_{n\to\infty}\lambda_n^{(k)} = \lim_{n\to\infty}\frac{1}{2}\int_{\Sigma_n-\Sigma_1}(\nabla\phi_n a\nabla\phi_n)\tilde{\phi}_n/\phi_n\,dx = 0$. Our first step is to show that in the recurrent case

$$\sup_{n\geq 1}\sup_{1\leq|x|\leq r}\tilde{\phi}_n(x) < \infty \quad \text{for all } r > 1. \tag{2.6}$$

The recurrence assumption guarantees the existence of a unique (up to multiplication by a positive constant) invariant σ-finite measure with strictly positive density $\theta(x)$ for the process $X(t)$. The strict positivity follows from the uniform ellipticity of L. By the smoothness of the coefficients, $\theta(x) \in C^2(R^d)$ and θ solves $\tilde{L}\theta = 0$. The result of (2.1) holds just as well for θ as for $\tilde{\phi}_n$; hence, similar to (2.2), we obtain $\theta(x) = \hat{E}_x\theta(\hat{X}(\sigma_1\wedge\sigma_n))\exp(-\int_0^{\sigma_1\wedge\sigma_n}(\nabla\cdot b)(\hat{X}(s))\,ds)$. Then, from (2.2), we have for $1 \leq |x| \leq r$,

$$\begin{aligned}
\tilde{\phi}_n(x) &\leq \Big(\sup_{|y|=1} e^{2k(y)}\Big)\hat{E}_x\Big(\exp\Big(-\int_0^{\sigma_1}(\nabla\cdot b)(\hat{X}(s))\,ds;\ \sigma_1 < \sigma_n\Big)\Big)\\
&\leq \Big(\inf_{|y|=1}\theta(y)\Big)^{-1}\Big(\sup_{|y|=1} e^{2k(y)}\Big)\hat{E}_k\theta(\hat{X}(\sigma_1\wedge\sigma_n))\\
&\quad\times\exp\Big(\int_0^{\sigma_1\wedge\sigma_n}(\nabla\cdot b)(\hat{X}(s))\,ds\Big)\\
&= \Big(\inf_{|y|=1}\theta(y)\Big)^{-1}\Big(\sup_{|y|=1} e^{2k(y)}\Big)\theta(x).
\end{aligned} \tag{2.7}$$

Now (2.6) follows from (2.7). It should be mentioned that both (2.6) and the positivity of $\tilde{\phi}_n(x)$ and its monotonicity in n, which followed as consequences of (2.2), can be obtained by nonprobabilistic arguments using a generalized maximum principle for operators with negative spectrum bounded away from zero.

Define $L_n = L + a(\nabla\phi_n/\phi_n)\cdot\nabla$ and its formal adjoint $\tilde{L}_n = \tilde{L} - a(\nabla\phi_n/\phi_n)\cdot\nabla - \nabla\cdot(a(\nabla\phi_n/\phi_n))$. One may verify that $L_n(\log\phi_n) = \frac{1}{2}(\nabla\phi_n a\nabla\phi_n)/\phi_n^2$ and that $\tilde{L}_n(\phi_n\tilde{\phi}_n) = 0$. Now let ψ be a C^∞-function satisfying $\psi = 0$ on $|x| = 1$, $\psi \equiv 1$ for $|x| \geq 2$ and $0 \leq \psi \leq 1$ for $|x| \geq 1$. Let $\gamma_n = \tilde{L}_n(\psi\phi_n\tilde{\phi}_n)$. Note that $\gamma_n \equiv 0$ for $|x| \geq 2$.

We now come to the key identity

$$\text{(2.8)}\quad \frac{1}{2}\int_{\Sigma_n-\Sigma_1}(\nabla\phi_n a\nabla\phi_n)\frac{\tilde{\phi}_n}{\phi_n}\psi\,dx = \int_{\Sigma_n-\Sigma_1}\gamma_n\log\phi_n\,dx = \int_{\Sigma_2-\Sigma_1}\gamma_n\log\phi_n\,dx.$$

This is verified as follows. Since

$$L_n(\log\phi_n) = \frac{1}{2}\frac{\nabla\phi_n a\nabla\phi_n}{\phi_n^2},$$

we have

$$\begin{aligned}\frac{1}{2}\int_{\Sigma_n-\Sigma_1}(\nabla\phi_n a\nabla\phi_n)\frac{\tilde{\phi}_n}{\phi_n}\psi\,dx &= \int_{\Sigma_n-\Sigma_1}\phi_n\tilde{\phi}_n\psi L_n(\log\phi_n)\,dx\\ &= \int_{\Sigma_n-\Sigma_1}(\log\phi_n)\tilde{L}_n(\phi_n\tilde{\phi}_n\psi) + \text{boundary terms}\\ &= \int_{\Sigma_n-\Sigma_1}\gamma_n\log\phi_n\,dx + \text{boundary terms.}\end{aligned}$$

All but one of the boundary terms that arise in the preceding integration by parts clearly vanish. The remaining boundary integral is

$$\lim_{\varepsilon\to 0}\int_{\partial\Sigma_{n-\varepsilon}}\psi\tilde{\phi}_n(\log\phi_n)\nu_\varepsilon a\nabla\phi_n\,d\sigma,$$

where ν_ε is the outward unit normal to $\Sigma_{n-\varepsilon}$ on $\partial\Sigma_{n-\varepsilon}$. To show that this limit is zero, it is enough to show that $\nabla\phi_n\cdot\nu_0 < 0$ and $\nabla\tilde{\phi}_n\cdot\nu_0 < 0$ on $\partial\Sigma_n$. For ϕ_n, this is the Hopf maximum principle. For $\tilde{\phi}_n$ (recall that $\tilde{L}$ has the zero-order term $-\nabla\cdot b$), this follows from a generalized version of the Hopf maximum principle which can be deduced from Theorem 10 in [8], Theorem 6.15 in [3] and the fact that $\mathrm{Re}(\sigma(\tilde{L})) = \mathrm{Re}(\sigma(L)) < 0$.

We have

$$\begin{aligned}\gamma_n = \tilde{L}_n(\psi\phi_n\tilde{\phi}_n) &= \phi_n\tilde{\phi}_n\tilde{L}_n\psi + \psi\tilde{L}_n(\phi_n\tilde{\phi}_n) + \left(\nabla\cdot b + \nabla\cdot a\frac{\nabla\phi_n}{\phi_n}\right)\psi\phi_n\tilde{\phi}_n\\ &\quad + \nabla\psi a\nabla(\phi_n\tilde{\phi}_n)\\ &= \phi_n\tilde{\phi}_n\left(\tilde{L}\psi - \frac{\nabla\psi a\nabla\phi_n}{\phi_n} - \psi\nabla\cdot a\frac{\nabla\phi_n}{\phi_n}\right) + \left(\nabla\cdot b + \nabla\cdot a\frac{\nabla\phi_n}{\phi_n}\right)\psi\phi_n\tilde{\phi}_n\\ &\quad + (\nabla\psi a\nabla\phi_n)\tilde{\phi}_n + (\nabla\psi a\nabla\tilde{\phi}_n)\phi_n\\ &= \phi_n\tilde{\phi}_n\hat{L}\psi + (\nabla\psi a\nabla\tilde{\phi}_n)\phi_n,\end{aligned}$$

where, as before, $\hat{L} = \frac{1}{2}\nabla\cdot a\nabla - b\cdot\nabla$. Thus we may rewrite the right-hand side of (2.8) as

$$\begin{aligned}\int_{\Sigma_2-\Sigma_1}\gamma_n\log\phi_n\,dx &= \int_{\Sigma_2-\Sigma_1}\phi_n\tilde{\phi}_n(\log\phi_n)\hat{L}\psi\,dx + \int_{\Sigma_2-\Sigma_1}(\nabla\psi a\nabla\tilde{\phi}_n)\phi_n\log\phi_n\,dx\\ &= \int_{\Sigma_2-\Sigma_1}\phi_n\tilde{\phi}_n(\log\phi_n)\hat{L}\psi\,dx - \int_{\Sigma_2-\Sigma_1}\phi_n\tilde{\phi}_n(\log\phi_n)\nabla\cdot a\nabla\psi\,dx\\ &\quad - \int_{\Sigma_2-\Sigma_1}(\nabla\psi a\nabla\phi_n)\tilde{\phi}_n\log\phi_n\,dx - \int_{\Sigma_2-\Sigma_1}(\nabla\psi a\nabla\phi_n)\tilde{\phi}_n\,dx,\end{aligned}\tag{2.9}$$

where we have used the fact that $\phi_n = 1$ on $\partial\Sigma_1$ and $\nabla\psi = 0$ on $\partial\Sigma_2$. Now (2.8) gives us

$$\begin{aligned}\lambda_n^{(k)} &= \frac{1}{2}\int_{\Sigma_n-\Sigma_1}(\nabla\phi_n a\nabla\phi_n)\frac{\tilde{\phi}_n}{\phi_n}\,dx\\ &\le \frac{1}{2}\int_{\Sigma_2-\Sigma_1}(\nabla\phi_n a\nabla\phi_n)\frac{\tilde{\phi}_n}{\phi_n}\,dx + \int_{\Sigma_2-\Sigma_1}\gamma_n\log\phi_n\,dx.\end{aligned}\tag{2.10}$$

By assumption, $\lim_{n\to\infty}\log\phi_n(x) = 0$ for $|x| \ge 1$ and by (2.6), $\tilde{\phi}_n$ is bounded independent of n on $\Sigma_2 - \Sigma_1$. It is also clear that

$$\inf_{n\ge n_0}\inf_{x\in\Sigma_2-\Sigma_1}\phi_n(x) = \inf_{x\in\Sigma_2-\Sigma_1}\phi_{n_0}(x) > 0 \quad \text{for } n_0 > 2. \tag{2.11}$$

From these facts and from (2.9), we obtain

$$\begin{aligned}\lim_{n\to\infty}\int_{\Sigma_2-\Sigma_1}\gamma_n\log\rho_n\,dx = -\lim_{n\to\infty}\int_{\Sigma_2-\Sigma_1}&(\nabla\psi a\nabla\phi_n)\\ &\times\left(\tilde{\phi}_n\log\phi_n + \tilde{\phi}_n\right)dx.\end{aligned}\tag{2.12}$$

In order to show that $\lim_{n\to\infty}\lambda_n^{(k)} = 0$, it suffices from (2.6) and (2.10)–(2.12) to show that $\lim_{n\to\infty}\int_{\Sigma_2-\Sigma_1}|\nabla\phi_n|^2\,dx = 0$, or equivalently, that

$$\lim_{n\to\infty}\int_{\Sigma_2-\Sigma_1}(\nabla\rho_n)^2\,dx = 0, \tag{2.13}$$

where $\rho_n = 1 - \phi_n$. Thus, to complete the proof, we will prove (2.13). Fix $m > 2$ and let $n \ge m$. Since $\rho_n \in C(\overline{\Sigma_m - \Sigma_1})$ and $L\rho_n = 0$ in $\Sigma_m - \Sigma_1$ with $\rho_n = 0$ on $\partial\Sigma_1$, it follows from standard elliptic regularity results about boundary behavior of solutions that $|\nabla\rho_n(x)| \le c\sup_{|y|=m}\rho_n(y)$ for some $c > 0$ and for $1 \le |x| \le 2$. Since $\rho_n(y)$ decreases monotonically to zero, it follows from Dini's theorem that $\lim_{n\to\infty}|\nabla\rho_n(x)| = 0$ for all $1 \le |x| \le 2$. Now (2.13) follows from this and the bounded convergence theorem. □

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