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ON THE MAXIMUM SEQUENCE IN A CRITICAL BRANCHING PROCESS[1]

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If $\{Z_n\}_0^\infty$ is a critical branching process such that $E_1 Z_1^2 < \infty$, then $(\log n)^{-1} E_i M_n \to i$, where E_i refers to starting with $Z_0 = i$ and $M_n = \max_{0 \le j \le n} Z_j$. This improves the earlier results of Weiner [9] and Pakes [7].

1. Introduction. Let $\{Z_n\}_0^\infty$ be a critical branching process. (See [1] for a complete definition and basic results.) Let $M_n = \max_{0 \le j \le n} Z_j$. This paper studies the growth rate of EM_n.

Here is a quick argument to show that $EM_n \to \infty$. By monotone convergence theorem $EM_n \to EM$, where $M = \max_{j \ge 0} Z_j$. Clearly,

$$E\{Z_n \colon Z_n > 0\} \le E(M \colon Z_n > 0),$$

and if $EM < \infty$, then this goes to zero since $P(Z_n > 0) \to 0$. On the other hand, $E\{Z_n \colon Z_n > 0\} = EZ_n = EZ_0$ for all n.

We establish the following:

Theorem 1. *If* $E_1 Z_1^2 < \infty$, *then* $(\log n)^{-1} E_i M_n \to i$, *where* E_i *refers to starting with* $Z_0 = i$.

This is an improvement over the results of Weiner [9] and Pakes [7]. The former showed that if there exists a finite K such that $P_1(Z_1 < K) = 1$, then there exists a finite positive constant α_K such that $EM_n \ge \alpha_K \log n$ with $\alpha_K \to 0$ as $K \to \infty$, and that if $E_1 Z_1^2 < \infty$, then there exists a β such that $EM_n \le \beta \log n$. Pakes [7] improved Weiner's upper bound and showed that

$$\limsup_n (EM_n)(\log n)^{-1} \le 1, \quad \text{if } E_1(Z_1^2) < \infty.$$

Here is an outline of the proof of Theorem 1.

Step 1. Pakes [7] has shown that for any nonnegative martingale $\{Z_n\}$ such that $E(Z_n(\log Z_n)^+) \to \infty$, Doob's maximal inequality ([2], page 263) can be improved to get

$$\limsup_n EM_n \left(EZ_n (\log Z_n)^+ \right)^{-1} \le 1.$$

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Since a critical branching process $\{Z_n\}$ is a nonnegative martingale for which $EZ_n(\log Z_n)^+$ does go to ∞ (see Proposition 1) this result is available to us without any additional moment conditions other than $E_1Z_1 = 1$.

Step 2. Using the conditional limit law of Yaglom (see [1], page 20) we establish (see Proposition 1) that if $E_1Z_1^2 < \infty$, then

$$\lim_n \left(E_iZ_n(\log Z_n)^+ - i \log n \right) = \mu_i, \qquad 0 < \mu_i < \infty,$$

exists for each $i \geq 1$.

Step 3. Using a stopping time argument we establish a bound for the lim inf (see Proposition 4).

If $E_1Z_1^2 < \infty$, then

$$\liminf_n E_iM_n(\log n)^{-1} \geq i.$$

Clearly, the above three steps imply Theorem 1.

2. Proofs.

PROPOSITION 1. *Let* $\{Z_n\}_0^\infty$ *be a critical branching process with offspring distribution* $\{p_j\}$. *Assume* $p_1 \neq 1$. *Then* (i) $E_iZ_n\log Z_n \to \infty$ *for each* $i \geq 1$, *and* (ii) *if, in addition,* $\sum j^2p_j < \infty$, *then for* $i \geq 0$, $\lim_n(E_iZ_n\log Z_n - i\log n) = i\mu$ *exists, where* $0 < \mu < \infty$.

PROOF. (i) It is enough to consider $i = 1$,

$$\begin{aligned} EZ_n\log Z_n &= E(Z_n\log Z_n|Z_n > 0)P(Z_n > 0) \\ &= E\left(a_n^{-1}Z_n\log\left(a_n^{-1}Z_n\right)|Z_n > 0\right) + \log a_n, \end{aligned}$$

if we choose $a_n = (P(Z_n > 0))^{-1}$.

Thus, $EZ_n\log Z_n - \log a_n \geq E(a_n^{-1}Z_n\log(a_n^{-1}Z_n)\colon\ 0 < Z_n < a_n|Z_n > 0)$. But since $\sup_{0<x<1}|x\log x| = C < \infty$, we get

$$\liminf(EZ_n\log Z_n - \log a_n) > -C,$$

proving (i) as $a_n \to \infty$.

(ii) Since $E_iZ_n = i$,

$$\begin{aligned} E_iZ_n\log Z_n - i\log n &= E_i(Z_n\log Z_n - Z_n\log n) \\ &= E_i\left(n^{-1}Z_n\log\left(n^{-1}Z_n\right)|Z_n > 0\right)nP_i(Z_n > 0). \end{aligned}$$

From (2) we see that $nP_i(Z_n > 0) \to i\lambda$. Now

$$\begin{aligned} E_i\left(n^{-2}Z_n^2|Z_n > 0\right) &= n^{-1}E_i\left(Z_n^2\colon Z_n > 0\right)\left(nP_i(Z_n > 0)\right)^{-1} \\ &= n^{-1}(in\sigma^2 + i^2)\left(nP_i(Z_n > 0)\right)^{-1}, \end{aligned}$$

and this converges to $2^{-1}\sigma^4$ as $n \to \infty$. Thus, $n^{-1}Z_n\log(n^{-1}Z_n)|Z_n > 0$ are uniformly integrable and (3) now implies (ii) with $\mu = \lambda\int_0^\infty(x\log x)\lambda e^{-\lambda x}\,dx$. □

PROPOSITION 2. *Let* $\{S_k\}_0^\infty$ *be random walk generated by the probability distribution* $\{p_j\}_0^\infty$. *Let* $\Sigma jp_j = 1$ *and* $\Sigma j^2 p_j < \infty$. *Let* $\rho > 1$ *and* $\psi(k, l) = E(S_k\colon S_k > \rho l)$. *Then* $\Sigma_l l^{-1}\psi(l, l) < \infty$.

PROOF. It is enough to consider the case $\rho = (1 + k^{-1})$ when k is a positive integer. Clearly,

$$\begin{aligned} l^{-1}\psi(l, l) &= l^{-1}E(S_l\colon S_l > \rho l) \\ &= l^{-1}E(S_l - \rho l\colon S_l > \rho l) + \rho P(S_l > \rho l) \\ &= a_l + b_l, \quad \text{say}. \end{aligned}$$

Since $\Sigma_j j^2 p_j < \infty$, $\Sigma_l b_l \le \rho \Sigma_l P(|S_l - l| > l/k) < \infty$ by the criterion for complete convergence due to Hsu and Robbins [5].

Next,

$$\begin{aligned} kla_l &= kE(S_l - \rho l\colon S_l > \rho l) \\ &= E(S_l'\colon S_l' > 0), \end{aligned}$$

where $\{S_l'\}$ is a random walk generated by the random variable $S_1' = k(S_1 - 1) - 1$. From Spitzer [8], pages 180–181, we know that

$$\sum_l l^{-1}E(S_l'\colon S_l' > 0) = \frac{E(S_T'\colon T < \infty)}{P(T = \infty)},$$

where

$$T = \begin{cases} \inf\{n\colon n \ge 1,\ S_n' > 0\}, \\ \infty, \quad \text{if } S_n' \le 0 \text{ for all } n \ge 1. \end{cases}$$

Since $ES_1' = -1 < 0$, $p = P(T = \infty) > 0$. Also $E(S_T'\colon T < \infty) \le EM'$, where $M' = \sup\{S_n'\colon n \ge 0\}$. Kiefer and Wolfowitz [6] have shown that $EM' < \infty$ if $ES_1'^2 < \infty$. But $ES_1'^2 < \infty$ is the same as $\Sigma j^2 p_j < \infty$. Thus, $\Sigma_l a_l < \infty$ and we are done. □

REMARK 1. Actually, the converse to Proposition 2 is also valid. If $\Sigma_l l^{-1}\psi(l, \rho l) < \infty$ for some $\rho > 1$, then $\Sigma b_l < \infty$ implying $\Sigma j^2 p_j < \infty$ by a converse proved by Erdös [3] to the Hsu and Robbins result [5] on complete convergence.

PROPOSITION 3. *Let* $\{Z_n\colon n = 0, 1, 2 \dots\}$ *be a nonnegative martingale. Fix* $n \ge 1$, $l > 1$. *Let* $T = T_{n,l}$ *be a stopping time defined by*

$$T = \begin{cases} \min\{r\colon 1 \le r \le n,\ Z_r = 0 \text{ or } Z_r \ge l\}, \\ n, \quad \text{if } 0 < Z_r < l \text{ for } 1 \le r \le n. \end{cases}$$

Then $E(Z_T\colon Z_T \ge l) \ge E(Z_n\colon Z_n \ge l)$.

PROOF. Since T is a bounded stopping time for the martingale $\{Z_n\}$, the optional sampling theorem applies so that $E(Z_T) = E(Z_n)$. The event

$(0 < Z_T < l)$ is the same as $T = n$ and $0 < Z_n < l$. Thus, $E(Z_T: 0 < Z_T < l) = E(Z_n: 0 < Z_n < l, T = n) \le E(Z_n: 0 < Z_n < l)$. Since $E(Z_T) = E(Z_n)$ the given assertion follows. □

PROPOSITION 4. *Let $\{Z_n\}$ be a critical branching process with offspring distribution $\{p_j\}$ such that $\sum j^2 p_j < \infty$. Then*

$$\liminf_n (\log n)^{-1} E_i M_n \ge i.$$

PROOF. Now let $\{Z_n\}$ be a critical branching process. It is a nonnegative martingale and so Proposition 3 applies. For fixed integers l and n and positive number $\rho > 1$, we get

$$\begin{aligned} E(Z_n: Z_n \ge l) &\le E(Z_T: Z_T \ge l) \\ &= E(Z_T: Z_T \ge l, Z_T \le \rho l) + E(Z_T: Z_T \ge l, Z_T > \rho l) \\ &= A_{n,l} + B_{n,l}, \quad \text{say.} \end{aligned}$$

Clearly, $A_{n,l} \le \rho l P(M_n \ge l)$, where $M_n = \max_{0 \le j \le n}\{Z_j\}$. Next

$$\begin{aligned} B_{n,l} &= \sum_{r=1}^{n} E(Z_T: Z_T > \rho l; T = r) \\ &= \sum_{r=1}^{n} E(Z_r: Z_r > \rho l; Z_1 < l, \dots, Z_{r-1} < l) \\ &\le \sum_{r=1}^{n} E(\psi(Z_{r-1}, l); 0 < Z_{r-1} < l), \end{aligned}$$

where $\psi(k, l) = E(S_k: S_k > \rho l)$, $\{S_k\}_0^\infty$ being random walk generated by the offspring distribution $\{p_j\}$. Since $\psi(k, l)$ is increasing in k for fixed l, we have

$$B_{n,l} \le \psi(l, l) \sum_{r=1}^{n} C_{r,l},$$

where

$$C_{r,l} = P(0 < Z_{r-1} < l).$$

We shall now show that if $\sum_j j^2 p_j < \infty$, then

$$\limsup_n (\log n)^{-1} \sum_{l=1}^{\infty} l^{-1} B_{n,l} = 0. \tag{1}$$

To do this we need the following facts proved in [1]. If $1 < \sum_1^\infty j^2 p_j < \infty$, then for $i \ge 1$ as $r \to \infty$,

$$rP_i(0 < Z_r) \to i\lambda \tag{2}$$

and

$$\sup_{x>0} |P_i(0 < Z_n < nx | Z_n > 0) - \phi(x)| \to 0, \tag{3}$$

where $\phi(x) = 1 - e^{-\lambda x}$, $\lambda = 2\sigma^{-2}$, σ^2 being the variance of $\{p_j\}$. Since $C_{r,l} \le$

$P(0 < Z_{r-1})$, (2) implies

$$\sup_{\substack{n \geq 2 \\ l \geq 1}} (\log n)^{-1} \sum_{r=1}^{n} C_{r,l} < \infty. \tag{4}$$

Next

$$\sum_{r=1}^{n} C_{r,l} \leq Kl + \sum_{r=Kl+1}^{n} C_r |P(0 < Z_{r-1} < l | Z_{r-1} > 0) - \phi\big(l(r-1)^{-1}\big)| + \phi(K^{-1}) \sum_{r=Kl+1}^{n} C_r,$$

where $C_r = P(Z_{r-1} > 0)$.

Using (2) and (3) and the fact that $\phi(x) \to 0$ as $x \to 0$, we get by choosing K first and then letting $n \to \infty$ that for each fixed l,

$$(\log n)^{-1} \sum_{r=1}^{n} C_{r,l} \to 0, \quad \text{as } n \to \infty. \tag{5}$$

We know from Proposition 2 that

$$\sum_{1} l^{-1} \psi(l, l) < \infty, \quad \text{if } \sum_{j} j^2 p_j < \infty. \tag{6}$$

Now (1) follows from (4), (5), (6) and the dominated convergence theorem. Returning now to the inequality

$$E(Z_n\colon Z_n \geq l) \leq A_{n,l} + B_{n,l} \leq \rho l P(M_n \geq l) + B_{n,l},$$

we have on adding over l,

$$\rho \sum_{l} P(M_n \geq l) + \sum_{l} l^{-1} B_{n,l} \geq \sum_{l} l^{-1} E(Z_n\colon Z_n \geq l),$$

yielding the inequality

$$\rho (\log n)^{-1} E M_n + (\log n)^{-1} \sum_{l} l^{-1} B_{n,l} \geq (\log n)^{-1} E\left(Z_n \sum_{1}^{Z_n} l^{-1} \right).$$

We know from (ii) in Proposition 1 that $\Sigma j^2 p_j < \infty$ implies

$$(\log n)^{-1} E_i Z_n \log Z_n \to i.$$

This with (1) and the above inequality yield

$$\liminf_{n} \rho (\log n)^{-1} E_i M_n \geq i.$$

Since $\rho > 1$ is arbitrary, the proof of Proposition 4 is complete. □

3. Extensions.

(a) *Markov branching processes.* Our Theorem 1 extends to any continuous-time Markov branching process with finite second moments. To deduce this from Theorem 1, we proceed as follows. The upper bound of Step 1 of Section 2 can be shown to hold for any continuous-time nonnegative martingale $\{Z_t\colon t \in [0, \infty)\}$ such that $EZ_t \log Z_t \to \infty$ and thus is valid for a critical Markov branching process. Also Yaglom's theorem holds in the continuous case (see [1], Chapter 3) and so Proposition 1 extends here. Combining these two, we get $\limsup_t E_i M_t (\log t)^{-1} \le i$. Finally, for $n \le t < n+1$ we have

$$M_t \ge M_n^* = \max_{0 \le j \le n} Z_j,$$

yielding

$$\liminf_t E_i M_t (\log t)^{-1} \ge \liminf_n E_i M_n^* (\log n)^{-1} \ge i,$$

by applying Theorem 1 to $\{Z_j\colon j = 0, 1, 2 \dots\}$ to get the last inequality.

A special case of the above result, namely, that of a critical birth and death process has been proved by Hammerle and Schuh in [4]. They use their result to establish a special case of our Theorem 1, namely, that of a linear fractional offspring distribution by embedding it in a critical birth and death process.

(b) *Multitype Galton–Watson process.* The methods of proof of Theorem 1 carry over to the multitype case. We state the result without proof.

THEOREM 2. *Let $\{\mathbf{Z}_n\colon n \ge 0\}$ be a p-type positively regular critical branching process with mean matrix M and finite second moments. Let $\mathbf{u}$ and $\mathbf{v}$ be right and left eigenvectors of M for the eigenvalue one normalized so that $\mathbf{u} \cdot \mathbf{1} = 1$, $\mathbf{u} \cdot \mathbf{v} = 1$, where $\mathbf{1}$ is a vector with all coordinates equal to one and $\cdot$ denotes dot product. Let $M_n = \max_{0 \le j \le n} \mathbf{v} \cdot \mathbf{Z}_j$. Then $(\log n)^{-1} E_{\mathbf{i}} M_n \to \mathbf{i} \cdot \mathbf{v}$ as $n \to \infty$.*

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