*The Annals of Probability*
1988, Vol. 16, No. 2, 479–486

# RATES OF CONVERGENCE FOR DENSITIES IN EXTREME VALUE THEORY

BY E. OMEY

*Economische Hogeschool St.-Aloysius*

In this paper we prove a rate of convergence result for the density of normalized sample maxima to the appropriate limit density. Also a local limit theorem—with rates of convergence—for maxima of i.i.d. random variables is proved.

**1. Introduction and main results.** In extreme value theory, not much is known about the quality of convergence in the case of weak convergence of normalized maxima to one of the limit distributions. Recently Smith (1982) and Omey and Rachev (1986a, b) established uniform rates of convergence in univariate and multivariate extreme value theory. Our focus in this paper is on uniform rates of convergence of the density of normalized partial maxima to the appropriate limit density.

Suppose $X_1, X_2, \ldots, X_n, \ldots$ are i.i.d. random variables with common d.f. $F$ and let $M_n = \max(X_1, \ldots, X_n)$. If for some choice of $a_n$ and $b_n$,

$$\lim_{n \to \infty} P\{a_n^{-1} M_n + b_n \le x\} = G(x)$$

for all $x$, then $F$ is said to be in the max domain of attraction of $G$. When this happens, $G$ must be one of the following three extreme value types:

$$\begin{aligned} \varphi_\alpha(x) &= \exp(-x^{-\alpha}), \qquad x \ge 0,\ \alpha > 0, \\ \Psi_\alpha(x) &= \exp(-(-x)^\alpha), \qquad x \le 0,\ \alpha > 0, \\ \Lambda(x) &= \exp(-e^{-x}), \qquad x \in \mathbb{R}. \end{aligned} \tag{1.1}$$

In the case where $F$ satisfies von Mises' condition, de Haan and Resnick (1982) proved the following result for the density of $M_n$:

LEMMA 1. *Suppose $F$ is absolutely continuous with bounded density $f$, which is positive for all $x$ sufficiently large. Let $f_n(x)$ denote the density of $a_n^{-1} M_n$, where $a_n$ is defined by $n^{-1} = -\log F(a_n)$. If for some $\alpha > 0$, $\lim_{x \to \infty}[xf(x)]/[1 - F(x)] = \alpha$, then as $n \to \infty$, $f_n(x) \to \varphi'_\alpha(x)$, uniformly in $x$.*

In this paper we estimate the rate of convergence in

$$\lim_{n \to \infty} \sup_x |f_n(x) - \varphi'_\alpha(x)| = 0.$$

For convenience our results are formulated in the case where the limit d.f.

Received April 1986; revised October 1986.
*AMS* 1980 *subject classifications*. Primary 60F05; secondary 60F99.
*Key words and phrases*. Density convergence, uniform rates of convergence, extreme value theory, local limit theorem.

$G(x) = \varphi_1(x)$. Using a monotonic transformation, similar results are easily established in case $G$ is one of the types (1.1). We present two types of results, which complement each other. In our first result, we prove that if the difference $f(x) - g(x)$ is "small," then also $f_n(x) - g(x)$ is "small." Here and in the sequel, $g(x)$ is the density of $G(x)$, i.e., $g(x) = x^{-2}G(x)$. In the second result, we use second order regular variation to estimate $f_n(x) - g(x)$.

THEOREM 2. *Let $F$ and $f$ be as in Lemma 1 and let $F(0+) = 0$. If $K := \sup_{x \geq 0} x^{1+s}|f(x) - g(x)| < \infty$ for some $s > 1$, then*

$$\limsup_{n\to\infty} n^{s-1} \sup_{x\geq 0} |f_n(x) - g(x)| < \infty, \tag{1.2}$$

*where $f_n(x) = n^2F^{n-1}(nx)f(nx)$ denotes the density of $n^{-1}M_n$. Also*

$$\limsup_{n\to\infty} n^{s-1} \sup_{x\geq 0} x^{1+s}|f_n(x) - g(x)| < \infty. \tag{1.3}$$

Under the conditions of the theorem, we also obtain the rate of convergence in $F^n(nx) = P\{n^{-1}M_n \leq x\} \to G(x)$ $(n \to \infty)$.

COROLLARY 3 [Omey and Rachev (1986a), Theorem 2.4]. *Under the conditions of Theorem 2 we have*

$$\limsup_{n\to\infty} n^{s-1} \sup_{x\geq 0} |F^n(nx) - G(x)| < \infty.$$

REMARK. In general, the result (1.2) with $< \infty$ replaced by $= 0$ does not hold, since this would imply that $\lim_{n\to\infty} n^{s-1}(F^n(nx) - G(x)) = 0$. In Omey and Rachev (1986a, Corollary 2.8), however, we proved that

$$\lim_{n\to\infty} n^{s-1}(F^n(nx) - G(x)) = \mathscr{C}x^{-s}G(x)$$

provided $\lim_{x\to\infty} x^s(F(x) - G(x)) = \mathscr{C}$.

In our next corollary we provide a local limit theorem with rates, hereby extending some of the results of de Haan and Resnick (1982).

COROLLARY 4. *Under the conditions of Theorem 2, for any $h > 0$ and sequence $d_n \to 0$ $(n \to \infty)$, we have*

$$\limsup_{n\to\infty} \min\left(d_n^{-1}, n^{s-1}\right) \sup_{x\geq 0} |d_n^{-1}P\left\{x < n^{-1}M_n \leq x + d_nh\right\} - hg(x)| < \infty.$$

In our next result, we modify the conditions of Theorem 2 and we prove a result which applies to d.f. $F(x)$ for which $f(x) \sim x^{-2}u(x)$ $(x \to \infty)$, where $u(x)$ is slowly varying. Whereas $u(x) \to 1$ $(x \to \infty)$ under the conditions of Theorem 2, we now assume that $u(x)$ is slowly varying with remainder function $h$ (or $h$-sv for short). A function $L(x)$ is said to be $h$-sv if $h(x) \to 0$ $(x \to \infty)$ and if for each

$t > 0$,

$$\frac{L(xt)}{L(x)} = 1 + O(h(x)), \qquad x \to \infty. \tag{1.4}$$

In order to formulate our result, we recall the following extension of Corollary 2 obtained by Smith [(1982), Theorem 1].

LEMMA 5. *Suppose that $F$ is continuous and $F(x) < 1$ for all $x \in \mathbb{R}$. Assume that $L(x) := -x \log F(x)$ satisfies* (1.4) *for some positive functions $h$ satisfying*

$$Bx^{-\theta} \le \frac{h(tx)}{h(t)} \le \mathscr{C}, \qquad x \ge 1,\ t \ge t_0,\ B, \mathscr{C}, t_0, \theta > 0. \tag{1.5}$$

*Let $a_n$ be defined by $-\log F(a_n) = n^{-1}$. Then as $n \to \infty$,*

$$\sup_x |F^n(a_n x) - G(x)| = O(h(a_n)).$$

The previous lemma is crucial in the proof of the following theorem.

THEOREM 6. *Let $L$ and $a_n$ be defined as in Lemma 5 and assume that $F$ has a bounded density $f$. Let $u(x) := x^2 f(x)$ and assume that $L$ and $u$ satisfy* (1.4) *with $h$ satisfying* (1.5). *Let $f_n$ be the density of $a_n^{-1} M_{n+1}$. Then as $n \to \infty$,*

$$\sup_x |f_n(x) - g(x)| = O(h(a_n)) + O\left(\left|\frac{n+1}{a_n} u(a_n) - 1\right|\right). \tag{1.6}$$

REMARK 1. The question of the optimal choice of the normalizing constants $a_n$ has been discussed by Smith [(1982), Section 4]. In the proof of Theorem 6 it turns out that it was convenient to normalize $M_{n+1}$ by $a_n$ instead of by $a_{n+1}$.

REMARK 2. By using the regular variation of $f$ and the choice of $a_n$, we have

$$\frac{(n+1)u(a_n)}{a_n} = (n+1) a_n f(a_n) \to 1, \qquad n \to \infty, \tag{1.7}$$

whence the rate of convergence in (1.6) depends on that in (1.7).

## 2. Proofs.

PROOF OF (1.2). Obviously

$$P\{n^{-1} M_n \le x\} = F^n(nx) \quad \text{and} \quad f_n(x) = n^2 F^{n-1}(nx) f(nx).$$

Also $g(x) = n^2 G^{n-1}(nx) g(nx)$. Now we write

$$\begin{aligned} f_n(x) - g_n(x) &= n^2 (F^{n-1}(nx) - G^{n-1}(nx))(f(nx) - g(nx)) \\ &\quad + n^2 G^{n-1}(nx)(f(nx) - g(nx)) \\ &\quad + n^2 g(nx)(F^{n-1}(nx) - G^{n-1}(nx)) \\ &=: \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

We estimate I, II and III separately.

(a) First consider II: Using $G^{n-1}(nx) = G(nx/(n-1))$ and the inequality

$$G(x) \le x^{\tau}B(\tau), \qquad x \ge 0,\ \tau \ge 0, \tag{2.1}$$

where $B(\tau) = (\tau e^{-1})^{\tau}$, we obtain

$$|\mathrm{II}| \le n^2(nx)^{s+1}(n-1)^{-s-1}B(1+s)|f(nx) - g(nx)|,$$

so that

$$|\mathrm{II}| \le n^2(n-1)^{-s-1}KB(1+s). \tag{2.2}$$

(b) Next consider III: Using the inequality

$$|a^m - b^m| \le m|a-b|\max(a^{m-1}, b^{m-1}), \tag{2.3}$$

we have

$$|\mathrm{III}| \le n^2 g(nx)(n-1)|F(nx) - G(nx)|\max(G^{n-2}(nx), F^{n-2}(nx)).$$

First consider

$$\mathrm{III}_a := n^2 g(nx)(n-1)|F(nx) - G(nx)|G^{n-2}(nx).$$

Using (2.1) with $\tau = s+2$ and using $g(x) = x^{-2}G(x)$, we obtain

$$\mathrm{III}_a \le n^2(n-1)^{-s-1}(nx)^s|F(nx) - G(nx)|B(2+s).$$

Since

$$|F(x) - G(x)| \le \int_x^{\infty}|f(u) - g(u)|\,du \le \frac{K}{s}x^{-s},$$

it follows that

$$\mathrm{III}_a \le n^2(n-1)^{-s-1}\frac{KB(2+s)}{s}. \tag{2.4}$$

Next consider

$$\mathrm{III}_b := n^2 g(nx)(n-1)|F(nx) - G(nx)|F^{n-2}(nx).$$

Let us first assume that $\mu$ defined by

$$\mu := \sup_{x \ge 0} x^s|\log F(x) - \log G(x)|$$

is finite and let $\{\delta_n\}_{\mathbb{N}}$ denote a sequence of positive numbers to be determined later. Obviously we have

$$|\log F^{n-2}(nx) - \log G^{n-2}(nx)| \le (n-2)\mu(nx)^{-s}.$$

Hence, if $nx \ge \delta_n$, we obtain that

$$F^{n-2}(nx) \le \{\exp(n-2)\mu\delta_n^{-s}\}G^{n-2}(nx). \tag{2.5}$$

Combining (2.5) and (2.4) we obtain

$$\sup_{nx \ge \delta_n} \mathrm{III}_b \le \{\exp(n-2)\mu\delta_n^{-s}\}n^2(n-1)^{-s-1}\frac{KB(2+s)}{s}. \tag{2.6}$$

In the case where $nx \le \delta_n$, we have $F^{n-2}(nx) \le F^{n-2}(\delta_n)$. Using (2.5), $|F(x) - G(x)| \le 2$ and $g(nx) \le B(2)$, we obtain

$$\sup_{nx \le \delta_n} \mathrm{III}_b \le 2B(2)(n-1)n^2\{\exp(n-2)\mu\delta_n^{-s}\}G^{n-2}(\delta_n). \tag{2.7}$$

Choosing $\delta_n = (n-2)^\delta$ with $1/s < \delta < 1$, we obtain from (2.6) and (2.7) that

$$\limsup_{n\to\infty} n^{s-1} \sup_{x\ge 0} \mathrm{III}_b \le \frac{KB(2+s)}{s}. \tag{2.8}$$

(c) Now let us consider I: Using (2.3) we obtain

$$|\mathrm{I}| \le \max(\mathrm{I}_a, \mathrm{I}_b)$$

where

$$\mathrm{I}_a := n^2(n-1)|f(nx) - g(nx)|\,|F(nx) - G(nx)|G^{n-2}(nx)$$

and $\mathrm{I}_b$ equals $\mathrm{I}_a$ with $G^{n-2}(nx)$ replaced by $F^{n-2}(nx)$. Using (2.1) with $\tau = 2s + 1$, we have

$$\mathrm{I}_a \le n^2(n-1)(n-2)^{-2s-1}B(2s+1)(nx)^{2s+1}$$
$$\times|f(nx) - g(nx)|\,|F(nx) - G(nx)|,$$

so that

$$\mathrm{I}_a \le n^2(n-1)(n-2)^{-2s-1}\frac{B(2s+1)K^2}{s}. \tag{2.9}$$

As to $\mathrm{I}_b$, as in part (b), we obtain

$$\limsup_{n\to\infty} n^{2s-2} \sup_{x\ge 0} \mathrm{I}_b \le \frac{B(2s+1)K^2}{s}. \tag{2.10}$$

(d) Now we remove the restriction that $\mu < \infty$. To this end, define the r.v. $Z$ with d.f. $F_Z$ as follows:

$$F_Z(x) = \begin{cases} 0, & x < 0, \\ F(a), & 0 \le x \le a, \\ F(x), & x \ge a, \end{cases}$$

where $a$ is such that $0 < F(a) < 1$. Obviously we have

$$\sup_{x\ge 0} x^s|F_Z(x) - F(x)| < \infty,$$

$$\mu := \sup_{x\ge 0} x^s|\log F_Z(x) - \log G(x)| < \infty$$

and

$$\sup_{x\ge 0}|F_Z^{n-2}(nx) - F^{n-2}(nx)| \le 2F^{n-2}(a).$$

Hence in considering $\mathrm{I}_b$ or $\mathrm{III}_b$, we may replace $F^{n-2}(nx)$ by $F_Z^{n-2}(nx)$ at the expense of the term $2F^{n-2}(a)$. Since $F^{n-2}(a)$ converges to zero geometrically fast, the estimates (2.8) and (2.10) remain correct.

(e) Combining the estimates (2.2), (2.4) and (2.8)–(2.10), we obtain

$$\limsup_{n\to\infty} n^{s-1} \sup_{x\geq 0}|f_n(x) - g(x)| \leq KB(1+s) + \frac{KB(2+s)}{s} + \frac{B(2s+1)K^2}{s},$$

which proves (1.2). □

**Proof of (1.3).** As in the Proof of (1.2), we use the decomposition $f_n(x) - g(x) = \mathrm{I} + \mathrm{II} + \mathrm{III}$. First consider II: Using $G(nx) \leq 1$, we have

$$x^{1+s}|\mathrm{II}| \leq (nx)^{1+s}|f(nx) - g(nx)|n^{1-s}G^{n-1}(nx) \leq Kn^{1-s}.$$

Next consider $\mathrm{III}_a$: Using $g = x^{-2}G(x)$ and (2.1) with $\tau = 1$, we have

$$\begin{aligned} x^{1+s}\mathrm{III}_a &\leq n^{1-s}(n-1)(nx)^s|F(nx) - G(nx)|G\left(\frac{nx}{n-1}\right)(nx)^{-1} \\ &\leq n^{1-s}\frac{KB(1)}{s}. \end{aligned}$$

In a similar way $\mathrm{III}_b$ and I can be estimated and this yields the proof of the result. □

**Proof of Corollary 3.** From (1.3), we have for some constant $\mathscr{C}$ that

$$|F^n(nx) - G(x)| \leq \int_x^\infty |f_n(u) - g(u)|\,du \leq \mathscr{C}n^{1-s}x^{-s}.$$

Also from (1.2), we have for some constant $\mathscr{C}$ that

$$|F^n(nx) - G(x)| \leq \int_0^x |f_n(u) - g(u)|\,du \leq \mathscr{C}n^{1-s}.$$

Combining these two estimates yields the proof of Corollary 3. □

**Proof of Corollary 4.** From (1.2), we obtain for some constant $\mathscr{C}$ that

$$|P\{x < n^{-1}M_n \leq x + d_n h\} - G(x + d_n h) + G(x)| \leq \mathscr{C}hd_n n^{1-s}.$$

Now $|g'(u)|$ is bounded by, say $B$; hence,

$$\begin{aligned} |d_n^{-1}\{G(x + d_n h) - G(x)\} - hg(x)| &\leq d_n^{-1}\int_x^{x+hd_n}\int_x^s |g'(u)|\,du\,ds \\ &\leq \frac{Bh^2 d_n}{2}. \end{aligned}$$

Combining these two estimates, we obtain

$$|d_n^{-1}P\{x < n^{-1}M_n \leq x + hd_n\} - hg(x)| \leq \mathscr{C}hn^{1-s} + \frac{Bh^2 d_n}{2}$$

and the proof of Corollary 4. □

**Proof of Theorem 6.** We have

$$\begin{aligned} f_n(x) - g(x) &= (n+1)\big(F^n(a_n x) - G(x)\big) f(a_n x) a_n \\ &\quad + g(x)(n+1)\big(u(a_n x) - u(a_n)\big) a_n^{-1} \\ &\quad + g(x)\big((n+1)u(a_n)a_n^{-1} - 1\big) \\ &=: \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

First consider I: Since $f$ is bounded and since $f$ is regularly varying with index $-2$, for some positive constants $\mathscr{C}$, $\varepsilon$ and $t_0$ and all $t \geq t_0$, we have

$$\frac{f(tx)}{f(t)} \leq \begin{cases} \mathscr{C}, & x \geq 1, \\ \mathscr{C} x^{-2-\varepsilon}, & x \leq 1,\ tx \geq t_0. \end{cases} \tag{2.11}$$

Also $[f(tx)]/[f(t)] \to x^{-2}$ $(t \to \infty)$ uniformly in compact $x$-intervals of $(0, \infty)$. Assume first $x \geq 1$. From (2.11) and Lemma 5 we have

$$\mathrm{I} = O\big((n+1)h(a_n)f(a_n)a_n\big),$$

which, using (1.8), is $O(h(a_n))$.

Next assume that $x \to 0$. Smith [(1982), page 605] shows that for some $\delta$ $(0 < \delta < \frac{1}{4})$ and $t_\delta$ (which may be taken larger than $t_0$) and all $x \geq a_n^{-1}t_\delta$, one has

$$|F^n(a_n x) - G(x)| \leq K h(a_n) x^{-2\theta - 1 + 4\delta} \exp\{-x^{-(1-4\delta)}\}.$$

Using this estimate and (2.11) we obtain that $\mathrm{I} = O(h(a_n))$ as $x \to 0$, $n \to \infty$, $x \geq a_n^{-1}t_\delta$. Finally, consider $x \leq a_n^{-1}t_\delta$. In this case we have

$$|\mathrm{I}| \leq (n+1)\big(F^n(t_\delta) + G(a_n^{-1}t_\delta)\big) f(a_n x) a_n.$$

Now $f$ is bounded and $F^n(t_\delta) \to 0$, $G(a_n^{-1}t_\delta) \to 0$ geometrically fast. Since the conditions of the theorem guarantee that $h(a_n)$ decreases at most as some power of $n$, we also obtain the estimate $\mathrm{I} = O(h(a_n))$ here.

Next we consider $f_n(x) - g(x)$ when $x \leq a_n^{-1}t_\delta$. Using the boundedness of $f$, we have

$$f_n(x) \leq \mathscr{C}(n+1)a_n F^n(t_\delta)$$

and, as before, $f_n(x) = O(h(a_n))$ $(n \to \infty)$ uniformly in $x \leq a_n^{-1}t_\delta$. As to $g(x)$, using (2.1) we have $g(x) = x^{-2}G(x) \leq B(\tau)x^{\tau-2}$. If $\tau > 2$, we obtain that

$$g(x) \leq \mathscr{C} a_n^{2-\tau}$$

for some constant $\mathscr{C}$. Now from (1.5) we have $0 < \liminf_{x\to\infty} y^\theta h(y)$. Hence

$$\frac{g(x)}{h(a_n)} \leq \mathscr{C} a_n^{2-\tau} a_n^\theta$$

and with a suitable choice of $\tau$, we obtain that $g(x) = O(h(a_n))$ uniformly in $x \leq a_n^{-1}t_\delta$.

Now we consider II with $x \geq a_n^{-1}t_\delta$. Under the conditions of the theorem, Corollary 3.6 of Bingham and Goldie (1982) applies: For some positive constants

$\mathscr{C}$, $\varepsilon$ and $n_0$ and all $n \geq n_0$ there holds

$$\left|\frac{u(a_n x) - u(a_n)}{u(a_n)h(a_n)}\right| \leq \begin{cases} \mathscr{C}(1 + \log x), & x \geq 1, \\ \mathscr{C}x^{-\varepsilon}, & x \leq 1. \end{cases}$$

Since $g(x) \leq B(\tau)x^{\tau-2}$, we have

$$\mathrm{II} = O\left(\frac{(n+1)u(a_n)h(a_n)}{a_n}\right) \quad \text{as } n \to \infty.$$

Using (1.8) we obtain

$$\mathrm{II} = O(h(a_n)) \quad \text{as } n \to \infty.$$

To complete the proof of the theorem, using the boundedness of $g(x)$ it immediately follows that

$$\mathrm{III} = O\left(\left|\frac{n+1}{a_n}u(a_n) - 1\right|\right). \qquad \square$$

## REFERENCES

BINGHAM, N. H. and GOLDIE, C. M. (1982). Extensions of regular variation I, II. *Proc. Lond. Math. Soc.* **44** 473–496, 497–534.

DE HAAN, L. (1970). *On Regular Variation and Its Application to the Weak Convergence of Sample Extremes.* Mathematical Centre Tract **32**. Mathematical Centre, Amsterdam.

DE HAAN, L. and RESNICK, S. (1982). Local limit theorems for sample extremes. *Ann. Probab.* **10** 396–413.

OMEY, E. and RACHEV, S. T. (1986a). On the rate of convergence in extreme value theory. *Theory Probab. Appl.* To appear.

OMEY, E. and RACHEV, S. T. (1986b). Rates of convergence in multivariate extreme value theory. Report, EHSAL, Brussels.

SMITH, R. L. (1982). Uniform rates of convergence in extreme-value theory. *Adv. in Appl. Probab.* **14** 600–622.

ECONOMISCHE HOGESCHOOL ST.-ALOYSIUS
STORMSTRAAT
1000 BRUSSELS
BELGIUM