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A THEOREM OF FELLER REVISITED[1]

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In 1946 Feller proposed and proved a famous law of iterated logarithm. Unfortunately his proofs were found to be incorrect, although the main result was true. In this paper the author gives a new proof of the main result.

1. Introduction. Let $X_1, X_2, \dots$ be a sequence of iid random variables with mean 0 and variance 1 and let $S_n = \sum_{i=1}^n X_i$. Following Lévy (1931), a sequence of positive numbers $\{\phi_n\}$ is said to belong to the upper class (with respect to $\{S_n\}$) if with probability 1 the inequality $S_n > \phi_n$ holds for only finitely many n and is said to belong to the lower class if with probability 1 the same inequality holds for infinitely many n. Throughout this paper, these two cases will be denoted by $\{\phi_n\} \in \mathscr{U}$ and $\{\phi_n\} \in \mathscr{L}$, respectively. The law of iterated logarithm proved by Khintchine (1924) established that

$$\sqrt{an \log_2 n} \in \begin{cases} \mathscr{U}, & \text{if } a > 2, \\ \mathscr{L}, & \text{if } a < 2, \end{cases} \tag{1.1}$$

for the special case where $P(X_1 = -1) = P(X_1 = 1) = 1/2$. Hereafter, let us denote $\log_1 = \log$ and $\log_{k+1} = \log\log_k$.

Following Khintchine (1924), Lévy (1933) proved that

$$\sqrt{2n \log_2 n + an \log_3 n} \in \begin{cases} \mathscr{U}, & \text{if } a > 3, \\ \mathscr{L}, & \text{if } a < 3. \end{cases}$$

Later, Kolmogorov [see Lévy (1937)] and Erdös (1942) showed that

$$\sqrt{n}\,\phi(n) \in \begin{cases} \mathscr{U}, & \text{if } \int_1^\infty t^{-1}\phi(t)\exp\left[-\frac{1}{2}\phi^2(t)\right] dt < \infty, \\ \mathscr{L}, & \text{if } \int_1^\infty t^{-1}\phi(t)\exp\left[-\frac{1}{2}\phi^2(t)\right] dt = \infty, \end{cases}$$

for the same sequence S_n, where $\phi(\cdot)$ is an increasing and positive function.

In the iid case, Hartman and Wintner (1941) proved (1.1) under the minimal condition that $EX_1 = 0$ and $EX_1^2 = 1$. In Feller (1946), an attempt to extend the upper and lower function results of Lévy and Erdös to random variables satisfying only the minimal conditions was made. More precisely, Feller claimed that if, as $x \to \infty$,

$$\int_{|t|>x} t^2\, dF(t) = O(1/\log_2 x), \tag{F5}$$

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then the sequence $\{\sqrt{n}\,\phi(n)\}$ belongs to the upper (lower) class if and onlv if

$$\sum_{n=1}^{\infty} n^{-1}\phi(n)\exp\left\{-\tfrac{1}{2}\phi_n^2\right\} < \infty \quad (=\infty),$$

where $\{\phi(n)\}$ is a nondecreasing sequence of positive numbers and $F(t)$ is the distribution of X_1. Also, he pointed out that if (F5) is not true, then the statement is false. His proof is based on Theorem 2 in the same paper. But Feller's proofs were found to be incorrect. This error was implicitly pointed out by Robbins and Siegmund (1970) and K. L. Chung communicated to Feller about this error. However, Feller was too ill to take up this project.

The main mistake in Feller's paper is that, according to his definition of X_i', X_i'' and X_i''',

$$X_i \neq X_i' + X_i'' + X_i'''. \tag{1.2}$$

Also, the primed X's do not have vanishing expectation. In his notation, Lemma 2 and Lemma 4 are not correct. A similar way to make (1.2) hold and to impose zero expectation for the new variables is to define them in the following way:

$$\tilde{X}_i' = \begin{cases} X_i - \int_{|x|<\eta_i} x\,dF(x), & \text{if } |X_i| < \eta_i, \\ -\int_{|x|<\eta_i} x\,dF(x), & \text{otherwise,} \end{cases}$$

$$\tilde{X}_i'' = \begin{cases} X_i - \int_{\eta_i \le |x| < \sqrt{i}} x\,dF(x), & \text{if } \eta_i \le |x_i| < \sqrt{i}, \\ -\int_{\eta_i \le |x| < \sqrt{i}} x\,dF(x), & \text{otherwise,} \end{cases}$$

$$\tilde{X}_i''' = \begin{cases} X_i - \int_{|x| \ge \sqrt{i}} x\,dF(x), & \text{if } |X_i| \ge \sqrt{i}, \\ -\int_{|x| \ge \sqrt{i}} x\,dF(x), & \text{otherwise,} \end{cases}$$

where $\eta_i^2 = i/(\log_2 i)^4$.

If we use the new notation defined here, Feller's Lemmas 3 and 6 are false although the exchange of definition makes his Lemmas 2 and 4 true. We have the following counterexample.

EXAMPLE 1. Let

$$P\left(X_1 = -\exp\left\{\tfrac{1}{2}\exp_2 k\right\}\right) = c\exp\{-k - \exp_2 k\} = p_k,$$

$$P(X_1 = a) = 1 - \sum_{k=1}^{\infty} p_k,$$

where $\exp_2 = \exp\exp$ and $\exp_3 = \exp\exp\exp$, a and c are positive constants to be chosen such that $EX_1 = 0$ and $EX_1^2 = 1$. Let $X_1, X_2, \ldots$ be an iid sequence. Then it is obvious that (F5) holds. When $n/(\log_2 n)^3 \le \exp_3 k \le$

$10(n+1)/\log_2(n+1))^3$, we have

$$\sum_{i=[10n\log_2^{-3}n]}^{n}\left(-EX_i\operatorname{ind}\{\eta_i < |X_i| < \sqrt{i}\}\right) \ge \left(n - [10n\log_2^{-3}n]\right)\sqrt{\log_2 n/n}$$
$$\ge (1/2)\sqrt{n\log_2 n},$$

where $[x]$ denotes the greatest integer which is less than or equal to x and $\operatorname{ind}(A\}$ the indicator function of the set A. The above term cannot be ignored in the proofs of Lemmas 4 and 6 (Feller) if we use $\tilde{X}_i''$ and X_i'', respectively.

Now let, for $i > 3$,

$$Y_i' = \begin{cases} X_i, & \text{if } |X_i| < \sqrt{i\log_2 i}, \\ 0, & \text{otherwise,} \end{cases}$$
$$Y_i = Y_i' - EY_i'$$

and for $i = 1, 2, 3$, $P(Y_i = \pm 1) = p(Y_i' = \pm 1) = 1/2$. Write $\sigma_i^2 = EY_i^2$ and $B_n^2 = \sum_{i=1}^n \sigma_i^2$. We have the following theorem.

THEOREM 1. *If $\{X_n\}$ is a sequence of iid random variables with mean* 0 *and variance* 1 *and if $\{\phi_n\}$ is an increasing sequence of positive numbers, then $\{B_n\phi_n\} \in \mathscr{U}$(or $\mathscr{L}$) if and only if*

$$\sum_{n=1}^{\infty} n^{-1}\phi_n \exp\{-\tfrac{1}{2}\phi_n^2\} < \infty \quad (or = \infty). \tag{1.3}$$

If assumption (F5) holds, i.e.,

$$\int_{|t|\ge x} t^2\,dF(t) = O(1/\log_2 x), \quad \text{as } x \to \infty, \tag{1.4}$$

then we have the following theorem.

THEOREM 2. *Suppose* (1.4) *holds. Then $\{\sqrt{n}\,\phi_n\} \in \mathscr{U}$ (or $\mathscr{L}$), if and only if* (1.3) *holds (correspondingly).*

One may notice that our Theorem 1 is different from Feller's Theorem 2 due to the different definition of the B_n's. But this difference is not essential. In Section 3, we shall give a corollary to Theorem 1, Theorem 3, which includes our Theorem 1 and Feller's Theorem 2 as special cases. In limit theory, a well-known fact is that the truncation location can be arbitrary up to a multiple constant. Our Theorem 3 shows that the range of this arbitrariness can be much larger, even up to an arbitrarily high power of $\log n$.

2. Lemmas.

LEMMA 1. *In proving Theorems* 1 *and* 2, *without loss of generality, we can assume that*

$$2\log_2 n + \log_3 n \le \phi_n^2 \le 2\log_2 n + 4\log_3 n, \quad for\ n \ge 3. \tag{2.1}$$

PROOF. Define, for $n \geq 3$,

$$\phi_n^* = \begin{cases} \phi_n, & \text{if } 2\log_2 n + \log_3 n < \phi_n^2 < 2\log_2 n + 4\log_3 n, \\ \sqrt{2\log_2 n + \log_3 n}, & \text{if } 2\log_2 n + \log_3 n \geq \phi_n^2, \\ \sqrt{2\log_2 n + 4\log_3 n}, & \text{if } \phi_n^2 \geq 2\log_2 n + 4\log_3 n, \end{cases}$$

and $\phi_i^* = 1$, $i = 1, 2, 3$.

Using the same approach as in Feller (1946), we can prove that the convergence of the series (1.3) does not change if the sequence $\{\phi_n\}$ is replaced by $\{\phi_n^*\}$. Therefore Lemma 1 holds. □

LEMMA 2. *If $X_1, X_2, \ldots, X_n$ are independent random variables with mean* 0 *$EX_i^2 = \sigma_i^2$, $E|X_i|^3 < \infty$, $i = 1, 2, \ldots, n$, then there exists an absolute constant C such that*

$$|F_n(x) - \Phi(x)| < C \sum_{i=1}^{n} E|X_i|^3 / \left[B_n^3(1 + |x|)^3\right],$$

where $B_n^2 = \sum_{i=1}^n \sigma_i^2$, $F_n(x) = P(S_n < B_n x)$ and $\Phi(x)$ is the standard normal distribution function.

The proof of this lemma can be found in Bikelis (1966).

LEMMA 3. *Let $X_1, X_2, \ldots$ be iid random variables with mean* 0 *and variance* 1 *and let the Y's be as defined in Section* 1. *Then we have*

(i) $\sum_{n=4}^{\infty} P(X_n \neq Y_n') < \infty$,
(ii) $|\sum_{i=1}^{n} EY_i'| = o(\sqrt{n/\log_2 n})$.

The proof is trivial and is omitted.

LEMMA 4. *If the sequence $\{\phi_n\}$ satisfies* (2.1), *then* (1.3) *is equivalent to*

$$\sum_{k=1}^{\infty} \log^{1/2} k \exp\left\{-\tfrac{1}{2}\phi^2(2^k)\right\} < \infty \quad (\text{or} = \infty, \text{correspondingly}). \tag{2.2}$$

PROOF. If the inequality of (1.3) is true, then we have

$$\sum_{n=1}^{\infty} n^{-1}\phi_n \exp\left\{-\tfrac{1}{2}\phi_n^2\right\} > \sum_{k=1}^{\infty} \sum_{n=2^k+1}^{2^{k+1}} 2^{-(k+1)}\phi(2^{k+1}) \exp\left\{-\tfrac{1}{2}\phi^2(2^{k+1})\right\}$$

$$> \tfrac{1}{4} \sum_{k=2}^{\infty} \log^{1/2} k \exp\left\{-\tfrac{1}{2}\phi^2(2^k)\right\}.$$

Hence the inequality of (2.2) holds. Conversely, suppose the equality of (1.3) is true. Then we have

$$\sum_{n=5}^{\infty} n^{-1}\phi_n \exp\left\{-\tfrac{1}{2}\phi_n^2\right\} < \sum_{k=2}^{\infty} \sum_{n=2^k+1}^{2^{k+1}} 2^{-k}\phi(2^k) \exp\left\{-\tfrac{1}{2}\phi^2(2^k)\right\}$$

$$< 2 \sum_{k=2}^{\infty} \log^{1/2} k \exp\left\{-\tfrac{1}{2}\phi^2(2^k)\right\},$$

which, together with the equality of (1.3), implies the equality of (2.2). The proof of Lemma 4 is complete. □

3. Proof of main results.

3.1. *Proof of Theorem* 1. By Lemma 1, without loss of generality, we can assume that the sequence $\{\phi_n\}$ satisfies (2.1). By (i) of Lemma 3 and the Borel–Cantelli lemma, we have $P\{X_n \neq Y_n', \text{ i.o.}\} = 0$, hence we have

$$\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} Y_i' = O(1), \quad \text{a.s.} \tag{3.1}$$

By (ii) of Lemma 3 and (3.1) we have

$$\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} Y_i = o\left(\sqrt{n/\log_2 n}\right), \quad \text{a.s.} \tag{3.2}$$

Note that the convergence of (1.3) does not change if we replace ϕ_n by $\phi_n + D/\phi_n$ for any real D. Thus if we can prove Theorem 1 for the sequence of $\{Y_n\}$, then Theorem 1 follows from (3.2). To this end, we need a result of Feller (1970). The following paragraph will be devoted to introducing Feller's notation and his result. The reader should note that some notations in the following paragraph may differ from those we used earlier, even though the same symbols are used.

Let $X_1, X_2, \ldots$ be a sequence of independent random variables with the common mean 0 and variances $\sigma_1^2, \sigma_2^2, \ldots$, respectively. Put $S_n = X_1 + \cdots + X_n$, $s_n^2 = \sigma_1^2 + \cdots + \sigma_n^2$. Also, let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences of positive numbers. Write $a_n = s_n\alpha_n$ and $b_n = s_n\beta_n$. We make the following assumptions on the two numerical sequences.

Assumption A (Condition on $\{\alpha_n\}$). There exist constants $\lambda > 1$ and C such that

$$s_k - s_i \leq C(a_k - a_i)/\alpha_i, \quad \text{if } 1 \leq a_k/a_i \leq \lambda. \tag{3.3}$$

Furthermore $\alpha_n \to \infty$, and $\{a_n\}$ is monotone.

It is easy to see that (3.3) is true when $\{\alpha_n\}$ is nondecreasing.

Assumption B (Condition on $\{\beta_n\}$). (i) There exists a constant $\omega > 0$ such that

$$\alpha_n \geq \beta_n \geq \omega\alpha_n^{-1}. \tag{3.4}$$

(ii) To each $\eta > 0$ there corresponds a $\delta > 0$ such that

$$1 - \eta < b_k/b_i < 1 + \eta, \quad \text{if } 1 \leq a_k/a_i \leq 1 + \delta. \tag{3.5}$$

(iii) To each $t > 1$ there exists a $\tau_t > 0$, such that

$$b_k/b_i > \tau_t, \quad \text{if } 1 \leq a_k/a_i \leq t. \tag{3.6}$$

Let $I = (L_1, L_2)$, with $-\infty < L_1 < L_2 \le \infty$, be a interval and denote

$$p(I) = P\{S_n^* \in I, \text{i.o.}\},$$

$$\Sigma(I) = \sum_{n=1}^{\infty} \min\{1, (a_{n+1} - a_n)/b_n\} P\{S_n^* \in I\},$$

where $S_n^* = (S_n - a_n)/b_n$. Also, for each real η, define $I_{(\eta)} = (L_1 - \eta, L_2 + \eta)$.
Feller (1970) proved the following theorem.

THEOREM (FELLER). *Under the assumptions stated in the above paragraph, we have*:

(i) *If* $\Sigma(I) < \infty$ *for some interval* I, *then* $p\{I_{(-\eta)}\} = 0$ *for every* $\eta > 0$.
(ii) *If* $I = (L_1, L_2)$ *with* $0 \le L_1 < L_2 < \infty$ *and* $\Sigma(I) = \infty$, *then* $p\{I_{(\eta)}\} = 1$, *for each* $\eta > 0$.

Now we proceed to our proof by using the above result. Define

$$\alpha_n = \phi_n \quad \text{and} \quad \beta_n = 1/\phi_n.$$

Since $\{\phi_n\}$ is nondecreasing, Assumption A is satisfied. (3.4) is automatically true by the definition of β_n. Also, by (2.1) and the fact that $n - B_n^2 = o(n)$ we can easily verify the validation of (3.5) and (3.6).
By Lemma 2, for any interval $I = (L_1, L_2)$, we have

$$\begin{aligned} \left| P\{S_n^* \in I\} - [\Phi\{L^*\} - \Phi\{L_*\}] \right| &\le C \sum_{i=1}^{n} E|Y_i|^3 B_n^{-3} L_*^{-3} \\ &\le C n^{-1/2} \log_2^{-3/2} n E|Y_n|^3, \end{aligned}$$

where $L_* = \alpha_n + \beta_n L_1 > \sqrt{2 \log_2 n}$ for all large n, $L^* = \alpha_n + \beta_n L_2$, and $\Phi(x)$ is the distribution function of the standard normal random variable. Moreover, we have

$$\begin{aligned} &\sum_{n=3}^{\infty} \min\{1, (a_{n+1} - a_n)/b_n\} n^{-1/2} \log_2^{-3/2} n E|Y_n|^3 \\ &\le C \sum_{k=1}^{\infty} \sum_{n=2^k+1}^{2^{k+1}} (a_{n+1}^2 - a_n^2) 2^{-3k/2} \log^{-3/2} k E|Y_{2^{k+1}}|^3 \\ &\le C \sum_{k=1}^{\infty} 2^{-k/2} \log^{-1/2} k \sum_{l=1}^{k} E|X_1|^3 \operatorname{ind}\{|X_1| \in [\Delta_l, \Delta_{l+1})\} \\ &\le C \sum_{l=1}^{\infty} E|X_1|^3 \operatorname{ind}\{|X_1| \in [\Delta_l, \Delta_{l+1})\} \sum_{k=l}^{\infty} 2^{-k/2} \log^{-1/2} k \\ &\le C \sum_{l=1}^{\infty} 2^{-l/2} \log^{-1/2} l E|X_1|^3 \operatorname{ind}\{|X_1| \in [\Delta_l, \Delta_{l+1})\} \\ &\le C \sum_{l=1}^{\infty} E|X_1|^2 \operatorname{ind}\{|X_1| \in [\Delta_l, \Delta_{l+1})\} \le C E|X_1|^2 < \infty, \end{aligned}$$

where $\Delta_l = \sqrt{2^l \log l}$ and $\operatorname{ind}\{A\}$ denotes the indicator function of the set A. Hence $\Sigma(I) <$ (or $=$)∞ is equivalent to

$$\sum_{n=3}^{\infty} \min\{1, (a_{n+1} - a_n)/b_n\}(\Phi\{L^*\} - \Phi\{L_*\}) < (\text{or} =)\infty, \tag{3.7}$$

for any interval $I = (L_1, L_2)$.

Now suppose that

$$\sum_{n=1}^{\infty} n^{-1}\phi_n \exp\{-\tfrac{1}{2}\phi_n^2\} < \infty.$$

We shall prove $\{B_n\phi_n\}$ belongs to the upper class with respect to $T_n = Y_1 + \cdots + Y_n$. We need only to verify that the inequality in (3.7) is true for the interval $I = (-1, \infty)$. In this case we have $L^* = \infty$ and $L_* = \phi(n) - \phi^{-1}(n) := \hat{\phi}(n)$. It is well known that

$$1 - \Phi(x) < (2x)^{-1}\exp\{-\tfrac{1}{2}x^2\}, \quad \text{as } x \to \infty.$$

Noticing that the convergence of (1.3) does not change by replacing ϕ_n as $\hat{\phi}(n)$, we have, by Lemma 4,

$$\begin{aligned}
&\sum_{n=3}^{\infty} \min\{1, (a_{n+1} - a_n)/b_n\}(1 - \Phi\{\hat{\phi}(n)\}) \\
&\quad \le C \sum_{n=3}^{\infty} n^{-1/2}(a_{n+1} - a_n)\exp\{-\tfrac{1}{2}\hat{\phi}^2(n)\} \\
&\quad \le C \sum_{k=1}^{\infty} \sum_{n=2^k+1}^{2^{k+1}} (a_{n+1} - a_n)2^{-k/2}\exp\{-\tfrac{1}{2}\hat{\phi}^2(2^k)\} \\
&\quad \le C \sum_{k=1}^{\infty} a(2^{k+1})2^{-k/2}\exp\{-\tfrac{1}{2}\hat{\phi}^2(2^k)\} \\
&\quad \le C \sum_{k=1}^{\infty} \log k \exp\{-\tfrac{1}{2}\hat{\phi}^2(2^k)\} < \infty.
\end{aligned} \tag{3.8}$$

Thus, by (3.7) and (3.8), $\Sigma(I) < \infty$. Taking $\eta = 1$, Feller's theorem yields that

$$P\{T_n \ge B_n\phi_n;\ \text{i.o.}\} = 0,$$

i.e., $\{B_n\phi_n\}$ belongs to the upper class of $\{T_n\}$ and consequently belongs to the upper class of $\{S_n\}$.

Finally, suppose that

$$\sum_{n=1}^{\infty} n^{-1}\phi_n \exp\{-\tfrac{1}{2}\phi_n^2\} = \infty. \tag{3.9}$$

Take $I = (1, 3)$ and write $L^* = \phi_n + 3/\phi_n$ and $L_* = \phi_n + 1/\phi_n$. Then we have, for large n,

$$(a_{n+1} - a_n)/b_n = (a_{n+1}^2 - a_n^2)/[(a_{n+1} + a_n)b_n] \ge (4n)^{-1}\phi^2(n). \tag{3.10}$$

From (3.9) and (3.10), applying Lemma 4, we have

$$\sum_{n=3}^{\infty} \min\{1, (a_{n+1} - a_n)/b_n\}(\Phi\{L^*\} - \Phi\{L_*\})$$
$$\geq \sum_{n=3}^{\infty} n^{-1}\phi^2(n+1)(2\pi)^{-1/2}\{L^* - L_*\}\exp\{-\tfrac{1}{2}L^{*2}\}$$
$$> C\sum_{k=1}^{\infty} \phi^2(2^{k+1})\log^{-1/2}k \exp\left\{-\tfrac{1}{2}\left(\phi(2^{k+1}) + 3/\phi(2^{k+1})\right)^2\right\}$$
$$> C\sum_{k=1}^{\infty} \log^{1/2}k \exp\left\{-\tfrac{1}{2}\left(\phi(2^{k+1}) + 3/\phi(2^{k+1})\right)^2\right\} = \infty,$$

which, together with Feller's theorem, implies that $p\{(0,4)\} = 1$ and hence implies that

$$P\{T_n \geq B_n\phi(n), \text{i.o.}\} = 1.$$

Here we use the fact that (3.9) holds for $L^* = \phi(n) + 3/\phi(n)$ instead of the original $\phi(n)$. The proof of Theorem 1 is complete. □

3.2. *Proof of Theorem* 2. Suppose that (1.4) holds. Then

$$\sigma_i^2 = 1 - \int_{\{|t| > \sqrt{i\log_2 i}\}} t^2\,dF(t) - \left(\int_{\{|t| > \sqrt{i\log_2 i}\}} t\,dF(t)\right)^2$$
$$> 1 - 2\int_{\{|t| > \sqrt{i\log_2 i}\}} t^2\,dF(t)$$
$$> 1 - D/\log_2 i, \quad \text{for some } D > 0.$$

Hence we have, for some $D > 0$,

$$n \geq B_n^2 \geq \sum_{i=1}^{n} \sigma_i^2 \geq \sum_{i=1}^{n} (1 - D/\log_2 i) \geq n(1 - D/\log_2 n). \tag{3.11}$$

Now suppose that

$$\sum_{n=1}^{\infty} n^{-1}\phi(n)\exp\{-\tfrac{1}{2}\phi^2(n)\} < \infty.$$

Then by Theorem 1, $\{B_n\phi(n)\} \in \mathscr{U}$. Since $B_n \leq \sqrt{n}$, we get $\{\sqrt{n}\,\phi(n)\} \in \mathscr{U}$.

Next, suppose that

$$\sum_{n=1}^{\infty} n^{-1}\phi(n)\exp\{-\tfrac{1}{2}\phi^2(n)\} = \infty. \tag{3.12}$$

By Lemma 1, we can assume that $\log_2 n \leq \phi^2(n) \leq 4\log_2 n$. By (3.11), we have $B_n \geq \sqrt{n}\,[1 - D/\phi^2(n)]$, which, together with (3.12), implies that

$$\sum_{n=1}^{\infty} n^{-1}\tilde{\phi}(n)\exp\{-\tfrac{1}{2}\tilde{\phi}^2(n)\} = \infty,$$

where $\tilde{\phi}(n) = \phi(n) + 2D/\phi(n)$. By Theorem 1, with probability 1, there are infinitely many n such that $S_n \geq B_n\tilde{\phi}(n)$. Hence Theorem 2 follows from the fact that $B_n\tilde{\phi}(n) > \sqrt{n}\,\phi(n)$, for large n. □

REMARK. To illustrate that if (1.4) is not true then Theorem 2 is false, we have the following example.

EXAMPLE. Let X_n, $n = 1, 2, \ldots,$ be a sequence of iid random variables with the common probability density function

$$p(x) = \begin{cases} c/\left\{|x|^3\log|x|\log_2|x|\log_3^2|x|\right\}, & \text{if } |x| > a, \\ 0, & \text{otherwise,} \end{cases}$$

where a is a large positive number such that $\log_2 a > 0$ and c is a positive constant such that, together with a, $p(x)$ becomes a density with unit variance. We can easily see that for large x $(x > a)$,

$$\int_{|t|>x} t^2 p(t)\,dt = 2c(\log_3 x)^{-1}.$$

Therefore for large n,

$$B_n^2 = \sum_{i=1}^{n} \sigma_i^2 = \sum_{i=[a^2]+1}^{n} (1 - 2/\log_3 i \log_2 i) \leq n\left(1 - 8(\log_3 n)^{-2}\right).$$

Hence

$$B_n \leq \sqrt{n}\left(1 - (\log_3 n)^{-2}\right). \tag{3.13}$$

Take $\tilde{\phi}_n = \sqrt{2\log_2 n + 4\log_3 n}$, for $n > 8$. Thus by Theorem 1, we have

$$\left\{B_n\tilde{\phi}(n)\right\} \in \mathscr{U}.$$

Let $\phi(n) = \tilde{\phi}(n)(1 - (\log_3 n)^{-2})$. Note that $\phi(n) \geq B_n\tilde{\phi}(n)/\sqrt{n}$, and from (3.13) we obtain that $\left\{\sqrt{n}\,\phi(n)\right\} \in \mathscr{U}$. On the other hand, one can easily verify that

$$\sum_{n=8}^{\infty} n^{-1}\phi(n)\exp\left\{-\tfrac{1}{2}\phi^2(n)\right\} = \infty.$$

3.3. *Proof of Feller's Theorem* 2. Let $\delta_n : \delta_{1n} \leq \delta_n \leq \delta_{2n}$ be a sequence of constants, where $\delta_{1n} = (\log n)^{-b}$, $\delta_{2n} = (\log n)^b$ and $b > 0$. Let

$$\overline{B}_n^2 = \sum_{i=4} \operatorname{var}\left\{X_1 \operatorname{ind}\left[|X_1| \leq \sqrt{i}\,\delta_i\right]\right\} + 3.$$

Then we have the following theorem which is more general than Feller's Theorem 2.

THEOREM 3. *Under the conditions of Theorem* 1, $\{\overline{B}_n\phi_n\} \in \mathscr{U}$ *(or* $\mathscr{L}$*) iff* (1.3) *holds.*

PROOF. As in proof of Theorem 1, we assume (2.1) holds and X_n is replaced by Y_n. Write

$$E = \{k\colon \text{there is } n \in \{2^{k-1}+1,\dots,2^k\} \text{ such that } |\overline{B}_n^2/B_n^2 - 1| \geq 1/\log_2 n\}$$

and write $\hat{B}_n = \overline{B}_n$, if $n \in \{2^{k-1}+1,\dots,2^k\}$ and $k \notin E$, $= B_n$, otherwise. Since for any fixed real constant D, replacing ϕ_n by $\phi_n + D/\phi_n$ does not change the convergence of the series in (1.3), Theorem 1 implies that $\{\hat{B}_n\phi_n\} \in \mathscr{U}$ (or $\mathscr{L}$) iff (1.3) holds. Thus, to prove Theorem 3, it suffices to show that

$$P\left[\bigcup_{n=2^{k-1}+1}^{2^k} \left(S_n \geq \tfrac{1}{2}B_n\phi_n\right), \text{ i.o. for } k \in E\right] = 0. \tag{3.14}$$

By Lévy's maximum inequality [see Loève (1977)] and Lemma 2, we have

$$\begin{aligned} P\left[\bigcup_{n=2^{k-1}+1}^{2^k} \left(S_n \geq \tfrac{1}{2}B_n\phi_n\right)\right] &\leq 2P\left[S_{2^k} \geq \tfrac{1}{8}B_{2^k}\phi_{2^k}\right] \\ &\leq 2\left[1 - \Phi\left(\tfrac{1}{8}\phi_{2^k}\right)\right] + C2^{-k/2}(\log k)^{-3/2}E|Y_{2^k}|^3 \\ &\leq C\left[k^{-1/32} + 2^{-k/2}(\log k)^{-3/2}E|Y_{2^k}|^3\right]. \end{aligned} \tag{3.15}$$

In the argument given above (3.7), we have actually proved

$$\sum_{k=1}^{\infty} 2^{-k/2}(\log k)^{-3/2}E|Y_{2^k}|^3 < \infty. \tag{3.16}$$

Using the same approach as used in Feller's Lemma 4, one can get

$$\sum_{i=1}^{\infty} i^{-1}(\log_2 i)^{-1}EX_1^2\operatorname{ind}\left[\sqrt{i}\,\delta_{1i} < |X_1| \leq \sqrt{i}\,\delta_{2i}\right] < \infty,$$

which implies by summation by parts that

$$\sum_{k=1}^{\infty} 2^{-k}(\log k)^{-1} \sum_{i=1}^{2^k} EX_1^2\operatorname{ind}\left[\sqrt{i}\,\delta_{1i} < |X_1| \leq \sqrt{i}\,\delta_{2i}\right] < \infty. \tag{3.17}$$

If $k \in E$, then there exists an $m \in \{2^{k-1}+1,\dots,2^k\}$ such that

$$\begin{aligned} &2^{-k}(\log k)^{-1} \sum_{i=1}^{2^k} EX_1^2\operatorname{ind}\left[\sqrt{i}\,\delta_{1i} < |X_1| \leq \sqrt{i}\,\delta_{2i}\right] \\ &\qquad \geq 2^{-k}(\log k)^{-1}|B_m^2 - \overline{B}_m^2| \geq C(\log k)^{-2} \geq k^{-1/32}. \end{aligned} \tag{3.18}$$

(3.15)–(3.18) imply that

$$\sum_{k\in E} P\left[\bigcup_{n=2^{k-1}+1}^{2^k} \left(S_n \geq \tfrac{1}{2}B_n\phi_n\right)\right] < \infty,$$

which, together with the Borel–Cantelli lemma, implies (3.14), and hence completes the proof of Theorem 3. □

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Note added in proof. An earlier version of this paper claimed that Theorem 2 in Feller (1946) is not correct. Theorem 3 was formulated after the author read a preprint by Uwe Einmahl. The author thanks the Editor for his communication about this matter.

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