The Annals of Probability
1989, Vol. 17, No. 1, 357–361

ON THE THREE SERIES THEOREM IN NUMBER THEORY[1]

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We obtain a generalization of part "if" of the Erdös–Wintner theorem to additive arithmetic functions with values in certain Banach spaces. Some applications in asymptotic density are considered.

1. Preliminary concepts and main result. Throughout this paper $\mathfrak{X}$ denotes a real separable Banach space with the norm $\|\cdot\|$. $\mathfrak{X}$ is of type r $(1 < r \le 2)$ if there exists a constant C such that if $\{X_i\}_{i=1,\dots,n}$ is any finite set of independent and centered $\mathfrak{X}$-valued random variables in $L_r(\mathfrak{X})$, then

$$E\left\|\sum_{i=1}^{n} X_i\right\|^r \le C\sum_{i=1}^{n} E\|X_i\|^r.$$

$\mathfrak{X}$ is said to be r-smoothable $(1 < r \le 2)$ if there exists an equivalent r-smooth norm $\|\cdot\|_1$, i.e., the modulus of smoothness

$$\rho_{\|\cdot\|_1}(\tau) \stackrel{\text{def}}{=} \sup\{(\|x+y\|_1 + \|x-y\|_1)/2 - 1\colon x, y \in \mathfrak{X}, \|x\|_1 = 1, \|y\|_1 = \tau\}$$

satisfies the inequality $\rho_{\|\cdot\|_1}(\tau) \le K\tau^r$ for some $K > 0$. It is a well known fact that a r-smooth norm is a type-r norm [cf. Hoffmann-Jørgensen and Pisier (1976)]. On the other hand, let $\mathbb{N}$, $\mathscr{P}$ be, respectively, the sets of natural numbers and prime numbers. For $p \in \mathscr{P}$ and $n \in \mathbb{N}$ we put $\beta_p(n) = k$ iff $p^k \| n$ and $\delta_p(n) = 1$ or 0 as $\beta_p(n) \ge 1$ or $\beta_p(n) = 0$; in connection with this we shall need independent integer-valued random variables b_p having the geometric distribution,

$$P(b_p = k) = (1 - 1/p)p^{-k}, \qquad k = 0, 1, 2, \dots,$$

and variables $d_p = I(b_p \ge 1)$, where $I(A)$ denotes the indicator function of the set A.

For $n = 1, 2, \dots$, let P_n be the probability measure on $\mathbb{N}$ assigning the weight $1/n$ to each $k \le n$; if $\{f_n\}_{n=1,2,\dots}$ is a sequence of $\mathfrak{X}$-valued functions defined on $\mathbb{N}$ and $P_n f_n^{-1}$ converges weakly to a probability measure μ on $\mathfrak{X}$, we write $f_n \Rightarrow \mu$ or $f_n \Rightarrow X$ if μ is the probability distribution of the random variable X. When $f_n \equiv f$, all n, we simply write $f \Rightarrow \mu$ or $f \Rightarrow X$, and μ is said to be the limit distribution of f.

The following theorem is the main result of this paper. It gives sufficient conditions for the existence of the limit distribution for an $\mathfrak{X}$-valued additive function, i.e., a function $f\colon \mathbb{N} \to \mathfrak{X}$ such that $f(mn) = f(m) + f(n)$ whenever m

Received July 1987; revised January 1988.
[1]Research supported by the University of the Basque Country.
AMS 1980 *subject classifications.* Primary 10K20; secondary 60B11.
Key words and phrases. Additive arithmetic function, distribution, density, Banach space type.

and n are relatively prime. For the special case $\mathfrak{X} = \mathbb{R}$, see Billingsley [(1974), Theorems 7.1 and 7.2].

THEOREM 1. *Suppose $\mathfrak{X}$ is r-smoothable* ($1 < r \le 2$) *and let f be an $\mathfrak{X}$-valued additive function. If the two series,*

$$\sum_{\|f(p)\| \ge c} 1/p \quad \text{and} \quad \sum_{\|f(p)\| < c} \frac{\|f(p)\|^r}{p}, \tag{1.1}$$

converge for some $c > 0$ and if

$$a_n = \sum_{\|f(p)\| < c,\ p \le n} \frac{f(p)}{p},$$

then

$$f - a_n \Rightarrow \sum_{\|f(p)\| < c} \left[f(p^{b_p}) - \frac{f(p)}{p} \right] + \sum_{\|f(p)\| \ge c} f(p^{b_p}). \tag{1.2}$$

Therefore, if a_n also converges (in the norm), then

$$f \Rightarrow \sum_p f(p^{b_p})$$

or, in case f is completely additive [i.e., $f(p^k) = f(p)$ for $k = 1, 2, \dots$],

$$f \Rightarrow \sum_p f(p) d_p.$$

PROOF. The sequences $\{f(p^{b_p})\}$ and $\{f(p)d_p\}$ being tail equivalent, the a.s. convergence of the two series on the right in (1.2) is ensured by the Borel–Cantelli lemma and Theorem 2.1 in Szulga (1977) if the two series in (1.1) converge. Then, in view of Theorem 4.2 in Billingsley (1968), the proof runs exactly as in the real case [see the proof of Theorem 7.2 in Billingsley (1974)] with norm instead of absolute value, and it will be complete as soon as we show that Lemma 1 holds.

LEMMA 1. *Let $\mathfrak{X}$ be of type r* ($1 < r \le 2$) *and let g be an $\mathfrak{X}$-valued function defined on $\mathscr{P}$. There is a constant C, independent of g and n, such that*

$$E_n \left\| \sum_{p \le n} g(p) \delta'_p \right\|^r \le C \sum_{p \le n} \frac{\|g(p)\|^r}{p}, \tag{1.3}$$

where $\delta'_p = \delta_p - 1/p$ and E_n is the P_n-expectation.

PROOF OF LEMMA 1. Use the method in Ruzsa (1984) to obtain

$$E_n \left\| \sum_{p \le n} g(p) \delta'_p \right\|^r \le C' E \left\| \sum_{p \le n} g(p) d'_p \right\|^r,$$

where C' is independent of g and n, $d'_p = d_p - 1/p$ and E is the expectation relative to the space of definition of the d_p. As $\mathfrak{X}$ is of type r and

$$E\|g(p)d_p'\|^r \le 2\frac{\|g(p)\|^r}{p},$$

the result follows. □

The proof of Theorem 1 is finished. □

We make two remarks.

REMARK 1. Lemma 1, which provides an inequality of the Turán–Kubilius type in (1.3), generalizes the real case of Lemma 2.1 in Billingsley (1974). Then, following the method in Billingsley [(1974), Section 2], it can be shown that if $\mathfrak{X}$ is of type r and $\{f_n\}$ is an array of $\mathfrak{X}$-valued additive functions, then the law of large numbers,

$$\frac{f_n - A_n}{\psi_n B_n} \Rightarrow 0,$$

holds for every sequence of real numbers $\psi_n \to \infty$ if

$$A_n = \sum_{p \le n} \frac{f_n(p)}{p}, \qquad B_n = \left(\sum_{p \le n} \frac{\|f_n(p)\|^r}{p} \right)^{1/r}$$

and

$$\sup_n \frac{\|f_n(m)\|}{B_n} < \infty, \qquad m = 1, 2, \dots.$$

REMARK 2. Take $\mathfrak{X} = \mathbb{R}^k$ and suppose that f is an $\mathfrak{X}$-valued additive function with limit distribution μ, i.e., $f \Rightarrow \mu$. Then $Tf \Rightarrow \mu T^{-1}$ for every coordinate functional T on $\mathbb{R}^k$. Therefore, by the Erdös–Wintner theorem, it follows that the series $\Sigma_p f(p)d_p$ is a.s. coordinatewise convergent and hence it is a.s. convergent (in the norm). By Theorem 2.1 in Szulga (1977), the three series

$$\sum_{\|f(p)\| \ge c} 1/p, \qquad \sum_{\|f(p)\| < c} \frac{\|f(p)\|^2}{p}, \qquad \sum_{\|f(p)\| < c} \frac{f(p)}{p}$$

converge for every $c > 0$. So in this case the converse to Theorem 1 holds. This same question for general r-smoothable Banach spaces remains open.

2. Applications. In this section we take $\mathfrak{X} = l_2$, i.e., the space of sequences $x = (x_1, \dots, x_k, \dots)$ of real numbers such that $\Sigma_k x_k^2 < \infty$, with the usual norm; l_2 is 2-smooth since it is a real separable Hilbert space [cf. Woyczyński (1975)].

Let $f(m)$ be an integer-valued additive function such that $\Sigma_{p \in Q} 1/p < \infty$, where Q is the set of primes satisfying the condition $f(p) \ne 0$ (note that by the Erdös–Wintner theorem this is both necessary and sufficient to ensure that f has limit distribution). We "spread" f by considering the l_2-valued additive function,

$$f^*(m) = \left(\gamma_{p_1}(m), \dots, \gamma_{p_k}(m), \dots\right),$$

where $\gamma_p(m) = f(p^{\beta_p(m)})$ and $p_1, p_2, \dots,$ is an arrangement of prime numbers. As $\|f^*(p)\| = 0$ or $\|f^*(p)\| \geq 1$, according to $p \notin Q$ or $p \in Q$, Theorem 1 applies and we have

$$f^* \Rightarrow X \stackrel{\text{def}}{=} \left(f\left(p_1^{b_{p_1}}\right), \dots, f\left(p_k^{b_{p_k}}\right), \dots \right).$$

Both f^* and X are S-valued, where S is the set of sequences of nonnegative integers with finite support. Since S is both countable and discrete in the l_2-norm, the equality

$$D(f^*(m) \in B) = P\left(\left(f\left(p_1^{b_{p_1}}\right), \dots, f\left(p_k^{b_{p_k}}\right), \dots \right) \in B\right) \tag{2.1}$$

holds true for every $B \subset S$, where $D(A) = \lim_n P_n(A)$; when this limit there exists, it is the asymptotic density of the set $A \subset \mathbb{N}$. This implies that each set in the σ-field $\sigma(\gamma_{p_1}, \dots, \gamma_{p_k}, \dots)$ generated by the γ_p has density; moreover, D is a probability measure on this σ-field.

By specializing f and B, (2.1) directly gives a great number of explicit formulas. We mention two examples.

Example 1. Let $\Omega_Q(m) \stackrel{\text{def}}{=} \sum_{q \in Q} \beta_q(m)$ be the number of prime divisors of m which lie in Q, counting multiplicity, where Q is a set of prime numbers such that $\sum_{q \in Q} 1/q < \infty$.

The sets in $\sigma(\gamma_{p_1}, \dots, \gamma_{p_k}, \dots)$ are the set of natural numbers definable in terms of divisibility by primes in Q. Now for each $q \in Q$, let r_q be a positive integer and let A be the set of natural numbers m for which the relations

$$\beta_q(m) \equiv 0 \pmod{r_q}, \qquad q \in Q,$$

are satisfied. By (2.1),

$$D(A) = \prod_{q \in Q} P\left(b_q \equiv 0 \pmod{r_q}\right)$$

$$= \prod_{q \in Q} \left(1 - \frac{q^{r_q - 1} - 1}{q^{r_q} - 1}\right).$$

Example 2. Now we consider the excess,

$$\Omega(m) - \omega(m) = \sum_p \beta_p(m) - \delta_p(m).$$

Here the sets in $\sigma(\gamma_{p_1}, \dots, \gamma_{p_k}, \dots)$ are the subsets of $\mathbb{N}$ definable in terms of the "partial excesses" $\gamma_p = \beta_p - \delta_p$.

Let B be the set of integers for which the excess is 2. Then B is the disjoint union of the sets,

$$U_p = \{m\colon \gamma_p(m) = 2, \gamma_q(m) = 0, q \neq p\}, \qquad p \in \mathcal{P},$$

$$V_{pq} = \{m\colon \gamma_p(m) = \gamma_q(m) = 1, \gamma_r(m) = 0, r \neq p, q\}, \qquad p, q \in \mathcal{P} \text{ and } p < q,$$

and we have

$$D(B) = \sum_p D(U_p) + \sum_{p<q} D(V_{pq})$$
$$= \frac{6}{\pi^2}\left[\sum_p \frac{1}{p^2(p+1)} + \sum_{p<q} \frac{1}{pq(p+1)(q+1)}\right].$$

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