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LAWS OF THE ITERATED LOGARITHM FOR THE EMPIRICAL CHARACTERISTIC FUNCTION[1]

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Let X be a real-valued random variable with distribution function $F(x)$ and characteristic function $c(t)$. Let $F_n(x)$ be the nth empirical distribution function associated with X and let $c_n(t)$ be the characteristic function of $F_n(x)$. Necessary and sufficient conditions in terms of $c(t)$ are obtained for $c_n(t) - c(t)$ to obey bounded and compact laws of the iterated logarithm in the Banach space of continuous complex-valued functions on $[-1,1]$.

1. Introduction. Let X be a real-valued random variable with distribution function $F(x)$ and characteristic function $c(t) = \int_{-\infty}^{\infty} e^{ixt}\,dF(x)$. Let $F_n(x)$ be the nth empirical distribution function associated with X. That is,

$$F_n(x) = \frac{1}{n}\sum_{k=1}^{n} 1_{[X_k,\infty)}(x), \tag{1}$$

where 1_A is the indicator of the set A and X_k, $k \geq 1$, are independent copies of X. $F_n(x)$ is a nondecreasing right-continuous stochastic process. Likewise, the nth empirical characteristic function

$$\begin{aligned} c_n(t) &= \int_{-\infty}^{\infty} e^{ixt}\,dF_n(x) \\ &= \frac{1}{n}\sum_{k=1}^{n} e^{iX_k t} \end{aligned} \tag{2}$$

is a stochastic process. We consider strong limits of

$$\begin{aligned} C_n(t) &= n^{1/2}(c_n(t) - c(t)) \\ &= n^{-1/2}\sum_{k=1}^{n} e^{iX_k t} - c(t) \end{aligned} \tag{3}$$

in the Banach space $C[-1,1]$ of continuous complex-valued functions on $[-1,1]$ with the usual sup norm ($\|\cdot\|_\infty = \sup_{-1\leq t\leq 1}|\cdot|$).

Let $\phi_n = (2\log\log n)^{1/2}$, $n > 27$. We say that $C_n(t)$ satisfies a bounded law of the iterated logarithm (BLIL) if

$$\limsup_{n\to\infty} \phi_n^{-1}\|C_n(t)\|_\infty < \infty \quad \text{a.s.}$$

The Kolomogorov 0–1 law implies that the preceding lim sup is a.s. a finite constant. This condition is equivalent to requiring that $E = E(\omega) =$

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$\{\phi_n^{-1}C_n(t,\omega): n \geq 27\}$ is a.s. a bounded set in $C[-1,1]$. If E is a.s. relatively compact in $C[-1,1]$ we say that $C_n(t)$ satisfies the compact law of the iterated logarithm (CLIL). In this case, as is well known, a functional law of the iterated logarithm holds. In a more general context this is proved in Kuelbs (1976).

The results that follow characterize the BLIL and CLIL for $C_n(t)$ in terms of $c(t)$. They are motivated by the paper of Marcus (1981) in which a similar characterization of the central limit theorem is obtained. Marcus' result is as follows. Let

$$\begin{aligned}\sigma^2(|s-t|) &= E|e^{iXs} - e^{iXt}|^2 \\ &= 4\int_{-\infty}^{\infty} \sin^2\frac{|s-t|}{2}\, dF(x) \\ &= 2(1 - \operatorname{Re}(c(|s-t|))) \leq 4\end{aligned} \tag{4}$$

and

$$m_\sigma(u) = m(\{-1 \leq x \leq 1: \sigma(x) < u\}), \tag{5}$$

where $m(\cdot)$ is Lebesgue measure. Then $C_n(t)$ converges weakly in $C[-1,1]$ to a Gaussian limit if and only if

$$I(\sigma) = \int_0^2 (\log 2/m_\sigma(u))^{1/2}\, du < \infty. \tag{6}$$

Also see Csörgő (1981).

Concerning the LIL, Marcus and Philipp (1981) showed in a general context that the central limit theorem implies an almost sure invariance principle, and so a CLIL. Hence (6) implies that $C_n(t)$ obeys a CLIL. Ledoux (1982) proved a CLIL assuming an integral condition more general than (6), but more restrictive than Theorem 1(iii). Yuckich (1988) showed that Ledoux's condition is essentially the weakest possible for the BLIL under the assumption that X has "convex tails." Additionally, the results of Kuelbs (1977) imply a characterization of the LILs in terms of the numbers $\phi_n^{-1}E\|C_n(t)\|_\infty$. Recently significant advances have been made on the LIL for empirical processes and Banach space-valued random variables. We refer the reader to Alexander (1987a, b), and Ledoux and Talagrand (1986, 1988).

Let

$$I_n(\sigma) = \int_{1/n}^2 (\log 2/m_\sigma(u))^{1/2}\, du.$$

We prove

THEOREM 1. *Let X be a nondegenerate real-valued random variable. Then the following are equivalent*:

(i) *There is a constant* $0 < L < \infty$ *so that*

$$\limsup_{n\to\infty} \phi_n^{-1}\|C_n(t)\|_\infty = L < \infty.$$

(ii)

$$\sup_n \phi_n^{-1}E\|C_n(t)\|_\infty < +\infty.$$

(iii)

$$\sup_n \phi_n^{-1}I_n(\sigma) < +\infty.$$

For the CLIL given X, define $K(X)$ by

$$K(X) = \left\{ g_t = \int_{-\infty}^{\infty} e^{ixt} g(x)\, dF(x) \colon \int_{-\infty}^{\infty} g(x)\, dF(x) = 0, \right.$$
$$\left. \int_{-\infty}^{\infty} g(x)\overline{g(x)}\, dF(x) \le 1 \right\}.$$

THEOREM 2. *The following are equivalent:*

(i) $\{\phi_n^{-1} C_n(t)\colon n > 27\}$ *is a.s. relatively compact in* $C[-1,1]$, *and the cluster points are a.s.* $K(X)$.

(ii) $$\lim_n \phi_n^{-1} E\|C_n(t)\|_\infty = 0.$$

(iii) $$\lim_n \phi_n^{-1} I_n(\sigma) = 0.$$

For square integrable Banach space-valued random variables, Alexander (1987b) has obtained a beautiful characterization of when the cluster points in (i) are a.s. the null set. As a consequence of this result and our proof, we obtain

THEOREM 3. *Assume* $C_n(t)$ *satisfies the BLIL. Then the following are equivalent:*

(i) *The cluster points of* $\{\phi_n^{-1} C_n(t)\}_{n \ge 27}$ *in* $C[-1,1]$ *are a.s. the null set.*

(ii) $$\liminf_n \phi_n^{-1} E\|C_n(t)\|_\infty > 0.$$

(iii) $$\liminf_n \phi_n^{-1} I_n(\sigma) > 0.$$

The equivalence between (i) and (ii) in Theorems 1 and 2 is due to Kuelbs (1977); that (i) and (ii) are equivalent in Theorem 3 is due to Alexander (1987). We shall prove (ii) if and only if (iii). The proof uses Gaussian and Rademacher randomization, which has been important in many recent developments in limit theory. We cite Giné and Zinn (1986) and Ledoux and Talagrand (1988) to give just two references. The proof of (ii) implies (iii) is very brief; that (ii) implies (iii) uses a convexity lemma of Fernique (1978), which has been used in many places, in particular Marcus (1981). However, the integrals $I_n(\sigma)$ are truncated away from 0, which introduces a difficulty not found in Marcus (1981).

2. Proof of theorems. We begin by recalling some facts about symmetrization. We denote by (Ω, A, P) the probability space on which the X_k, $k \in \mathbb{N}$, are defined. Let X' be an independent copy of X and let ε and g be independent Rademacher and Gaussian random variables (mean 0, variance 1). We assume that X, X', ε, g and the associated sequences of independent copies X_k, X_k', ε_k and g_k, $k \in \mathbb{N}$, are defined on separate probability spaces. We denote by E_X the expectation operator over the probability space for X_k, $k \in \mathbb{N}$, and likewise for

X', ε and g. Define

$$D_n(t) = n^{-1/2} \sum_1^n \varepsilon_k e^{iX_k t}$$

and

$$S_n(t) = n^{-1/2} \sum_1^n g_k e^{iX_k t}.$$

Symmetrization techniques imply the familiar lemma:

LEMMA 4. *The following are equivalent*:

(a) $$\lim_n \phi_n^{-1} E\|C_n(t)\|_\infty = 0.$$

(b) $$\lim_n \phi_n^{-1} E\|D_n(t)\|_\infty = 0.$$

(c) $$\lim_n \phi_n^{-1} E\|S_n(t)\|_\infty = 0.$$

Similar statements hold for the conditions (ii) of Theorems 1 and 3.

PROOF. That (c) implies (b) implies (a) follows from two applications of Jensen's inequality:

$$\begin{aligned} E\|C_n(t)\|_\infty &= E_X \|n^{-1/2} \sum_1^n e^{iX_k t} - E_{X'} e^{iX_k' t}\|_\infty \\ &\le E\|n^{-1/2} \sum_1^n e^{iX_k t} - e^{iX_k' t}\|_\infty \\ &= E\|n^{-1/2} \sum_1^n \varepsilon_k (e^{iX_k t} - e^{iX_k' t})\|_\infty \\ &\le 2E\|D_n(t)\|_\infty. \end{aligned}$$

This proves that (b) implies (a). For (c) implies (b), observe that with $C = (E|g|)^{-1}$,

$$\begin{aligned} E\|D_n(t)\|_\infty &= CE\|n^{-1/2} \sum_1^n \varepsilon_k E_g |g_k| e^{iX_k t}\|_\infty \\ &\le CE\|n^{-1/2} \sum_1^n \varepsilon_k |g_k| e^{iX_k t}\|_\infty \\ &= CE\|S_n(t)\|_\infty. \end{aligned}$$

To see that (a) implies (b), by Lemma 2.3 of Giné and Zinn (1986), we have for all $a > 0$,

$$\begin{aligned} P(\|D_n(t)\|_\infty > a\phi_n) &\le P(\|C_n(t)\| > a\phi_n/2) \\ &\le 2(a\phi_n)^{-1} E\|C_n(t)\|_\infty \\ &\to 0, \qquad n \to +\infty. \end{aligned}$$

D_n is a symmetric random variable, so that Hoffmann-Jørgensen's inequality [Giné and Zinn (1986), Lemma 2.6] now implies (b).

That (b) implies (c) follows from a surprising inequality, due to Pisier and Fernique. We have

$$\begin{aligned} E\|S_n(t)\|_\infty &\le 2n^{-1/2}E\max_{k\le n}|g_k| + M\max_{1\le l\le n}E\|D_n(t)\|_\infty \\ &\le 4(\log(n)/n)^{1/2} + M\max_{1\le k\le n}E\|D_n(t)\|_\infty \\ &\to 0, \qquad n\to+\infty, \end{aligned}$$

where M is a finite constant. For a proof of this, see Giné and Zinn [(1986), Lemma 2.9]. □

We shall prove Theorem 2, the details for the other results being quite close to those. Further, we shall only prove (ii) if and only if (iii), and refer the reader to Kuelbs (1977) and Alexander (1987b) for the proofs of (i) if and only if (ii).

Assume (ii). By Lemma 3, we have

$$\lim_n \phi_n^{-1}E\|S_n(t)\|_\infty = 0. \tag{7}$$

Observe that conditioned on X (X's fixed), $S_n(t,\omega)$, $\omega\in\Omega$, is a stationary Gaussian process on $[-1,1]$, with pseudometric

$$\begin{aligned} \tau_n^2(|t|,\omega) &= E_g|S_n(0,\omega) - S_n(t,\omega)|^2 \\ &= n^{-1}\sum_1^n|1 - e^{iX_k(\omega)t}|^2 \\ &= 2n^{-1}\sum_1^n 1 - \cos X_k(\omega)t \le 4. \end{aligned}\tag{8}$$

As (ii) holds, by Kuelbs (1977), $C_n(t)$ obeys the BLIL in $C[-1,1]$. Further, by stationarity, $C_n(t)$ obeys the BLIL over any compact interval, and in particular in $C[-2,2]$. Hence

$$\begin{aligned} &\limsup_n \sup_{|t|\le 2} n^{1/2}\phi_n^{-1}|\tau_n^2(t,\omega) - \sigma^2(t)| \\ &\quad\le \limsup_n \sup_{|t|<2} n^{1/2}\phi_n^{-1}\left|\frac{1}{n}\sum_1^n e^{iX_k(\omega)t} - c(t)\right| \\ &\quad= M < +\infty \quad \text{a.s.} \end{aligned}$$

Therefore,

$$\sup_{|t|\le 2}|\tau_n(t,\omega) - \sigma(t)| = o(n^{-1/6})\cdot \quad \text{a.s.} \tag{9}$$

We now condition on the X's and apply Fernique's minoration of stationary Gaussian processes to $D_n(t)$ [Fernique (1975), Theorem 8.1.1; Marcus and Jain (1978), Theorems 7.3 and 7.6]. Let

$$m_n(u,\omega) = m\{t\in[-1,1]:\ \tau_n(t,\omega) < u\}. \tag{10}$$

Then

$$\begin{aligned} E_g\|S_n(t,\omega)\|_\infty &\geq K\int_0^2 (\log 2/m_n(u,\omega))^{1/2}\,du \\ &\geq K\int_{2n^{-1/6}}^2 (\log 2/m_n(u,\omega))^{1/2}\,du \\ &\geq K\int_{n^{-1/6}}^2 (\log 2/m_\sigma(u))^{1/2}\,du. \end{aligned}$$

Here K is an absolute constant, the first line follows from Fernique's theorem and (9) implies that the last line holds for large n, with probability tending to 1. This proves that for large n,

$$E_X E_g\|S_n(t)\|_\infty \geq \tfrac{1}{2}K\int_{n^{-1/6}}^2 (\log 2/m_\sigma(u))^{1/2}\,du.$$

In view of (7), this completes the proof of (ii) implies (iii).

We turn to the reverse implication, (iii) implies (ii). By Lemma 4, it is enough to prove that

$$\lim_n \phi_n^{-1} E\|D_n(t)\|_\infty = 0. \tag{11}$$

Observe that conditioned on X, $D_n(t,\omega)$ is a sub-Gaussian process with pseudometric τ_n defined by (8). Let $T_n(\omega)$ be a random subset of $[-1,1]$ defined in the following way: $T_n(\omega)$ is a maximal subset of points in $[-1,1]$ such that for all $t, t' \in T_n(\omega)$,

$$\tau_n(|t-t'|,\omega) \geq 4n^{-1/2} \quad \text{a.s.}$$

We then have

$$\min_{t\in T_n} \sup_{s\in[-1,1]} \tau_n(|s-t|,\omega) \leq 8n^{-1/2} \quad \text{a.s.}$$

Then

$$\begin{aligned} &E_\varepsilon\|S_n(t,\omega)\|_\infty \\ &\quad\leq E_\varepsilon \sup_{\substack{s,t \\ \tau_n(|s-t|)<8n^{-1/2}}} |S_n(s,\omega)-S_n(t,\omega)| + E_\varepsilon \sup_{t\in T_n(\omega)} |S_n(t)| \\ &\quad:= \mathrm{I} + \mathrm{II}. \end{aligned} \tag{12}$$

As

$$\begin{aligned} |S_n(s,\omega)-S_n(t,\omega)| &= n^{1/2}\left(n^{-1}\sum_1^n |e^{iX_k s}-e^{iX_k t}|\right) \\ &\leq n^{1/2}\tau_n(|s-t|,\omega), \end{aligned}$$

we have

$$\mathrm{I} \leq 8. \tag{13}$$

To control II, we will apply Dudley's theorem [Dudley (1967)]. That Dudley's theorem applies to sub-Gaussian processes is well known; see, e.g., Jain and Marcus [(1978), Theorem 4.5.5]. For a general (pseudo-) metric space (T,d) and

$u > 0$, let $N(T, d, u)$ be the least number of d-balls of radius u needed to cover T. Note that

$$(14)\quad N(T_n, \tau_n(\omega), u) \le \begin{cases} N([-1,1], \tau_n(\omega), u), & u > 0, \\ N([-1,1], \tau_n(\omega), 8n^{-1/2}), & 0 < u < 4n^{-1/2}. \end{cases}$$

Recalling (10), it is easy to see that

$$(15)\qquad N([-1,1], \tau_n(\omega), u) \le 2/m_n(u/2, \omega).$$

Therefore, by Dudley's theorem, (14) and (15),

$$(16)\qquad \begin{aligned} \mathrm{II} &\le K\left(1 + \int_0^2 (\log N(T_n, \tau_n, u))^{1/2}\, du\right) \\ &\le K\Big(1 + 4n^{-1/2}\big(\log 2/m_n(4n^{-1/2}, \omega)\big)^{1/2} \\ &\qquad + 2\int_{2n^{-1/2}}^1 \big(\log 2/m_n(u, \omega)\big)^{1/2}\, du\Big) \\ &\le K\left(1 + 4\int_{2n^{-1/2}}^1 \big(\log 2/m_n(u, \omega)\big)^{1/2}\, du\right). \end{aligned}$$

We apply Fernique's convexity lemma [Fernique (1975), Proposition 1.4.2; Marcus and Pisier (1984), Lemma 2.4] to (16), but the application is not immediate, as the integral in (16) is truncated away from 0. To get around this, for $0 < \delta < 2$, let $\mu = (\log 2/\delta)^{1/2}$ and define

$$\psi_\delta(u) = \begin{cases} (\log 2/u)^{1/2}, & \delta \le u \le 2, \\ \mu + 1/2\mu - u/2\mu\delta, & 0 \le u \le \delta. \end{cases}$$

Let $\psi(u) = (\log 2/u)^{1/2}$, $0 \le u \le 2$. $\psi_\delta(u)$ is convex and decreasing on $[0,2]$; it satisfies $\psi_\delta \le \psi$ and

$$(17)\qquad \psi(\delta) \le \psi_\delta(0) \le 2\psi(\delta), \qquad 0 < \delta < 2/e.$$

Choose $\delta_n \downarrow 0$ so that

$$(18)\qquad \psi(\delta_n) = n\int_{1/n}^1 \psi \circ m_\sigma(u)\, du.$$

Set $\psi_n = \psi_{\delta(n)}$. Then

$$(19)\qquad \begin{aligned} E_X\int_{n^{-1/2}}^1 \psi_n \circ m_n(u, \omega)\, du &\le E_X\int_0^1 \psi_n \circ m_n(u, \omega)\, du \\ &\le \int_0^1 \psi_n \circ m\{t \in [-1,1]\colon E_X\tau_n(t, \omega) < u\}\, du \\ &\le \int_0^1 \psi_n \circ m_\sigma(u)\, du \\ &\le n^{-1}\psi_n(0) + \int_{1/n}^1 \psi_n \circ m_\sigma(u)\, du \\ &\le 2n^{-1}\psi(\delta_n) + \int_{1/n}^1 \psi \circ m_\sigma(u)\, du \\ &= 3\int_{1/n}^1 \psi_n \circ m_\sigma(u)\, du. \end{aligned}$$

Here, the second line follows from Fernique's convexity lemma, the third from

$$E_X \tau_n(t, \omega) \le \sigma(t)$$

and the fact that ψ_n is decreasing, the fifth from (17), δ_n decreasing, and $\psi_n \le \psi$, and the last line follows from (18).

Let

$$A_n = \left\{ m_n(2n^{-1/2}, \omega) > \delta_n \right\}.$$

For $\omega \in A_n$,

$$\int_{2n^{-1/2}}^1 \psi_n \circ m_n(u, \omega)\, du = \int_{1/n}^1 \psi \circ m_\sigma(u)\, du.$$

Hence, by (13) and (16),

$$E_X E_\varepsilon \|D_n(t)\|_\infty 1_{A_n} \le 8 + K\left(1 + 3\int_{1/n}^1 \psi \circ m_\sigma(u)\, du\right). \tag{20}$$

Moreover, by (18) and (19),

$$\begin{aligned} P(A_n^c) &= \psi(\delta_n)^{-1}\psi_n(\delta_n)P(A_n^c) \\ &\le \psi(\delta_n)^{-1}E_X\psi_n \circ m_n(2n^{-1/2}, \omega) \\ &\le n^{1/2}\psi(\delta_n)^{-1}E_X\int_{n^{-1/2}}^{2n^{-1/2}} \psi_n \circ m_n(u)\, du \\ &\le 3n^{1/2}\psi(\delta_n)^{-1}\int_{1/n}^1 \psi_n \circ m_\sigma(u)\, du \\ &= 3n^{-1/2}. \end{aligned} \tag{21}$$

We have assumed (iii); hence, (20) and (21) imply that

$$\phi_n^{-1}\|D_n(t)\|_\infty \to 0 \quad \text{in probability}.$$

The Hoffmann-Jørgensen inequality [Giné and Zinn (1986), Lemma 2.6] then proves (11).

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