The Annals of Probability
1989, Vol. 17, No. 1, 189–193

DIMENSIONAL PROPERTIES OF ONE-DIMENSIONAL BROWNIAN MOTION[1]

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For each closed set $F \subseteq [0,1]$, $\dim X(F+t) = \min(1, 2\dim F)$ for almost all $t > 0$. (X is one-dimensional Brownian motion). For each closed set $F \subseteq [0,1]$ of dimension greater than $1/2$, $m(X(F+t)) > 0$ for almost all $t > 0$. These statements are true outside a single null-set in the sample space.

Introduction. $X(t)$ is the standard one-dimensional Wiener process on $0 \le t < +\infty$. We are interested in the Hausdorff dimension $\dim X(F)$ for closed sets F in R^+. Since X is almost surely in every class $\mathrm{Lip}^{1/2-\varepsilon}$ on every bounded set, we obtain easily $\dim X(F) \le \min(1, 2\dim F)$ for all sets F, outside a single null set. For fixed closed sets F the inequality is an equality [Kahane (1968, 1986)], the exceptional set depends on F. Since $X^{-1}(0)$ has almost surely dimension $1/2$, it is clear that results valid for *all* closed sets F must have a different form. Theorems 1 and 2 name properties of X valid outside a single null set for all closed sets F. After presenting their proofs, we make some comments of a more speculative nature.

Theorem 1. *For each closed set* $F \subseteq [0,1]$, $\dim X(F+t) = \min(1, 2\dim F)$ *for almost all* $t > 0$.

Theorem 2. *For each closed set* $F \subseteq [0,1]$ *of dimension greater than* $1/2$, $m(X(F+t)) > 0$ *for almost all* $t > 0$.

Proof of Theorem 1. It is convenient to define $H(u) = 1$ if $|u| < 1$, $H(u) = 0$ otherwise and

$$I(x, y, R) = \int_0^1 H(RX(x+t) - RX(y+t))\,dt$$

provided $R > 0$, $0 \le x < y \le 1$.

Lemma 1. $E(I(x, y, R)^p) \le p!\,3^p R^{-p}(y-x)^{-p/2}$ *for* $p = 1, 2, \ldots$, $0 \le x < y \le 1$, $R > 1$.

Proof. The pth moment is a multiple integral,

$$p! \int \cdots \int^* P\big(|X(x+t_i) - X(y+t_i)| < R^{-1}, 1 \le i \le p\big)\,dt_1 \cdots dt_p,$$

where the integral is extended over the set defined by $0 \le t_1 \le t_2 \le \cdots \le t_p \le 1$.

Received May 1987; revised January 1988.
[1]Supported in part by National Science Foundation Grant DMS-84-20561.
AMS 1980 *subject classifications.* 60J65, 28A75.
Key words and phrases. Brownian motion, dimension, capacity.

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We estimate the conditional probability

$$P\big(|X(x+t_j) - X(y+t_j)| < R^{-1} | X(s), 0 \le s < y + t_{j-1}\big),$$

for $2 \le j \le p$. Now $P \le 1$ always,

$$P \le R^{-1}(t_j - t_{j-1})^{-1/2}, \quad \text{when } R^{-2} \le t_j - t_{j-1} < y - x,$$

$$P \le R^{-1}(y-x)^{-1/2}, \quad \text{when } y - x < t_j - t_{j-1}.$$

(We assume that $R^{-2} < y - x$, since the inequality is trivial otherwise.) Integration on t_j yields an upper bound $3R^{-1}(y-x)^{-1/2}$, and iteration of this yields the inequality of Lemma 1. □

For use in Theorem 2, we observe that the integral of the *square* of the probability has magnitude $O(R^{-2}\log R) + O(R^{-2})(y-x)^{-1}$ and that this too is trivial if $y - x < R^{-2}$.

To prove Theorem 1 we use Lemma 1 with $R_n = 2^{-n}$ $(n = 1, 2, 3, \dots)$ and x, y all possible choices from the set T_n of rationals $k8^{-n}$ in $[0,1]$. The number of pairs $x < y$ in question is at most 8^{2n+1}. Hence for $A > 1$,

$$P\big(I(x, y, 2^n) > nA2^{-n}(y-x)^{-1/2} \text{ for some } x \in T_n,\ y \in T_n\big)$$
$$< 8^{2n+1}3^p p!(An)^{-p}.$$

The optimal estimation of $P(\cdot)$ is easily estimated by Stirling's formula and is summable for large A (e.g., $A > 24\log 2$).

We claim now that $I(x, y, 2^n) < A'n2^{-n}(y-x)^{-1/2}$ for all x, y and $n > n_0(\omega)$, almost surely. This is trivial unless $A'n^2 4^{-n} < y - x$, which we therefore assume to be true. Let $\bar{x}$ and $\bar{y}$ be the closest points in T_{n-1} to x and y, respectively. For $n > n_0(\omega)$, we get from Lévy's modulus of continuity $I(x, y, 2^n) \le I(\bar{x}, \bar{y}, 2^{n-1})$. But

$$I(\bar{x}, \bar{y}, 2^{n-1}) < An2^{1-n}(\bar{y} - \bar{x})^{-1/2} < 4An2^{-n}(y-x)^{-1/2} \quad \text{for } n > n_0(\omega).$$

Let now $e < \dim F$ and $0 < \eta < 1$, $0 < \eta < 2e$. By a theorem of Frostman [see Carleson (1967), page 28 or Kahane and Salem (1962), page 62], F carries a probability measure μ such that $\mu(S) \le c(\operatorname{diam} S)^e$ for every measurable set S. Let λ_t be the transform of μ by the mapping $x \to X(x+t)$ $(0 < x < 1$, $0 < t < 1)$. A further theorem of Frostman [Carleson (1967), page 28 or Kahane and Salem (1962), page 34] shows that $X(F+t)$, the support of λ_t, has dimension at least η if

$$\int\!\!\int |s_1 - s_2|^{-\eta}\lambda_t(ds_1)\lambda_t(ds_2) \equiv \int\!\!\int |X(x+t) - X(y+t)|^{-\eta}\mu(dx)\mu(dy)$$

is finite. The second formula for "energy of λ_t in dimension η" can be transformed (using the function H introduced above) into

$$\eta\int\!\!\int\!\!\int H(RX(x+t) - RX(y+t))R^{\eta-1}\mu(dx)\mu(dy)\,dR$$

$$\le 1 + \int_1^\infty \int\!\!\int H(RX(x+t) - RX(y+t))R^{\eta-1}\mu(dx)\mu(dy)\,dR.$$

To prove that the integral on the right is finite for almost all $t \in (0,1)$, we integrate on $(0,1)$ obtaining

$$2\iint_{x<y}\int_1^{\infty} I(x, y, R)R^{\eta-1}\mu(dx)\mu(dy).$$

The product measure of the set defined by $0 < y - x < R^{-2}$ is $O(R^{-2e})$, and the consequent estimation converges because $-2e + \eta - 1 < -1$. On the complementary domain we have $(y - x)^{-1/2} < R$ and then we have $I(x, y, R) < B(\omega)\log(e + R)R^{-1}(y - x)^{-1/2}$ (with B depending only on the path). Integrating with respect to R first, we obtain $O(y - x)^{-\eta/2}\log(e + |y - x|^{-1})$, and the integral converges because $\eta < 2e$. This completes the proof of Theorem 1. □

PROOF OF THEOREM 2. The argument applies to sets E of positive h-measure, where $h(u) = u^{1/2}\log^{-3}(e + u^{-1})$, $0 < u < 1$. Obviously the method used for Theorem 1 must fail, since the energy in dimension 1 is always infinite. The standard technique involves the Plancherel formula; we employ the notations $e(t) \equiv \exp 2\pi i t$, $\hat{\mu}(u) \equiv \int \overline{e(us)}\mu(ds)$. In proving that $X(F + t)$ has positive measure for almost all $t \in (0,1)$, it is natural to consider

$$\int_{-\infty}^{\infty}\int_0^1 |\hat{\lambda}_t(u)|^2\,dt\,du = \int_{-\infty}^{\infty}\iint\left[\int_0^1 e(-uX(x + t) + uX(y + t))\,dt\right] \times \mu(dx)\mu(dy)\,du,$$

for an appropriate measure μ on F, determined by Frostman's theorem. The inner integral, however, cannot be brought down to $o(u^{-1})$ even for $x = 0$, $y = 1$, and so this method, too, seems to fail. To overcome this difficulty, we choose and fix a smooth, even function $\psi \geq 0$, such that $\psi(u) = 1$ when $1 \leq |u| \leq 2$ and $\psi(u) = 0$ outside $1/2 < |u| < 5/2$. Then for any function $g(u)$,

$$\int_{|u|>1} |g(u)|^2\,du < \sum_0^{\infty}\int \psi(2^{-n}u)|g(u)|^2\,du.$$

Writing $g(u) = \hat{\lambda}_t(u)$, we find a formula for the nth integral on the right $(n = 0, 1, 2, 3, \ldots)$,

$$2^n\iint\hat{\psi}(2^nX(x + t) - 2^nX(y + t))\mu(dx)\mu(dy).$$

Bearing in mind that this integral is positive, we see that Theorem 2 can be proved by verifying the convergence of

$$(1)\qquad \sum_1^{\infty} 2^n\iint\left|\int_0^1\hat{\psi}(2^nX(x + t) - 2^nX(y + t))\,dt\right|\mu(dx)\mu(dy)$$

for all measures μ on $(0,1)$ with the appropriate Lipschitz-type property. From the nth integral in (1) we remove the set defined by $|x - y| < 4^{-n}(n + 1)^{-2}$, allowing thereby an error $O(n^{-2})$. For the remaining points (x, y), we define

$$J(x, y, n) = \int_0^1 \hat{\psi}(2^n X(x + t) - 2^n X(y + t))\, dt$$

and state

LEMMA 2. *For* $n \geq n(\omega)$ *and* $|y - x| \geq 4^{-n} n^2$, $|J(x, y, n)| \leq (2 + c)^{-n}(y - x)^{-1/2}$, *for some* $c > 1/2$.

Taking into account the Hölder-continuity of X and the smoothness of $\hat{\psi}$, we see that it will be sufficient to prove an inequality

$$E\left(J(x, y, n)^{2p}\right) \leq A_p(2 + c_1)^{-2pn}(y - x)^{-p}$$

with a constant $c_1 > 1/2$. (J is real because ψ is even.) The moment is the expected value of a multiple integral,

$$\int \cdots \int \prod_1^{2p} \hat{\psi}(2^n X(x + t_K) - 2^n X(y + t_K))\, dt_1 \cdots dt_{2p}.$$

We can assume that $0 < x < y$ and claim that the expected value is exceedingly small if, for a certain K, $|t_K - t_j| \geq 4^{-n}(n + 1)^2$ for $j \neq K$ and $|y + t_K - x - t_j| \geq 4^{-n}(n + 1)^2$ for $j \neq K$. To verify this we let $r_n = 4^{-n}(n + 1)^2$ so that the interval $(t_K + y - r_n, t_K + y + r_n)$ is entirely contained in $(0, +\infty)$ and contains none of the $4p$ values appearing in the product $\prod$ except $y + t_K$. Thus $X(y + t_K - r_n) - 2X(y + y_K) + X(y + t_K - r_n)$ is orthogonal to all values $X(\cdot)$ appearing there, except $X(y + t_K)$, with which it has inner product $-r_n$, its variance being $2r_n$. Hence $X(y + t_K) = h + Z$, where h is measurable over the σ-field of the remaining values $X(\cdot)$, and Z is Gaussian and *independent* of those values, $\sigma^2(Z) \geq (r_n/2)$, $\sigma^2(2^n Z) \geq 4^n r_n/2 = (n + 1)^2/2$. Here we invoke a formula from Fourier analysis: When $\psi \in L^1(R)$ and Y is a random variable, $E(\hat{\psi}(Y)) = \int_{-\infty}^{\infty} \psi(s) \overline{E(e(sY))}\, ds$. We use the requirement that $\psi(u) = 0$ when $|u| < 1/2$, and first take the expected value with respect to the variable Z. The expectation is indeed minuscule, being bounded by $c_1 \exp(-c_2 n^2)$ $(c_1 > 0,\ c_2 > 0)$. This argument is valid for $K = 1, 2, \ldots, 2p$; a bit of combinatorics shows that it applies to all values $t_1, \ldots, t_{2p}$ except a set of product measure $A_p r_n^p = A_p' 2^{-np}(n + 1)^{2p}$, which we call $T_n(x, y)$.

$$\int \cdots \int_{T_n} E\left(\prod |\hat{\psi}(2^n X(x + t_K) - 2^n X(y + t_K))|\right) dt_1 \cdots dt_{2p}$$

by means of the Cauchy–Schwarz inequality and a remark made in the proof of Lemma 1. Let B be any (large) positive number; since $\hat{\psi}$ is a rapidly decreasing function, the product $\prod$ is bounded by $C(B)2^{-nB}$ outside the set defined by the inequalities $|X(x + t_K) - X(y + t_K)| \leq 2^{-7n/8}$. The Cauchy–Schwarz inequality, the estimate for the measure of T_n and the remark cited above therefore yield (with $R = 2^{-9n/10}$) an estimate

$$A_p''\left((y-x)^{-2np}2^{-7np/2}2^{-np}\right)^{1/2}n^{8p} < A_p'''\left((y-x)^{-np}\right)(2\cdot 1)^{-2np}.$$

The nth integral in the sum (1) has magnitude

$$O(n^{-2}) + (2+c)^{-n}2^n\int\int^*|y-x|^{-1/2}\mu(dx)\mu(dy),$$

where the integral $\int\int^*$ extends over the subset $|x-y| \geq 4^{-n}(n+1)^2$. Since $\int_0^1 h(t)t^{-3/2}\,dt < +\infty$, the sum (1) converges. □

Remarks and problems. For Brownian motion (X_1, X_2) with range in R^2, Theorem 1 has no interest in view of Kaufman (1969b) and Hawkes (1970). The following problem analogous to Theorem 1 seems very difficult.

For each closed set F, a number θ in $[0, \pi]$ is *exceptional* if $X_1\cos\theta + X_2\sin\theta$ maps F onto a linear set of dimension less than $\min(1, 2\dim F)$. Is there a random closed set F whose exceptional set of angles has positive dimension?

Returning to one-dimensional Brownian motion X, t is *exceptional* if $\dim X(F+t) < \min(1, 2\dim F)$. What about the exceptional sets? On these topics compare Kaufman (1968, 1969a) and Kaufman and Mattila (1975).

When F is a fixed set of dimension greater than $1/2$, then $X(F)$ has almost surely an interior point [Kahane (1986)]. Is it true that for every closed set F of dimension greater than $1/2$, $X(F+t)$ has an interior point for some t?

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