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STOCHASTIC DIFFERENCE EQUATIONS AND GENERALIZED GAMMA DISTRIBUTIONS.

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We study the asymptotic growth rates of discrete-time stochastic processes (X_n), where the first two conditional moments of the process depend only on the present state. Such processes satisfy a stochastic difference equation $X_{n+1} = X_n + g(X_n) + R_{n+1}$, where g is a positive function and (R_n) is a martingale difference sequence. It is known that a large class of such processes diverges with positive probability, and when properly normalized converges almost surely or converges in distribution to a normal or a lognormal distribution. Here we find a class of processes that when properly normalized converges in distribution to a generalized gamma distribution. Applications of this result to state dependent random walks and population size-dependent branching processes yield new results and reprove some of the known results.

Introduction. This paper is concerned with the asymptotic behavior of discrete-time stochastic processes that satisfy stochastic difference equations of the form

$$X_{n+1} = X_n + g(X_n) + R_{n+1}, \tag{1}$$

where $g(x)$ is a positive function and (R_n) is a square-integrable martingale difference sequence, the second conditional moments of which depend only on the present state of the process (X_n):

$$E(R_{n+1}|X_0, X_1, \dots, X_n) = 0,$$
$$E\left(R_{n+1}^2|X_0, X_1, \dots, X_n\right) = v(X_n),$$

for some positive function $v(x)$.

It is possible to describe the asymptotic behavior of a large class of stochastic processes (X_n) that satisfy some weak Markov property in terms of the conditional expectation $E(X_{n+1}|X_n = x)$ and the conditional second moment $E(X_{n+1}^2|X_n = x)$ functions; see, for example, Keller, Kersting and Rosler (1987), Kuster (1985), Klebaner (1986) and Cohn and Klebaner (1986).

Stochastic difference equations of the form (1) arise in various stochastic models. Any discrete time Markov chain can be written in the form (1) by letting $g(x)$ and (R_n) be defined by relations

$$E(X_{n+1}|X_n = x) = x + g(x), \qquad R_{n+1} = X_{n+1} - E(X_{n+1}|X_n). \tag{2}$$

State dependent random walks, population size dependent branching processes and branching processes with dependent offspring provide some of the examples.

In this paper we are interested in the asymptotic growth rates of the process on the set where the process diverges to infinity. This problem was recently

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studied by Keller, Kersting and Rosler (1987) and by Kuster (1985). We will not consider the problem of recurrence or transience of (X_n). This problem has received attention in many papers. To quote a few: Zubkov (1974), Levy (1979), Klebaner (1984) and Hopfner (1985) for branching processes; Keller, Kersting and Rosler (1987) and Kersting (1986) for stochastic difference equations. Kersting (1986) generalized results of Lamperti (1960) and obtained most general and nearly necessary and sufficient conditions for the process (X_n) to diverge with positive probability. It is known from the results of Lamperti (1960), Keller, Kersting and Rosler (1987) and Kersting (1986) that if the $2+\delta$ $(\delta > 0)$ conditional moment of the increments of the process satisfies a certain growth condition, which will be given later, then in order for the process (X_n) to be transient $g(x)/v(x)$ is not allowed to tend to 0 too fast. The boundary case is $g(x)/v(x) \sim Cx^{-1}$, $x \to \infty$, where both types of behavior (transient or recurrent) are possible. The processes we consider here are the divergent processes that live on this boundary.

To study asymptotic growth rates on the set of divergence, Keller, Kersting and Rosler (1987) and Kuster (1985) compared the growth in the stochastic equation to the growth of the unperturbed deterministic equation

(3) $$x_{n+1} = x_n + g(x_n).$$

Kuster (1985) found that if $g(x)$ satisfies certain growth restrictions and some regularity properties and $v(x), g(x)$ satisfy a rather involved joint growth condition that comes close to $v(x) = O(g(x)x^{1-\delta})$ as $x \to +\infty$ for some $\delta > 0$, then X_n/x_n converges almost surely (a.s.) on the set $X_n \to +\infty$ to a positive random variable. Moreover if some additional assumptions hold, then this random variable is a.s. 1. See Kuster (1985) for details. A comprehensive study of stochastic difference equations with $g(x)$ and (R_n) satisfying a set of assumptions was done by Keller, Kersting and Rosler (1987). They assumed $g(x) = o(x)$ as $x \to \infty$ and a joint growth condition on $g(x), v(x)$ similar to but somewhat weaker than Kuster's. They gave necessary and sufficient conditions for the convergence of X_n/x_n to the log normal distribution, for the convergence in probability and a.s. of X_n/x_n to 1 and for the convergence to the standard normal distribution of a suitably normalized process $(X_n - x_n)$.

Here we study a class of processes X_n that satisfy (1) with $g(x) = cx^\alpha + o(x^\alpha)$, $\alpha < 1$, $v(x) \sim vx^{1+\alpha}$, as $x \to +\infty$. In this case for a given g the variance function $v(x)$ grows too fast, so that Kuster's and Keller, Kersting and Rosler's joint growth conditions placed on the functions g and v mentioned previously are not fulfilled and their results do not apply. We shall see that if some additional assumptions on (R_n) hold then such processes when properly normalized converge in distribution to a generalized gamma distribution.

In what follows C will denote an unspecified positive constant. All the asymptotic relations are given at $+\infty$ unless stated otherwise.

Assumptions and results. Let a stochastic process (X_n) satisfy relation (1). We shall assume that:

(A1) $$X_n \geq 0 \quad \text{for all } n.$$

(A2a) $$X_n \to \infty \quad \text{a.s.}$$

Or 0 is the only absorbing state and

(A2b) $$I(X_n > 0) \to I(X_n \to \infty) \quad \text{a.s.}$$

$g(x)$ is a positive function defined on $(0, +\infty)$ that satisfies

(A3) $$g(x) = cx^\alpha + o(x^\alpha),$$

with $c > 0$, $\alpha < 1$.

Let the σ-fields F_n form a filtration to which (X_n) is adapted. Let (R_n) be a martingale difference sequence with

$$E(R_{n+1}|F_n) = 0 \quad \text{a.s.}$$

and

$$E\left(R_{n+1}^2|F_n\right) = v(X_n) \quad \text{a.s.}$$

$v(x)$ is a positive function defined on $(0, +\infty)$ that satisfies

(A4) $$v(x) = vx^{1+\alpha} + o(x^{1+\alpha}), \qquad v > 0.$$

Suppose further that (R_n) possess moments of all orders and there exist functions $M_k(x)$ such that for $k \geq 3$,

$$E\left(|R_{n+1}|^k|F_n\right) \leq M_k(X_n) \quad \text{a.s.}$$

and $M_k(x)$ satisfy

(A5) $$M_k(x) = o(x^{k+\alpha-1}).$$

Theorem. *Assume* (A1)–(A5) *with* (A2a) *or* (A2b), *and that* $2c > v$. *Then as* $X_n \to \infty$, $X_n^{1-\alpha}/n$ *converges in distribution to a gamma distribution with parameters* $(2c - v\alpha)/(v - v\alpha)$ *and* $2/(v(1-\alpha)^2)$.

Remark. A power of a gamma distribution is known in the literature as a generalized gamma distribution. We show that x_n satisfy $x_n \sim Cn^{1/(1-\alpha)}$, with $C = (c - c\alpha)^{1/(1-\alpha)}$. Thus X_n/x_n converges in distribution to a generalized gamma distribution.

We comment on the assumptions. The fact that (X_n) is nonnegative is not very restrictive and (A1) could be replaced by $X_n > -C$. In this case we would consider the process $X_n + C$, so that we are only dealing with the processes with state-space bounded away from $-\infty$. Many growth models satisfy this. (A2a) is satisfied by transient Markov chains on the nonnegative integers. (A2b) is typically satisfied by branching processes. $g(x) = o(x)$ is a standard assumption as in Keller, Kersting and Rosler (1987) and Kuster (1985). That is why $\alpha < 1$. In the case $\alpha = 1$ an exponential growth occurs as in supercritical Galton–Watson processes. Before commenting on (A4) and (A5), we discuss a sufficient condition for the transience of (X_n). We are considering processes with $P(X_n \to \infty) > 0$. Condition (A6) guarantees

(A6) $$E\left(|R_{n+1}|^{2+\delta}|F_n\right) < Cv^{1+\delta/2}(X_n) \quad \text{for some } \delta > 0 \text{ a.s.}$$

See Kersting (1986) and Keller, Kersting and Rosler (1987). Lamperti (1960) uses a more stringent condition of boundedness of the preceding moments. By a result of Kersting (1986) if some additional assumptions on growth of $g(x)$ hold, then (A6) together with

$$\liminf_{x\to+\infty} xg(x)v^{-1}(x) > \tfrac{1}{2}$$

is a sufficient condition for transience. In view of the last condition we demand $2c > v$.

We shall comment on what happens if we let $v(x) = vx^{\beta}$ with $\beta \neq 1 + \alpha$ in (A4). If (A6) holds and $\beta > 1 + \alpha$, then $P(X_n \to \infty) = 0$ by Theorem 1 of Kersting (1986). If $\beta < 1 + \alpha$, then Keller, Kersting and Rosler's (1987) joint growth condition on $g(x)$, $v(x)$ is fulfilled. If in addition $g(x)$ and $v(x)$ satisfy some regularity assumptions placed there, then the following holds: If $3\alpha - 1 < \beta < 1 + \alpha$, then X_n/x_n converges to 1 in probability and $X_n - x_n$ suitably normalized converges to the standard normal distribution. If $\beta \leq 3\alpha - 1$, then X_n/x_n converges to 1 a.s. and $(X_n - x_n)/x_n^{\alpha}$ converges a.s.

Condition (A5) for *some* $k > 2$ is weaker than (A6) in our case since with $k = 2 + \delta$, we have

$$v^{1+\delta/2}(x) = v^{k/2}(x) \sim vx^{(1+\alpha)k/2} = o(x^{k+\alpha-1})$$

since $\alpha < 1$. The demand for (A5) for higher moments is due to technical reasons, since the method of proof is the method of moments. To prove the theorem we shall use the following lemma and its corollary which are of interest in their own right.

LEMMA. *Suppose f, g are nonnegative on $[0,\infty)$, f is bounded on bounded intervals, $g(x) \to \infty$ as $x \to \infty$, and* $\limsup f(x)/g(x) = C < \infty$. *If $X_n \to \infty$ in probability and $Eg(X_n) < \infty$ for all n [this implies $Ef(X_n) < \infty$], then we have*

$$\limsup_{n\to\infty} Ef(X_n)/Eg(X_n) \leq C.$$

PROOF. Let $\varepsilon > 0$ and pick M large enough so that $f(x)/g(x) \leq C + \varepsilon$ for $x > M$. Then

$$\begin{aligned} Ef(X_n) &= E(f(X_n)I(X_n \leq M)) + E(f(X_n)I(X_n > M)) \\ &\leq \sup_{x\leq M} f(x) + (C+\varepsilon)Eg(X_n). \end{aligned}$$

Dividing both sides by $Eg(X_n)$ and noticing that $Eg(X_n) \to \infty$ gives the desired result since ε is arbitrary. □

COROLLARY. *If f, g and X_n satisfy the conditions of the lemma and $f(x) \sim g(x)$, then $Ef(X_n) \sim Eg(X_n)$.*

Proofs.

PROOF OF THE THEOREM. The proof is given by the method of moments. We show by induction on l that

$$\mu_l = \lim_{n\to\infty} E\left(X_n^{l(1-\alpha)}n^{-l}\right)$$

exists and is finite; moreover, the following recurrence relation holds:

$$\mu_l = (c(1-\alpha) + v(1-\alpha)(l - \alpha l - 1)/2)\mu_{l-1}, \qquad \mu_0 = q. \tag{4}$$

The starting point is the following binomial expansion. For any $k \geq 0$, $x \geq -1$, integer m,

$$(1+x)^k = \sum_{j=0}^{m-1} \binom{k}{j} x^j + r_m(x), \tag{5}$$

where $r_m(x)$ is the remainder.

We show that if $k < m$, then for all $x \geq -1$,

$$|r_m(x)| < C|x|^m \tag{6}$$

and if $k \geq m$, then for all $x \geq -1$,

$$|r_m(x)| < C(|x|^m + |x|^k). \tag{7}$$

To see (6) and (7), observe that $r_m(x)$ is continuous on $[-1, +\infty)$ since it is the difference of two continuous functions. $|r_m(x)|/|x|^m$ and $|r_m(x)|/(|x|^m + |x|^k)$ are continuous functions on $[-1, +\infty)$, except perhaps at 0. Continuity at 0 is established by considering the Lagrange form of the remainder,

$$r_m(x) = \binom{k}{m} x^m (1+\theta x)^{k-m} \quad \text{for some } 0 < \theta < 1.$$

Thus $r_m(x)/x^m \sim \binom{k}{m}$, $x \to 0$. Since $|r_m(x)| < C(1 + |1+x|^k + |x| + \cdots + |x|^{m-1})$,

$$\limsup_{x \to +\infty} |r_m(x)|/|x|^m = 0 \quad \text{for } k < m.$$

This implies that $|r_m(x)|/|x|^m$ is bounded on $[-1, +\infty)$, which is (6).

For $k > m$, $|r_m(x)|/(|x|^m + |x|^k) \sim \binom{k}{m}$ as $x \to 0$ and it is bounded away from ∞ as $x \to +\infty$; therefore, it is bounded on $[-1, +\infty)$, which is (7).

Denote by $\delta X_n = (X_{n+1} - X_n)/X_n$ for $X_n > 0$. Then $\delta X_n \geq -1$ and we can write, by using (5) with $m = 3$,

$$X_{n+1}^k = X_n^k (1 + \delta X_n)^k = X_n^k + k X_n^k \delta X_n + \binom{k}{2} X_n^k (\delta X_n)^2 + X_n^k r(\delta X_n).$$

Using (1) we obtain

$$\begin{aligned} X_{n+1}^k = {} & X_n^k + k X_n^{k-1} g(X_n) + \binom{k}{2} X_n^{k-2} R_{n+1}^2 \\ & + \left[k X_n^{k-1} + 2\binom{k}{2} X_n^{k-2} g(X_n) \right] R_{n+1} \\ & + \binom{k}{2} X_n^{k-2} g^2(X_n) + X_n^k r(\delta X_n). \end{aligned} \tag{8}$$

If (A2a) holds, then without loss of generality we can assume $X_n \geq 1$; otherwise consider the process $X_n + 1$. In this case (8) holds for all X_n. If (A2b) holds, then (8) holds for all $X_n > 0$. Now take expectations in (8) and use $E(R_{n+1}|F_n) = 0$ to obtain

$$E\left(X_{n+1}^k|F_n\right) = X_n^k + kX_n^{k-1}g(X_n) + \binom{k}{2}X_n^{k-2}E\left(R_{n+1}^2|F_n\right)$$
$$+\binom{k}{2}X_n^{k-2}g(X_n)^2 + X_n^kE\left(r(\delta X_n)|F_n\right).$$

Using (A3) and (A4), we have

$$E(X_{n+1}|F_n) = X_n^k + ckX_n^{k+\alpha-1} + v\binom{k}{2}X_n^{k+\alpha-1}$$
$$+X_n^kE\left(r(\delta X_n)|F_n\right) + o\left(X_n^{k+\alpha-1}\right), \tag{9}$$

if (A2a) holds or if (A2b) holds and $X_n > 0$. If (A2b) holds and $X_n = 0$, then $E(X_{n+1}|F_n) = 0$. We estimate the expectation of the remainder next.

For $0 < k < 3$ use the bound in (6) with $m = 3$ together with the inequality $|a + b|^m < C(|a|^m + |b|^m)$ to obtain

$$X_n^k|r(\delta X_n)| < CX_n^k|\delta X_n|^3 < CX_n^{k-3}\left(g(X_n)^3 + |R_{n+1}|^3\right). \tag{10}$$

For $k \geq 3$ use the bound in (7) together with the abovementioned inequality to have

$$X_n^k|r(\delta X_n)| < CX_n^{k-3}\left(g(X_n)^3 + |R_{n+1}|^3\right) + C\left(g(X_n)^k + |R_{n+1}|^k\right). \tag{11}$$

Thus for $0 < k < 3$ with (A3) and (A5) being $M_3(x) = o(x^{2+\alpha})$ we have from (10),

$$X_n^k|E\left(r(\delta X_n)|F_n\right)| < o\left(X_n^{k-3(1-\alpha)}\right) + o\left(X_n^{k+\alpha-1}\right) = o\left(X_n^{k+\alpha-1}\right).$$

For $k \geq 3$ with (A3) and (A5) we have from (11),

$$X_n^k|E\left(r(\delta X_n)|F_n\right)| < o\left(X_n^{k+\alpha-1}\right) + o\left(X_n^{k\alpha}\right) = o\left(X_n^{k+\alpha-1}\right).$$

Thus we can write from (9) for all $k > 0$,

$$E\left(X_{n+1}^k|F_n\right) = \left[X_n^k + \left(ck + v\binom{k}{2}\right)X_n^{k+\alpha-1} + D_{k,n}\right]I(X_n > 0), \tag{12}$$

where $D_{k,n} = o(X_n^{k+\alpha-1})$. Now take expectations in (12) and iterate to obtain

$$EX_{n+1}^k = \left(ck + v\binom{k}{2}\right)\sum_{j=0}^{n} EX_j^{k+\alpha-1}I(X_j > 0)$$
$$+\sum_{j=0}^{n} D_{k,j}I(X_j > 0) + EX_0^k. \tag{13}$$

Let $k = l(1-\alpha)$ in (13), $l = 1, 2, \ldots$, and write $\Delta_{l,n}$ for $D_{l(1-\alpha),n}$. Then for $l = 1$,

$$EX_{n+1}^{1-\alpha} = c_0(n+1) + \sum_{j=0}^{n} E\Delta_{1,j} + EX_0^{1-\alpha} \tag{14}$$

under (A2a) and

$$EX_{n+1}^{1-\alpha} = c_0 \sum_{j=0}^{n} P(X_j > 0) + \sum_{j=0}^{n} E\Delta_{1,j} I(X_j > 0) + EX_0^{1-\alpha} \tag{15}$$

under (A2b) with $c_0 = c(1-\alpha) + v\binom{1-\alpha}{2}$.

$\Delta_{1,n} = o(1)$ on $X_n \to +\infty$. If (A2a) holds, then by the dominated convergence $E\Delta_{1,n} \to 0$ and $\sum_{j=0}^{n} \Delta_{1,j} = o(n)$. Thus from (14),

$$EX_n^{1-\alpha} = \mu_1 n + o(n). \tag{16}$$

If (A2b) holds, then $E\Delta_{1,n} I(X_n > 0) = o(1)$, $P(X_n > 0) \to q = P(X_n \to \infty)$ and from (15) we have (16). Thus (4) holds for $l = 1$. Suppose now that (4) holds for all integers up to and including l. $\Delta_{k,n} I(X_n > 0) = o(X_n^{k+\alpha-1})$ a.s.; hence, for $k > 1 - \alpha$ we have from (13) and the lemma,

$$EX_{n+1}^{k} = c_k \sum_{j=0}^{n} EX_j^{k+\alpha-1} + o\left(\sum_{j=0}^{n} EX_j^{k+\alpha-1}\right), \tag{17}$$

with $c_k = ck + v\binom{k}{2}$. Using the supposition of induction,

$$\sum_{j=1}^{n} EX_j^{l(1-\alpha)} \sim \mu_l \sum_{j=1}^{n} j^l \sim \mu_l (l+1)^{-1} n^{l+1}.$$

Therefore we can see from (17) that

$$EX_{n+1}^{(l+1)(1-\alpha)} n^{-(l+1)} \sim c_{(l+1)(1-\alpha)} (l+1)^{-1} \mu_l.$$

Hence (4) holds for $l+1$ and thus for all l.

The moments of the gamma distribution with parameters $(2c - v\alpha)/(v - v\alpha)$ and $2/(v(1-\alpha)^2)$ satisfy relation (4). Since a gamma distribution is uniquely determined by its moments, it follows that $X_n^{1-\alpha}/n$ converges in distribution to the gamma distribution with the parameters given previously and the theorem is proved. □

PROOF OF THE REMARK. It is clear from (3) and (A3) that $x_n \to \infty$. Using (5) with $m = 2$ and the Lagrange form of the remainder, we obtain $x_{n+1}^{1-\alpha} = x_n^{1-\alpha} + c(1-\alpha) + o(1)$. Hence $x_n^{1-\alpha} = c(1-\alpha)n + o(n)$. The rest is obvious. □

Applications.

1. *State dependent random walk.* (X_n) is a random walk on the nonnegative integers with transition probabilities

$$p_{i,i+1} = p(i), \qquad p_{i,i-1} = 1 - p(i) \quad \text{for } i > 0, \qquad p_{01} = 1.$$

It can be seen easily that

$$g(i) = 2p(i) - 1, \qquad v(i) = 1 - g^2(i), \qquad i > 0.$$

From this follows that the only case covered by our theorem is

$$g(i) \sim ci^{\alpha} \quad \text{with } \alpha = -1.$$

In this case

$$p(i) = 1/2 + c/2i^{-1} + o(i^{-1}). \tag{18}$$

If the sequence $(p(i))$ satisfies (18), then $P(X_n \to \infty) = 1$ if and only if $c > \frac{1}{2}$, which can be seen from a well-known criterion for transience of Markov chains given later. If $c > \frac{1}{2}$, then conditions (A1)–(A5) with (A2a) of our theorem are fulfilled. Hence $X_n/n^{1/2}$ converges in distribution to a generalized gamma distribution with density

$$2^{1/2-c}\Gamma^{-1}(c + 1/2)x^{2c}\exp(-x^2/2).$$

This is a known result which is given in Guivarc'h, Keane and Roynette (1977) and also in Keller, Kersting and Rosler (1987).

In the preceding example of a random walk our theorem was applicable only when $p(i)$ approached $\frac{1}{2}$ at the rate i^{-1} resulting in the growth rate $n^{1/2}$. But if we allow p_{ii} to be positive, we can treat cases producing the convergence rates n^{β}, $0 < \beta \leq \frac{1}{2}$.

Let (X_n) be a random walk with transition probabilities

$$p_{ii+1} = p(i), \qquad p_{ii} = r(i), \qquad p_{ii-1} = q(i), \qquad i \geq 1, \qquad p_{01} = 1.$$

$p(i) + q(i) + r(i) = 1$, $p(i), q(i), r(i) > 0$. It can be seen that

$$g(i) = p(i) - q(i) \quad \text{and} \quad v(i) = p(i) + q(i) - g^2(i), \qquad i \geq 1.$$

Let $p(i), q(i), r(i)$ satisfiy

$$r(i) = 1 - vi^{1+\alpha} + o(i^{1+\alpha}), \qquad p(i) - q(i) = ci^{\alpha} + o(i^{\alpha}),$$

with $c, v > 0$, $2c > v$, $\alpha \leq -1$.

It is easy to see from the preceding relations that $p(i) \sim v/2i^{1+\alpha}$ and $q(i)/p(i) = 1 - 2c/vi^{-1} + o(i^{-1})$. The criterion for transience and recurrence of Markov chains [see Karlin and Taylor (1975), page 108] gives $P(X_n \to \infty) = 1$ if and only if $\sum_{n=1}^{\infty}\prod_{i=1}^{n} q(i)/p(i) < \infty$. Since $\log\prod_{i=1}^{n} q(i)/p(i) \sim (-2c/v)\log n$, this series converges if and only if $2c > v$. It is easily checked now that assumptions (A1)–(A5) with (A2a) are fulfilled and the conclusion holds.

Notice that the first part of our example is a particular case of the second part corresponding to $\alpha = -1$ and $r(i) = 0$. If we let $r(i) \neq 0$ while $\alpha = -1$, then we obtain another case resulting in $n^{1/2}$ growth: $p(i) = p + c_1 i^{-1} + o(i^{-1})$, $q(i) = p + c_2 i^{-1} + o(i^{-1})$, $r(i) = 1 - 2p - (c_1 + c_2)i^{-1} + o(i^{-1})$ with $p = v/2 \leq \frac{1}{2}$ and $c_1 - c_2 > p$.

If $\alpha < -1$, then $r(i) \to 1$ and $p(i), q(i) \to 0$. It is seen that $p(i)/(1 - r(i)) = \frac{1}{2} + (c/v)i^{-1} + o(i^{-1})$ so that this case is a "time change" of the previous one.

2. *Generalized state dependent random walk.* Let $(Y_n(i))$, $n, i \geq 0$, be a double array of integer valued independent random variables with identically distributed rows, i.e., the distribution of $Y_n(i)$ depends on i and does not depend

on n. Let (X_n) satisfy

$$X_{n+1} = X_n + Y_{n+1}(X_n). \tag{19}$$

All Markov chains on the nonnegative integers satisfy (19) with a suitable choice of $(Y_n(i))$. Equation (19), however, arises naturally in the generalized birth and death processes. Embedded Markov chains of the generalized birth and death processes satisfy (19), where $Y_n(i) + i$ has the interpretation of the total number of offspring when the population size is i. For processes satisfying (19) the conditions of the theorem translate into conditions on the moments of the defining array $(Y_n(i))$. By using (2), we find

$$g(i) = EY_n(i), \qquad v(i) = \operatorname{Var}(Y_n(i)), \qquad R_{n+1} = Y_{n+1}(X_n) - g(X_n).$$

Assume $P(Y_n(i) \geq -i) = 1$ for all i. Then $X_n \geq 0$ a.s. for all n. If we assume $P(Y_n(0) = 1) = 1$ then the state 0 is reflecting. If we assume $P(Y_n(0) = 0) = 1$ and $P(Y_n(i) = 0) < 1$ for $i \geq 1$, then 0 is the only absorbing state.

Let $EY_n(i) \sim ci^{\alpha}$, $\operatorname{Var}(Y_n(i)) \sim vi^{\alpha+1}$, $2c > v$ and for some $k > 2$, $EY_n^k(i) < Ci^{(1+\alpha)k/2}$. Then by the results of Kersting (1986) $P(X_n \to \infty) = 1$ if 0 is reflecting or $P(X_n \to \infty) + P(X_n \to 0) = 1$ if 0 is absorbing. If we assume further that for $k \geq 3$, $EY_n^k(i) = o(i^{k+\alpha-1})$, then the assumptions of the theorem are fulfilled and the conclusion holds. This result appears to be new. The preceding conditions, of course, can be written in terms of the transition probabilities of a Markov chain (X_n).

To illustrate this result, here is an example. For $\alpha < 1$ let $Y_n(i)$ be integer valued and equal to $v^{1/2}i^{(1+\alpha)/2} + ci^{\alpha} + o(i^{\alpha})$ and $-v^{1/2}i^{(1+\alpha)/2} + ci^{\alpha} + o(i^{\alpha})$ with probability $\frac{1}{2}$ for $i \geq 1$ and $P(Y_n(0) = 1) = 1$. One can check that for all $k > 2$ $EY_n^k(i) < Ci^{(1+\alpha)k/2}$, hence $X_n^{1-\alpha}/n$ converges to a gamma distribution. To illustrate the theorem for the case when 0 is absorbing one can use an example given in the next section.

3. *Population size dependent branching processes.* Let (X_n) be defined recursively by

$$X_{n+1} = \sum_{j=1}^{X_n} Y_{n,j}(X_n), \qquad X_0 > 0. \tag{20}$$

Given $X_n = i$, $Y_{n,j}(i)$ are i.i.d. distributed as $Y(i)$ and independent of $(X_j, j < n)$. The sum is taken to be 0 if $X_n = 0$. Here X_n stands for the population size at time n, $Y_{n,j}(X_n)$ is the number of offspring of the jth member of the nth generation when the population size is X_n. Here the data of the process are the offspring distributions $(Y(i))$, ($Y(i)$ is the offspring distribution when population size is equal to i). Hence conditions of the theorem should be formulated in terms of the moments of $(Y(i))$.

Suppose $P(Y(i) = 0) + P(Y(i) = 1) < 1$ for $i > 0$. It was shown in Fujimagari (1976) and Klebaner (1984) that $P(X_n \to 0) + P(X_n \to \infty) = 1$, hence (A2b) holds. Let $m(i) = EY(i)$ and $\sigma^2(i) = \mathrm{Var}(Y(i))$. Then

$$g(i) = i(m(i) - 1), \qquad v(i) = i\sigma^2(i), \qquad R_{n+1} = \sum_{j=1}^{X_n} \left(Y_{n,j}(X_n) - m(X_n)\right).$$

Assume

$$m(i) = 1 + ci^{\alpha-1} + o(i^{\alpha-1}), \qquad \sigma^2(i) \sim vi^{\alpha}. \tag{21}$$

Then (A3) and (A4) hold.

Denote $m_{ki} = E|Y_{n,1}(i) - m(i)|^k$. Assume $m_{ki} < \infty$. Moreover

$$m_{ki} = o(i^{k/2+\alpha-1}) \quad \text{for } k \geq 3.$$

Applying the Marcinkiewicz–Zygmund inequality, we have

$$E\left(|R_{n+1}|^k | X_n = i\right) = E\left|\sum_{j=1}^{i} Y_{n,j}(i) - m(i)\right|^k < Ci^{k/2}m_{ki} = o(i^{k+\alpha-1}).$$

Thus (A5) is satisfied. If we assume $2c > v$, then the assumptions of the theorem are fulfilled and the conclusion holds.

To illustrate the result we provide an example. Let $\alpha < \frac{1}{2}$ and $Y(i)$ take values a_i and 0 with probabilities p_i and $1 - p_i$, respectively, $i \geq 1$. Take $a_i = [vi^{\alpha}]$, where $[a]$ denotes the integer part of a, and $p_i = a_i^{-1} + ca_i^{-1}i^{\alpha-1} + o(a_i^{-1}i^{\alpha-1})$. Then, clearly, (21) holds. For $k > 2$, $EY^k(i) = a_i^k p_i \sim a_i^{k-1} \sim v^{k-1}i^{(k-1)\alpha} = o(i^{k/2+\alpha-1})$. If we let $\alpha = 0$, then we obtain an example similar to the binary splitting case considered in Klebaner (1983).

A gamma limit for the process X_n/n with general offspring distributions satisfying (21) with $\alpha = 0$ was obtained by Klebaner (1984) and Hopfner (1985). The result obtained here is new and is a generalization that covers other rates of convergence of offspring means $m(i)$ to 1.

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