The Annals of Probability
1989, Vol. 17, No. 1, 58–73

LARGE DEVIATIONS FOR l^2-VALUED ORNSTEIN–UHLENBECK PROCESSES

BY I. ISCOE[1] AND D. MCDONALD[2]

Case Western Reserve University and University of Ottawa

A stationary l^2-valued Ornstein–Uhlenbeck process given formally by $dX(t) = -AX(t)\,dt + \sqrt{2a}\,dB(t)$, where A is a positive self-adjoint constant operator on l^2 and $B(t)$ is a cylindrical Brownian motion on l^2, is considered. An upper bound on $P(\sup_{t\in[0,T]}\|X(t)\| > x)$ is established and the asymptotics for the given bound, as $x \to \infty$, is derived.

1. Introduction. Let A be a constant, positive definite, self-adjoint operator on a real, separable Hilbert space $\mathbb{H}$ (in practice l^2). We assume that A has a complete orthonormal family of eigenvectors ϕ_k corresponding to a set of positive eigenvalues λ_k,

$$A\phi_k = \lambda_k\phi_k, \qquad k = 1, 2, \dots .$$

We study the stationary, weakly continuous solution $X(t)$ of the equation

$$dX(t) = -AX(t)\,dt + \sqrt{2a}\,dB(t), \tag{1.0}$$

where $B(t)$ is a cylindrical Brownian motion on $\mathbb{H}$ [see Yor (1974)] and a is a constant, positive operator such that $(\phi_k, \sqrt{a}\,\phi_k) = \sqrt{a_k}$; $(\phi_i, \sqrt{a}\,\phi_j) = 0$, $i \neq j$. Equation (1.0) with $a = I$ has been studied by Dawson (1972) (see Proposition 5). Also consult Kotelenez (1984a), (1984b) and Dawson (1975). It is worth noting that the solution $X(t)$ satisfies (1.0) only in the mild sense [see Dawson (1972)] since $B(t) \notin \mathbb{H}$ and $X(t) \notin \mathscr{D}(A)$.

The principal result in this article is the estimate given in Theorem 1.

THEOREM 1. *Let $T > 0$. Assume that $\sum_{k=1}^{\infty} a_k/\lambda_k < \infty$, $\sum_{k=1}^{\infty} a_k^2/\lambda_k < \infty$ and that σ^2, the maximum ratio among $(a_k/\lambda_k)_{k=1}^{\infty}$, occurs with multiplicity m. Then*

$$P\Big(\sup_{t\in[0,T]} \|X(t)\| > x\Big) \le \left[2eK \sum_{k:\,(a_k/\lambda_k)=\sigma^2} a_k x^m \exp\left(-\frac{x^2}{2\sigma^2}\right)\right]\left[T\left(1 + O(x^{-2})\right) + O(x^{-2})\right],$$

where

$$K = \left[(2\sigma^2)^{m/2+1}\Gamma(m/2+1)\right]^{-1} \prod_{k:\,(a_k/\lambda_k)\neq\sigma^2}^{\infty} \left[1 - a_k/(\lambda_k\sigma^2)\right]^{-1/2},$$

and the two O-terms are with respect to $x \to +\infty$ and are uniform in $T > 0$.

Received February 1987; revised January 1988.

[1]Research supported in part by Natural Sciences and Engineering Research Council of Canada Grant A9444, while visiting McGill University.

[2]Research supported in part by Natural Sciences and Engineering Research Council of Canada Grant A4551.

AMS 1980 *subject classifications*. Primary 60H10, 60G17; secondary 60G15.

Key words and phrases. Ornstein–Uhlenbeck, Hilbert space, large deviations, Dirichlet form.

It should be noted that the exponential rate is sharp since it is well known [see, e.g., Donsker and Varadhan (1976)] that for at least fixed $T > 0$,

$$\lim_{x\to\infty} x^{-2}\log P\Bigl(\sup_{t\in[0,T]}\|X(t)\| > x\Bigr) = -\frac{1}{2\sigma^2}.$$

The asymptotics for the corresponding fixed-time deviation is given in Lemma 4.12 [see (4.14)], where it is shown that

$$P(\|X(t)\| > x) = 2mK\sigma^4 x^{m-2}e^{-x^2/2\sigma^2}\bigl[1 + O(x^{-2})\bigr], \quad \text{as } x \to \infty,$$

where the O-term is independent of t.

Theorem 1 should be compared with the result of Borell (1975) (see Theorem 5.2 therein) which implies that if $X(t)$ is l^2-continuous, then

$$P\Bigl(\sup_{t\in[0,T]}\|X(t)\| > x\Bigr) \le C_T e^{cx}\exp\Bigl(-\frac{x^2}{2\sigma^2}\Bigr),$$

where c is determined by the median of the distribution of $\sup_{t\in[0,T]}\|X(t)\|$. This is obtained by observing that

$$\sup_{t\in[0,T]}\|X(t)\| = \sup_{(\phi,t)}\{\langle\phi, X(t)\rangle: \|\phi\| = 1,\ t\in[0,T]\}.$$

It should also be noted that the probability $P(\sup_{t\in[0,T]}\|X(t)\| > x)$ must depend not only on $E\|X(t)\|^2 = \sum_{k=1}^{\infty} a_k/\lambda_k$, but also on the covariance $E[X(0)X(t)] = \sum_{k=1}^{\infty}(a_k/\lambda_k)\exp(-\lambda_k t)$. This dependence appears in the condition $\sum_{k=1}^{\infty} a_k^2/\lambda_k < \infty$; in case the latter series diverges, the O-terms in the theorem are actually infinite for every positive x and T. It is shown in Iscoe, Marcus, McDonald, Talagrand and Zinn (1988) that if $\sup_k(a_k/\lambda_k)(\log a_k)^r < \infty$ for some $r > 1$, then $X(\cdot)$ is continuous in $\mathbb{H}$ (in fact a sharper result is given). Hence Borell's inequality, though weaker, holds under milder hypotheses than those of the above theorem.

The idea of the proof of the theorem is summarized as follows. We project $X(t)$ onto $\mathbb{R}^n$, equipped with Euclidean norm $|\cdot|$, by considering the "truncated" process $X^n(t) = (X_1(t),\dots, X_n(t))$, where $X_k(t) = \langle\phi_k, X(t)\rangle$. The coordinates $X_k(t)$ are stationary, independent Ornstein–Uhlenbeck processes satisfying

$$dX_k(t) = -\lambda_k X_k(t)\,dt + \sqrt{2a_k}\,dB_k(t),$$

where $X_k(0) \sim N(0, a_k/\lambda_k)$, a centered normal random variable with variance a_k/λ_k, and are independent; and $\{B_k(\cdot)\}_{k=1}^n$ are independent standard Brownian motions. Denote the product-multivariate normal density of the law of $X^n(0)$ by w_n,

$$w_n(y) = \prod_{k=1}^{n}\bigl[\lambda_k/(2\pi a_k)\bigr]^{1/2}\exp\{-[\lambda_k/(2a_k)]\,y_k^2\}, \qquad y\in\mathbb{R}^n, \tag{1.1}$$

and define

$$\tau = \inf\{t \ge 0: |X^n(t)| > x\}, \qquad \theta = T^{-1}. \tag{1.2}$$

Then

$$
\begin{aligned}
P\Big(\sup_{t\in[0,T]}|X^n(t)| > x\Big) &= P(\tau < T) = P(\theta\tau < 1)\\
&\le eEe^{-\theta\tau}\\
&= e\int_{\mathbb{R}^n} w_n(y)E_y[e^{-\theta\tau}]\,dy\\
&\equiv e\int_{\mathbb{R}^n} w_n(y)u(y)\,dy,
\end{aligned}
$$

where

(1.3) $$u(y) = E_y[e^{-\theta\tau}].$$

In Section 2 we note that u satisfies the boundary value problem

$$
\begin{cases}
L(u) \equiv \sum_{k=1}^{n} a_k \dfrac{\partial^2 u}{\partial y_k^2} - \sum_{k=1}^{n} \lambda_k y_k \dfrac{\partial u}{\partial y_k} = \theta u, & |y| < x,\\
u(y) = 1, & |y| \ge x,
\end{cases}
$$

where L is the infinitesimal generator associated with the Markov process X^n. If we define a bilinear form on the Sobolev space $H^1(B_x)$ ($B_x \equiv \{y \in \mathbb{R}^n\colon |y| \le x\}$) by

$$a(f,g) = \theta\int_{B_x} f(y)g(y)w_n(y)\,dy + \int_{B_x}\sum_{k=1}^{n} a_k \frac{\partial f}{\partial y_k}\frac{\partial g}{\partial y_k} w_n(y)\,dy,$$

then by Lemma 2.4, u minimizes $a(v,v)$ among $\{v \in H^1(B_x) | v(y) = 1$ for $y \in \partial B_x\}$. Also by Lemma 2.6, $\theta\int_{B_x} w_n(y)u(y)\,dy = a(u,u)$. Hence

(1.4)
$$
\begin{aligned}
P\Big(\sup_{t\in[0,T]}|X^n(t)| > x\Big) &\le e\left[\int_{B_x^c} w_n(y)\,dy + Ta(u,u)\right]\\
&\le e\left[\int_{B_x^c} w_n(y)\,dy + Ta(v,v)\right]
\end{aligned}
$$

for any $v \in H^1(B_x)$ such that $v = 1$ on ∂B_x.

The next step is to guess a suitable function v. This is done in Section 2. The asymptotic properties of the resulting estimate (1.4) as $n \to \infty$ and then $x \to \infty$ are given in Sections 3 and 4, respectively.

2. The Ansatz. In this section we study and characterize the function u introduced at (1.3) as the solution to a variational problem, and then estimate $\int_{\{|y| \le x\}} w_n(y)u(y)\,dy$, with w_n as in (1.1). Also we denote

$$B_x := \{y \in \mathbb{R}^n\colon |y| \le x\}, \qquad B_x^0 := \{y \in \mathbb{R}^n\colon |y| < x\},$$
$$\partial B_x \equiv S_x := \{y \in \mathbb{R}^n\colon |y| = x\};$$

$H^1(B_x)$, $H_0^1(B_x)$ denote the usual Sobolev spaces and $C^{2,\alpha}(B_x)$ the usual Hölder spaces [see, e.g., Gilbarg and Trudinger (1983)].

DEFINITION 2.1. For $v \in C^{2,\alpha}(B_x)$,

$$(2.1)_1 \quad (Lv)(y) := \sum_{k=1}^{n} \left[a_k\left(\partial^2 v(y)/\partial y_k^2\right) - \lambda_k y_k\left(\partial v(y)/\partial y_k\right)\right], \qquad y \in B_x^0.$$

For $v_1, v_2 \in H^1(B_x)$,

$$(2.1)_2 \quad \begin{aligned} a(v_1, v_2) := \theta \int_{B_x} v_1(y) v_2(y) w_n(y)\, dy \\ + \int_{B_x} \sum_{k=1}^{n} a_k w_n(y) \frac{\partial v_1(y)}{\partial y_k} \frac{\partial v_2(y)}{\partial y_k}\, dy. \end{aligned}$$

LEMMA 2.2. *Let* $u(y) := E_y e^{-\theta\tau}$, *where* τ *is defined by* (1.2). *Then* $u \in C^{2,\alpha}(B_x)$ *for any* $\alpha \in (0,1)$ *and is the unique solution to*

$$(2.3) \quad \begin{cases} Lu - \theta u = 0, & \text{in } B_x^0, \\ u = 1, & \text{on } S_x. \end{cases}$$

PROOF. The existence of a unique smooth solution to (2.3) is given for example in Corollary 6.9 of Gilbarg and Trudinger (1983). That $u(y) \equiv E_y e^{-\theta\tau}$ solves (2.3) is well known [see Dynkin (1965), Theorem 13.16]. □

LEMMA 2.4. *Let* $u(y) = E_y[e^{-\theta\tau}]$. *Then*

$$a(u, u) = \min\left\{a(v, v) \colon v \in H^1(B_x),\ v - 1 \in H_0^1(B_x)\right\}.$$

PROOF. Let $\mathbb{K} = \{v \in H^1(B_x)\colon v - 1 \in H_0^1(B_x)\}$. Then $\mathbb{K}$ is a closed convex subset of $H^1(B_x)$. Now $a(\cdot,\cdot)$, defined in $(2.1)_2$, is clearly a continuous bilinear form on $H^1(B_x)$ since w is bounded; $a(\cdot,\cdot)$ is also coercive, i.e., $a(v, v) \geq C\|v\|_{H^1}^2$ for some constant $C > 0$ and all $v \in H^1(B_x)$, since w_n is bounded below away from zero on B_x and the a_k's are positive. In other words, $a(\cdot,\cdot)$ serves as an inner product on $H^1(B_x)$ (equivalent to the usual one). By the well-known projection theorem [see, e.g., Theorem 1 of Section 4 of Chapter 1 of Aubin (1979) with x there taken as 0], $\mathbb{K}$ contains a unique element, say $\bar{u}$, of minimal $a(\cdot,\cdot)$-norm:

$$a(\bar{u}, \bar{u}) = \min\{a(v, v) \colon v \in \mathbb{K}\};$$

$\bar{u}$ is characterized by the condition $a(\bar{u}, v - \bar{u}) \geq 0$ for all $v \in \mathbb{K}$. Multiply the equation $Lu - \theta u = 0$ by $w_n(v - u)$. [note: $v - u = (v - 1) - (u - 1) \in H_0^1(B_x)$.] Integrating over B_x and then integrating by parts, using the boundary condition, yields

$$(2.5) \quad a(u, v - u) = 0 \quad \text{for } v \in \mathbb{K}.$$

Since $u \in \mathbb{K}$, $\bar{u} = u$ and we are done. □

LEMMA 2.6. $\theta\int_{B_x} u(y) w_n(y)\, dy = a(u, u)$.

PROOF. Simply observe that $v \equiv 1 \in \mathbb{K}$, as defined in the proof of Lemma 2.4. Hence by setting $v \equiv 1$ in (2.5), we obtain the result. □

We conclude this section by (over)estimating $a(u, u)$ by a suitable choice of candidate v as in Lemma 2.4. Guided by a generalization of some calculations in Newell (1962), we choose v to be the function

$$(2.7)\qquad \begin{aligned} v_n(y) &= \nu_n(x)^{-1} \int_0^{|y|} [f_n(r)]^{-1} M_n(r)\, dr, \qquad 0 \le |y| \le x \\ &\equiv \nu_n(|y|)/\nu_n(x), \end{aligned}$$

where

$$(2.8)\qquad M_n(r) = \int_{B_r} w_n(y)\, dy,$$

$$(2.9)\qquad f_n(r) = \int_{S_r} w_n(y) \sum_{k=1}^{n} a_k y_k^2 |y|^{-2}\, d\sigma_r(y),$$

("$d\sigma_r$" is the usual surface measure)

$$(2.10)\qquad \nu_n(r) = \int_0^r [f_n(s)]^{-1} M_n(s)\, ds, \qquad 0 \le r \le x;$$

for which

$$\begin{aligned} a(v_n, v_n) &= \theta \int_{B_x} v_n(y)^2 w_n(y)\, dy \\ &\quad + \int_0^x \int_{S_r} \sum_k a_k w_n(y) \nu_n(x)^{-2} \left[f_n(|y|)^{-1} y_k/|y| \right]^2 M_n(r)^2\, d\sigma_r(y)\, dr \\ &= \theta \int_{B_x} v_n(y)^2 w_n(y)\, dy + \nu_n(x)^{-2} \int_0^x f_n(r)^{-2} f_n(r) M_n(r)^2\, dr \\ (2.11)\qquad &\le \theta \int_{B_x} v_n(y)^2 w_n(y)\, dy + \nu_n(x)^{-1}. \end{aligned}$$

Then by Lemma 2.4 and the estimate (2.11),

$$(2.12)\qquad a(u, u) \le \theta \int_{B_x} v_n(y)^2 w_n(y)\, dy + \nu_n(x)^{-1}. \qquad \square$$

Combining (1.4) and (2.12) we obtain the following estimate.

LEMMA 2.13.

$$P\Big(\sup_{t \in [0,T]} |X^n(t)| > x \Big) \le e \left\{ \int_{B_x^c} w_n(y)\, dy + \int_{B_x} v_n(y)^2 w_n(y)\, dy + T \nu_n(x)^{-1} \right\}.$$

REMARK 2.14. With a more clever choice of trial function v_n, it may be possible to relax the hypotheses of the theorem (when $\{a_k\}_{k=1}^{\infty}$ is unbounded) through an improvement of the estimate at (1.4) and (2.12). It is precisely the nature of the term $\nu_n(x)$ [see (2.11) and (2.10)] which leads to the hypothesis: $\sum_{k=1}^{\infty} a_k^2/\lambda_k < +\infty$ [see (3.3)].

REMARK 2.15. On the other hand we see below that v_n is the best choice among radial functions,

$$(2.16)\quad \begin{aligned} a(u,u) &= \min\{a(v,v) : v \in H^1(B_x),\ v - 1 \in H_0^1(B_x)\} \\ &\le \min\{a(v^* \circ F, v^* \circ F) : v^* \in H^1((0,x)),\ v^*(x) = 1\}, \end{aligned}$$

where $F(y) = |y|$. By direct calculation

$$\begin{aligned} a^*(v^*, v^*) &:= a(v^* \circ F, v^* \circ F) \\ &= \int_{B_x} \sum_{k=1}^{n} a_k \frac{y_k^2}{|y|^2} \left(\frac{dv^*}{dr} \circ F(y) \right)^2 w_n(y)\, dy \\ &= \int_0^x f_n(r) \left(\frac{dv^*}{dr} \right)^2 dr, \quad \text{using (2.9)}. \end{aligned}$$

Defining $a_\theta{}^*(v^*, v^*) := a^*(v^*, v^*) + \theta \int_0^x (v^*(r))^2 w^*(r)\, dr$, where $w^*(r) = d/dr \int_{B_r} w_n(y)\, dy$, we see by (2.16) that to obtain an upper bound for $a(u,u)$ it suffices to consider

$$(2.17)\quad C(x) := \min\{a_\theta^*(v^*, v^*) : v^* \in H^1((0,x)),\ v^*(x) = 1\}.$$

This process of inducing a form like a_θ^* on a simpler (lower dimensional) space is studied in Iscoe and McDonald (1987) in the context of Dirichlet spaces. In a one-dimensional setting the asymptotics for $C(x)$ are shown to satisfy

$$\lim_{x \to \infty} \nu_n(x) C(x) = 1;$$

so the function $v_n(r; x) := \nu(r)/\nu(x)$ asymptotically achieves the minimum in (2.17). Consequently $v_n(y) = v_n(|y|; x)$ is the best choice among radial functions for minimizing (2.16). For more general F, or other reversible Hunt processes on $\mathbb{R}^n$, the induced one-dimensional form provides a prescription for writing down an Ansatz.

Independently, Ichihara (1978) used a similar Ansatz to give conditions for recurrence of symmetric, n-dimensional diffusions.

3. An estimate for fixed x: $n \to \infty$. Throughout this and the next section, we denote the Fourier transform of an L^1-function $\psi: \mathbb{R}_+ \to \mathbb{R}$ by $\hat{\psi}$ with the convention

$$\hat{\psi}(t) = \int_0^\infty e^{itr} \psi(r)\, dr.$$

LEMMA 3.1. *If $\sum_{k=1}^{\infty} a_k/\lambda_k < \infty$ and $\sum_{k=1}^{\infty} a_k^2/\lambda_k < \infty$, then with f_n given by* (2.9), *$f := \lim_{n\to\infty} f_n$ exists (pointwise) and is continuous; and $f(r) = 2r^{-1} g(r^2)$,* *where $g(r)$ is determined by its Fourier transform,*

$$\hat{g}(t) = \sum_{k=1}^{\infty} \frac{a_k^2}{\lambda_k}\left(1 - 2i\left(\frac{a_k}{\lambda_k}\right)t\right)^{-1} \prod_{k=1}^{\infty}\left(1 - 2i\left(\frac{a_k}{\lambda_k}\right)t\right)^{-1/2}. \tag{3.2}$$

(Here the square root is defined with a branch cut along the negative real axis.)

PROOF. Let $g_n(r) = \sqrt{r}/2 f_n(\sqrt{r})$. Then with σ_r denoting the usual surface measure on S_r, the sphere of radius r centered at $0 \in \mathbb{R}^n$,

$$\begin{aligned}
\hat{g}_n(t) &= \int_0^\infty e^{itr} g_n(r)\, dr \\
&= \int_0^\infty e^{itr^2} \int_{S_r} w_n(y) \sum_{k=1}^{n} a_k y_k^2 d\sigma_r(y)\, dr \quad \text{(by a change of variables)} \\
&= E\left\{\exp\left(it\sum_{j=1}^{n} Y_j^2\right)\left[\sum_{k=1}^{n} a_k Y_k^2\right]\right\} \quad \left(\text{where } Y_k \sim N\left(0, \frac{a_k}{\lambda_k}\right) \text{ are independent}\right) \\
&= \sum_{k=1}^{n} a_k E\left[Y_k^2 \exp(itY_k^2)\right] \prod_{j\neq k} Ee^{itY_j^2} \\
&= (-i)\sum_{k=1}^{n} a_k\left[\frac{d}{dt}E\left(\exp(itY_k^2)\right)\right]\prod_{j\neq k} Ee^{itY_j^2} \\
&= (-i)\sum_{k=1}^{n} a_k\left[\frac{d}{dt}\left(1 - 2i\frac{a_k}{\lambda_k}t\right)^{-1/2}\right]\prod_{j\neq k}\left(1 - 2i\frac{a_k}{\lambda_k}t\right)^{-1/2}
\end{aligned}$$

$$= \left[\sum_{k=1}^{n} \frac{a_k^2}{\lambda_k}\left(1 - 2i\frac{a_k}{\lambda_k}t\right)^{-1}\right]\prod_{j=1}^{n}\left(1 - 2i\frac{a_k}{\lambda_k}t\right)^{-1/2}. \tag{3.3}$$

Now since $|1 - 2i(a_k/\lambda_k)t| = (1 + 4(a_k^2/\lambda_k^2)t^2)^{1/2}$ the above sum converges pointwise in t as $n \to \infty$, using the hypothesis that $\sum_{k=1}^{\infty} a_k^2/\lambda_k < \infty$. The partial product also converges pointwise in t as $n \to +\infty$ since $\sum_{k=1}^{\infty} a_k/\lambda_k < +\infty$ [see, e.g., Theorem (15.4) of Rudin (1966) with the estimate $|u_j(t)| \equiv |2ia_jt/\lambda_j| \leq 2a_jt/\lambda_j$, in the notation of the cited reference]. Also the right-hand side of (3.3) is dominated in absolute value by the integrable function $(\sum_{k=1}^{\infty} a_k^2/\lambda_k)\prod_{j=1}^{3}(1 + 4a_j^2t^2/\lambda_j^2)^{-1/4}$. By Lebesgue's dominated convergence theorem, $L^1\text{-}\lim_{n\to+\infty}\hat{g}_n$ exists and is given by

$$\left[\sum_{k=1}^{\infty}(a_k^2/\lambda_k)(1 - 2ia_kt/\lambda_k)^{-1}\right]\left[\prod_{j=1}^{\infty}(1 - 2ia_jt/\lambda_j)^{-1/2}\right]. \tag{3.4}$$

By the Fourier inversion formula

$$\begin{aligned}
|g_n(r) - g_m(r)| &= \left|(2\pi)^{-1}\int_{-\infty}^{\infty} e^{-irt}[\hat{g}_n(t) - \hat{g}_m(t)]\, dt\right| \\
&\leq (2\pi)^{-1}\int_{-\infty}^{\infty} |\hat{g}_n(t) - \hat{g}_m(t)|\, dt \qquad (3.5)\\
&\to 0 \quad \text{as } m, n \to +\infty.
\end{aligned}$$

Therefore $g := \lim_{n\to+\infty} g_n$ exists uniformly on $\mathbb{R}^1$ and is bounded [replace g_m in (3.5) by 0 and let $n \to +\infty$]. Moreover L^1-$\lim_{n\to+\infty} g_n = g$. To see this we estimate

$$
\begin{aligned}
r^2 g_n(r) &= (2\pi)^{-1}\int_{-\infty}^{\infty}\left[-\frac{d^2}{dt^2}e^{-irt}\right]\hat{g}_n(t)\,dt \\
&= (2\pi)^{-1}\int_{-\infty}^{\infty} - e^{irt}\frac{d^2}{dt^2}\hat{g}_n(t)\,dt \qquad (3.6)\\
&\le (2\pi)^{-1}\int_{-\infty}^{\infty}\left|\frac{d^2}{dt^2}\hat{g}_n(t)\right|dt.
\end{aligned}
$$

As in the previous paragraph, it can be shown that the right-hand side of (3.6) is bounded with respect to n. This, along with the pointwise convergence and uniform boundedness of the $\{g_n\}_{n\in\mathbb{N}}$, fulfills the hypotheses of the dominated convergence theorem, allowing us to conclude (3.2) from (3.3). □

Define for $r \ge 0$

$$\eta_n(r) := \int_{S_r} w_n(y)\,d\sigma_r(y); \tag{3.7}$$

$$v_n(r;x) := \nu_n(r)/\nu_n(x) \quad [\text{see } (2.10)]. \tag{3.8}$$

Lemma 3.9. *Assuming that* $\sum_{k=1}^{\infty} a_k/\lambda_k < \infty$, *then* $\eta := \lim_{n\to\infty}\eta_n$ *exists pointwise and in* $L^1(\mathbb{R}_+; dr)$; η *is continuous and is characterized by*

$$\eta(r) = 2rh(r^2), \tag{3.10}$$

where

$$\hat{h}(t) = \prod_{k=1}^{\infty}(1 - 2i(a_k/\lambda_k)t)^{-1/2}, \tag{3.11}$$

Proof. Set $h_n(r) = (2\sqrt{r})^{-1}\eta_n(\sqrt{r})$ and calculate

$$
\begin{aligned}
\hat{h}_n(t) &= \int_0^{\infty} e^{itr}h_n(r)\,dr = \int_0^{\infty} e^{itr^2}\eta_n(r)\,dr \\
&= E\left[\exp\left(t\sum_{k=1}^{n} Y_k^2\right)\right] \quad (\text{where } Y_k \sim N(0, a_k/\lambda_k) \text{ are independent}) \\
&= \prod_{k=1}^{n}(1 - 2i(a_k/\lambda_k)t)^{-1/2}.
\end{aligned}
$$

By the same argument as in Lemma 3.1 we conclude that

$$\lim_{n\to\infty}\hat{h}_n(t) = \hat{h}(t) = \prod_{k=1}^{\infty}(1 - 2i(a_k/\lambda_k)t)^{-1/2},$$

the limit being uniform and in L^1. Then as in (3.5) and (3.6) we obtain the pointwise and L^1-convergence, as $n \to \infty$, of h_n and η_n to h and η, respectively, yielding (3.10) and (3.11). □

LEMMA 3.12. *If* $\sum_{k=1}^{\infty} a_k/\lambda_k < \infty$ *and* $\sum_{k=1}^{\infty} a_k^2/\lambda_k < \infty$, *then*

(i) $M := \lim_{n\to\infty} M_n$ *exists and is differentiable,* $M(r) = \int_0^r \eta(s)\,ds$ *and* $M(\infty) = 1$;

(ii) $M_n(r)/f_n(r)$ *is uniformly bounded in* n *and* $r \in [0, R]$, *for each* R;

(iii) $\nu = \lim_{n\to\infty} \nu_n$ *exists and is differentiable,* $\nu(r) = \int_0^r [M(s)/f(s)]\,ds$;

(iv) *Define* $v(r; x) := \nu(r)/\nu(x)$, *then* $v(r; x) = \lim_{n\to\infty} v_n(r; x)$ *and is differentiable in* r.

PROOF. (i) $M_n(r) = \int_0^r \eta_n(s)\,ds \to \int_0^r \eta(s)\,ds$, as $n \to \infty$, by Lemma 3.9; the L^1-convergence of η_n to η, as $n \to \infty$, also yields the result $M(\infty) = 1$.

(ii) Denote the factorization of $g_n(t)$ in (3.3) into a series and a product by $\hat{g}_n(t) = \hat{\varphi}_n(t)\hat{h}_n(t)$ [consistent with (3.11)]. Then

$$\begin{aligned} M_n(r)/f_n(r) &= M_n(r)r/[2g_n(r^2)] \\ &= r\int_0^r s h_n(s^2)\,ds \Big/ \int_0^{r^2} \varphi_n(r^2 - s)h_n(s)\,ds \\ &= r\int_0^r s h_n(s^2)\,ds \Big/ \int_0^r 2s\varphi_n(r^2 - s^2)h_n(s^2)\,ds \\ &\le r\int_0^r s h_n(s^2)ds \Big/ \int_0^r 2s\varphi_1(r^2)h_n(s^2)\,ds \\ &\le r/[2\varphi_1(r^2)], \end{aligned}$$

since $\varphi_n \ge \varphi_1$ and φ_1 is a multiple of an exponential density; consequently $\varphi_1(0) > \varphi_1(r) > \varphi_1(R) > 0$ for $r \in (0, R]$.

(iii) $\nu_n(r) = \int_0^r [M_n(s)/f_n(s)]\,ds \to \int_0^r [M(s)/f(s)]\,ds$, as $n \to \infty$, by (i), (ii), Lemma 3.1 and the bounded convergence theorem. Also the integrand in ν is continuous.

(iv) This is an immediate consequence of (iii). □

COROLLARY 3.13.

$$P\Big(\sup_{t\in[0,T]} \|X(t)\| > x\Big) \le e\Big\{T\nu(x)^{-1} + \int_0^x v(r; x)^2\eta(r)\,dr + \int_x^\infty \eta(r)\,dr\Big\}. \tag{3.14}$$

PROOF. By (3.7), (3.8) and Lemma (2.13),

$$P\Big(\sup_{t\in[0,T]} \|X^n(t)\| > x\Big) \le e\Big\{T\nu_n(x)^{-1} + \int_0^x v_n(r; x)^2\eta_n(r)\,dr + \int_0^x \eta_n(r)\,dr\Big\}.$$

Then (3.14) results upon letting $n \to \infty$, using Lemma (3.12)(iii), (iv), the boundedness of v and the $L^1(\mathbb{R}_+; dr)$-convergence of η_n to η as $n \to \infty$ (Lemma 3.9). □

The asymptotic behaviour in $x \to \infty$, of each of the three terms on the right-hand side of (3.14) will be calculated in Section 4.

PROOF OF THEOREM 1. Combining (3.14) with the results of Lemmas 4.12, 4.16 and 4.23, we obtain (recalling the notational simplification $\max_{k>m} a_k/\lambda_k < a_m/\lambda_m = \cdots = a_1/\lambda_1 \equiv \sigma^2$):

$$P\Big(\sup_{t\in[0,T]} \|X(t)\| > x\Big) \le e x^m e^{-x^2/(2\sigma^2)}$$
$$\times\left\{T\left(2K\sum_{k=1}^{m} a_k\right)[1 + O(x^{-2})] + O(x^{-2}) + O(x^{-2})\right\}$$
$$\le \left[2eK\left(\sum_{k:\,(a_k/\lambda_k)=\sigma^2} a_k\right)\right] x^m e^{-x^2/(2\sigma^2)}$$
$$\times\left\{T\left[1 + O(x^{-2})\right] + O(x^{-2})\right\},$$

where K is given by (4.3). □

4. Asymptotics as $x \to \infty$. In this section we calculate the asymptotic behaviour of each of the three terms in the expression at (3.14), as $x \to \infty$, in Lemmas 4.12, 4.16 and 4.23. We assume for notational simplicity that

$$\max_{k>m} \frac{a_k}{\lambda_k} < \frac{a_m}{\lambda_m} = \frac{a_{m-1}}{\lambda_{m-1}} = \cdots = \frac{a_1}{\lambda_1} \equiv \sigma^2$$

and we set $z_0 = -1/(2\sigma^2)$. Throughout this section we assume that

$$\sum_{k=1}^{\infty} a_k/\lambda_k < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} a_k^2/\lambda_k < \infty.$$

The main tool which we use to calculate the asymptotics is the following theorem taken from Olver [(1974), Theorem 2 therein]; see also Section 3 of the survey article by Wong (1980).

THEOREM 4.0. *Let $\delta \in C^\infty((0,\infty))$ and $\alpha > 0$ such that*

(i) $\delta(t) \sim \sum_{n=0}^{\infty}\delta_n t^{n+\alpha-1}$, *as $t \to 0^+$;*
(ii) *the asymptotic expansion in* (i) *is infinitely differentiable;*
(iii) *each of the integrals $\int e^{irt}\delta^{(n)}(t)\,dt$, $n = 0,1,2,\dots$, converges uniformly (as an improper integral) at ∞, and at 0 if $n = 0$, for sufficiently large $r > 0$. Then*

$$\int_0^\infty e^{irt}\delta(t)\,dt \sim \sum_{n=0}^{\infty}\left[\delta_n \exp\{(n+\alpha)\pi i/2\}\right]\Gamma(n+\alpha) r^{-(n+\alpha)}, \quad \text{as } r \to \infty.$$

Lemmas 4.12, 4.16 and 4.23 derive from the following key lemma.

LEMMA 4.1. *Let* $\varphi\colon \mathbb{R}_+ \to \mathbb{R}$ *have the Fourier transform*

$$\hat{\varphi}(t) \equiv \int_0^\infty e^{itr}\varphi(r)\,dr = \sum_{k=1}^\infty \frac{b_k}{\lambda_k}(1 - 2i(a_k/\lambda_k)t)^{-1} \prod_{k=1}^\infty (1 - 2i(a_k/\lambda_k)t)^{-1/2},$$

where $a_k, b_k, \lambda_k > 0$ *and* $\Sigma_{k=1}^\infty a_k/\lambda_k < \infty$, $\Sigma_{k=1}^\infty b_k/\lambda_k < \infty$. *Then*

$$\varphi(r) = K\left(\sum_{k=1}^m \frac{b_k}{\lambda_k}\right) r^{m/2}\exp(z_0 r) + \Delta(r)\exp(z_0 r), \tag{4.2}$$

where

$$K = \left[(2\sigma^2)^{m/2+1}\Gamma\left(\frac{m}{2}+1\right)\right]^{-1} \prod_{k>m}\left(1 - \frac{a_k}{\lambda_k}\frac{\lambda_1}{a_1}\right)^{-1/2} \tag{4.3}$$

and as $r \to \infty$,

$$\Delta(r) = O(r^{m/2-1}), \tag{4.4}$$

$$\dot{\Delta}(r) = O(r^{m/2-2}) \tag{4.4$'$}$$

[*in* (4.4)$'$: $\dot{\Delta}(r) \equiv d\Delta(r)/dr$].

PROOF. Set $C = \Sigma_{k=1}^m b_k/\lambda_k$ and note that $z - z_0 = (-z_0)(1 + 2(a_k/\lambda_k)z)$, $k = 1, 2, \ldots, m$. By the Mellin inversion formula,

$$\varphi(r) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{rz}\Phi(z)\,dz,$$

where

$$\Phi(z) := \int_0^\infty e^{-zr}\varphi(r)\,dr = \sum_{k=1}^\infty \frac{b_k}{\lambda_k}\left(1 + 2\frac{a_k}{\lambda_k}z\right)^{-1} \prod_{k=1}^\infty \left(1 + 2\frac{a_k}{\lambda_k}z\right)^{-1/2}, \quad z \in C \setminus (-\infty, z_0], \tag{4.5}$$

the principal branch of the square root being understood at (4.5). Write

$$\Phi(z) = (z - z_0)^{-m/2-1}A_1(z) + (z - z_0)^{-m/2}A_2(z),$$

where

$$\begin{aligned} A_1(z) &= C(-z_0)^{m/2+1}\prod_{k>m}\left(1 + 2\frac{a_k}{\lambda_k}z\right)^{-1/2}, \\ A_2(z) &= (-z_0)^{m/2}\sum_{k>m}\frac{b_k}{\lambda_k}\left(1 + 2\frac{a_k}{\lambda_k}z\right)^{-1}\prod_{k>m}\left(1 + 2\frac{a_k}{\lambda_k}z\right)^{-1/2} \end{aligned} \tag{4.6}$$

and note that A_1 and A_2 are analytic in an open neighbourhood of the strip $\mathscr{S} = \{z \in C\colon z_0 \le \mathrm{Re}(z) \le 0\}$. Therefore, by expanding A_1 and A_2 into the beginning of their Taylor expansions about z_0 plus analytic remainders, we obtain

$$\Phi(z) = \Psi(z) + \Phi_1(z),$$

where

$$\Psi(z) = \sum_{n=0}^{l} c_n(z - z_0)^{-m/2-1+n}, \qquad l = \text{integer part of } m/2, \tag{4.7}$$

and

$$\Phi_1(z) = (z - z_0)^{-p}A(z), \qquad p = \begin{cases} \frac{1}{2}, & m \text{ odd}, \\ 0, & m \text{ even}. \end{cases} \tag{4.8}$$

In (4.8), A is analytic in an open neighbourhood of $\mathscr{S}$ for the appropriate choice of $(c_n)_{n=0}^{l}$ in (4.7); in particular $c_0 = A_1(z_0) = \Gamma(m/2 + 1)KC$. Inverting the transform Ψ by inspection yields

$$\begin{aligned} \varphi(r) &= \sum_{n=0}^{l} \left[\frac{c_n}{\Gamma(m/2 - n + 1)} \right] r^{m/2-n} e^{z_0 r} + \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{rz}\Phi_1(z)\,dz \\ &\equiv \Psi(r) + \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{rz}\Phi_1(z)\,dz. \end{aligned} \tag{4.9}$$

In case m is odd (or even but $c_l = 0$) the terms in (4.7) are absolutely integrable near $z_0 \pm i\infty$ so we can deform the vertical contour of integration $\{\mathrm{Re}(z) = 0\}$ [at (4.9)] onto $\{\mathrm{Re}(z) = z_0\}$ by Cauchy's theorem [and Lebesgue's dominated convergence theorem for odd m]:

$$\begin{aligned} \varphi(r) &= \psi(r) + \frac{1}{2\pi i} \int_{z_0-i\infty}^{z_0+i\infty} e^{rz}\Phi_1(z)\,dz \\ &= \psi(r) + \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{irt}\Phi_1(z_0 + it)\,dt \right\} e^{z_0 r} \end{aligned} \tag{4.10}$$

$$\equiv \psi(r) + \Delta_1(r)e^{z_0 r}. \tag{4.11}$$

Since $\Phi_1(z_0 + it) \sim \sum_{n=0}^{\infty} \alpha_n^{\pm} t^{n-p}$, as $t \to 0^{\pm}$ for some sequences $(\alpha_n^{\pm})_{n=0}^{\infty}$, then Theorem 4.0 yields with $\alpha = 1 - p$ [after transforming the integral $\int_{-\infty}^{0}$ at (4.9) to one of the form $\int_0^{\infty}$ through the change of variables: $t \mapsto -t$] that $\Delta_1(r) = O(r^{p-1})$ as $r \to \infty$. Also $\dot{\Delta}_1(r) = (2\pi)^{-1}\int_{-\infty}^{\infty} e^{irt} t \Phi_1(z_0 + it)\,dt$ and $t\Phi_1(z_0 + it) \sim \sum_{n=0}^{\infty} \alpha_n^{\pm} t^{n-p+1}$ as $t \to 0^{\pm}$; which implies, again by Theorem 4.0 that $\dot{\Delta}_1(r) = O(r^{p-2})$, as $r \to \infty$.

Set $\Delta(r) = \Delta_1(r) + e^{-z_0 r}(\psi(r) - [c_0/\Gamma(m/2 + 1)]r^{m/2})$. Then we obtain the decomposition at (4.2), such that (4.4) and (4.4)′ hold, using (4.9) and (4.11), the evaluation $c_0 = A_1(z_0) = \Gamma(m/2 + 1)KC$ and the estimates of the previous paragraph

$$\begin{aligned} \Delta(r) &= O(r^{p-1}) + O(r^{m/2-1}) \quad [\text{see } (4.9)] \\ &= O(r^{m/2-1}), \quad \text{as } r \to \infty, \end{aligned}$$

and

$$\begin{aligned} \dot{\Delta}(r) &= O(r^{p-1}) + O(r^{m/2-1}) \quad [\text{see } (4.9)] \\ &= O(r^{m/2-2}), \quad \text{as } r \to \infty. \end{aligned}$$

In case m is even and $c_l \neq 0$, it may not be possible to deform the contour of integration to $\{\mathrm{Re}(z) = z_0\}$ due to the presence of the term $c_l(z - z_0)^{-1}$ in the decomposition (4.7) of Ψ, such term not being absolutely integrable near $z_0 \pm i\infty$. To alleviate this problem we first integrate by parts,

$$\varphi(r) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} [r^{-1}e^{rz}]'\Phi(z)\,dz$$
$$= \frac{-r^{-1}}{2\pi i}\int_{-i\infty}^{i\infty} e^{rz}\Phi'(z)\,dz.$$

We can now repeat the previous analysis with $\Phi' = \Psi' + \Phi_1'$; $\Psi(z)$ does not contain a summand of the form const.$(z - z_0)^{-1}$,

$$\varphi(r) = -r^{-1}\sum_{n=0}^{l}[(-m/2 + n - 1)/\Gamma(m/2 - n + 2)]\,c_n r^{m/2-n+1}e^{z_0 r}$$
$$-r^{-1}e^{z_0 r}/2\pi\int_{-\infty}^{\infty} e^{irt}\Phi_1'(z_0 + it)\,dt$$
$$= \psi(r) + e^{z_0 r}\Delta_2(r).$$

Defining $\Delta(r)$ as before, with $\Delta_2(r)$ replacing $\Delta_1(r)$ we again arrive at the desired result since $\Phi_1(z_0 + it)$ is analytic at $t = 0$ and hence, from Theorem 4.0, $\Delta_2(r) = O(r^{-2})$ and $\dot{\Delta}_2(r) = O(r^{-3})$, as $r \to \infty$. The remainder of the proof in this case is the same as that of the previous case. □

We can now proceed to the analysis of each of the terms in the expression (3.14). We begin with the term $\int_x^\infty \eta(r)\,dr$. Note that it is the probability $P(\|X(t)\| > x)$ at a fixed instant $t \geq 0$. For this reason we state the asymptotics in a somewhat finer form than is strictly necessary for Theorem 1 for the sake of comparison.

LEMMA 4.12. *With the constant K given by* (4.3),

(4.13) $$\eta(r) = 2m\sigma^2 K r^{m-1}e^{z_0 r^2}[1 + \Delta_0(r)],$$

where $\Delta_0(r) = O(r^{-2})$ as $r \to \infty$ and

(4.14) $$\int_x^\infty \eta(r)\,dr = 2mK\sigma^4 x^{m-2}e^{-x^2/(2\sigma^2)}[1 + O(x^{-2})], \quad as\ x \to \infty.$$

PROOF. In the notation of Lemma 3.9 [see (3.10)], $\int_x^\infty \eta(r)\,dr = \int_x^\infty 2rh(r^2)\,dr$. Now by (3.11),

$$[rh(r)]\hat{\ }(t) = -i(\hat{h})'(t) = \sum_{k=1}^{\infty}\frac{a_k}{\lambda_k}\left(1 - 2i\left(\frac{a_k}{\lambda_k}\right)t\right)^{-1}\prod_{k=1}^{\infty}\left(1 - 2i\left(\frac{a_k}{\lambda_k}\right)t\right)^{-1/2}$$

By Lemma (4.1), with $b_k = a_k$ and $\varphi(r) = rh(r)$, we have $(\sum_{k=1}^m a_k/\lambda_k = m\sigma^2)$

(4.15) $$rh(r) = m\sigma^2 K r^{m/2}e^{z_0 r} + \Delta(r)e^{z_0 r},$$

where $\Delta(r)$ satisfies (4.4). Since $\eta(r) = 2rh(r^2)$, (4.13) is an immediate consequence of (4.15), with $\Delta_0(r) := (m\sigma^2 K)^{-1} r^{-m} \Delta(r^2)$.

Integrating by parts yields $[z_0 = -1/(2\sigma^2)]$

$$\begin{aligned}\int_x^\infty \eta(r)\,dr &= 2m\sigma^4 K x^{m-2} e^{z_0 x^2} + 2mK\sigma^4 \int_x^\infty e^{z_0 r^2}(r^{m-2})'\,dr \\ &\quad + 2m\sigma^4 K \int_x^\infty r^{m-1} e^{z_0 r^2} \Delta_0(r)\,dr \\ &= 2m\sigma^4 K x^{m-2} e^{z_0 x^2}[1 + \Delta_1(x) + \Delta_2(x)],\end{aligned}$$

where

$$\begin{aligned}\Delta_1(x) &= x^{2-m} e^{-z_0 x^2} \int_x^\infty e^{z_0 r^2}(r^{m-2})'\,dr, \\ \Delta_2(x) &= \sigma^{-2} x^{2-m} e^{-z_0 x^2} \int_x^\infty r^{m-1} e^{z_0 r^2} \Delta_0(r)\,dr.\end{aligned}$$

By l'Hôpital's rule $\Delta_1(x) = O(x^{-2})$ and $\Delta_2(x) = O(x^{-2})$, as $x \to \infty$. □

Next we consider the dominant term, $\nu(x)^{-1}$.

LEMMA 4.16. *With the constant K defined by* (4.3),

$$\nu(x)^{-1} = \left(2K \sum_{k=1}^m a_k\right) x^m e^{-x^2/(2\sigma^2)}[1 + O(x^{-2})], \quad \text{as } x \to \infty. \tag{4.17}$$

PROOF. In the notation of Lemma 3.1, $\nu(r) = \int_0^r M(s)/f(s)\,ds$, where $M(r) = \int_0^r \eta(s)\,ds$ and $f(r) = [2/r]g(r^2)$, where

$$\hat{g}(t) = \sum_{k=1}^\infty (a_k^2/\lambda_k)(1 - 2i(a_k/\lambda_k)t)^{-1} \prod_{k=1}^\infty (1 - 2i(a_k/\lambda_k)t)^{-1/2}.$$

By Lemma 4.1, with $b_k = a_k^2$ and $\varphi(r) = g(r)$, we have that

$$g(r) = \left(\sum_{k=1}^m a_k^2/\lambda_k\right) K r^{m/2} e^{z_0 r} + \Delta(r) e^{z_0 r}, \tag{4.18}$$

where, by (4.4), $\Delta(r) = O(r^{m/2-1})$, as $r \to \infty$. Note that $\sum_{k=1}^m a_k^2/\lambda_k = \sigma^2 \sum_{k=1}^m a_k$ since $a_k/\lambda_k = \sigma^2$ for $1 \le k \le m$; we set $C := \sigma^2 \sum_{k=1}^m a_k K$ for brevity. From (4.18) we derive

$$\begin{aligned}1/f(s) &= [s/2]/g(s^2) \\ &= [s/2] C^{-1} s^{-m} e^{-z_0 s^2}/[1 + O(s^{-2})], \quad \text{as } s \to \infty \\ &= [C^{-1}/2] s^{-m+1} e^{-z_0 s^2}[1 + O(s^{-2})], \quad \text{as } s \to \infty.\end{aligned} \tag{4.19}$$

Also, from (4.14) of Lemma 4.12 we derive

$$\begin{aligned}M(r) &= \int_0^r \eta(s)\,ds \\ &= 1 - \int_r^\infty \eta(s)\,ds \\ &= 1 - e^{z_0 r^2} O(r^{m-2}), \quad \text{as } r \to \infty.\end{aligned} \tag{4.20}$$

Combining (4.19) and (4.20) we see that

$$M(s)/f(s) = [C^{-1}/2] s^{-m+1} e^{-z_0 s^2}[1 + O(s^{-2})], \quad \text{as } s \to \infty.$$

Therefore

$$\begin{aligned} \nu(r) &= \int_0^r [M(s)/f(s)]\, ds \\ (4.21) \qquad &= [C^{-1}/2] \int_1^r s^{-m+1} e^{-z_0 s^2}\, ds \\ &\quad + [C^{-1}/2] \int_1^r s^{-m-1} e^{-z_0 s^2} O(1)\, ds + \int_0^1 [M(s)/f(s)]\, ds. \end{aligned}$$

Now, integrating by parts,

$$\begin{aligned} \int_1^r s^{-m+1} e^{-z_0 s^2}\, ds &= \int_1^r s^{-m}(-z_0^{-1}/2)(e^{-z_0 s^2})'\, ds \\ (4.22) \qquad &= r^{-m}(-z_0^{-1}/2) e^{-z_0 r^2} - \int_1^r s^{-m-1}(z_0^{-1}/2) e^{-z_0 s^2}\, ds. \end{aligned}$$

By l'Hôpital's rule we see that $\lim_{r\to\infty}[\int_1^r s^{-m-1} e^{-z_0 s^2}\, ds / r^{-m-2} e^{-z_0 r^2}]$ exists and is finite and nonzero. This, along with (4.21) and (4.22), implies that

$$\nu(r) = \int_0^r [M(s)/f(s)]\, ds = [\sigma^2 C^{-1}/2] r^{-m} e^{r^2/(2\sigma^2)}[1 + O(r^{-2})], \quad \text{as } r \to \infty,$$

since $z_0 = -1/(2\sigma^2)$. The result given by (4.17) follows immediately upon inversion, since $C = \sigma^2 \sum_{k=1}^m a_k K$. □

Finally, we analyze the term $\int_0^x v(r; x)^2 \eta(r)\, dr$.

LEMMA 4.23.

$$\int_0^x v(r; x)^2 \eta(r)\, dr = x^m e^{-x^2/(2\sigma^2)} O(x^{-2}), \quad \text{as } x \to \infty.$$

PROOF. Writing $a(r) \sim b(r)$ when $\lim_{r\to\infty} a(r)/b(r) = 1$,

$$\nu(r) \sim C_1 r^{-m} e^{r^2/(2\sigma^2)} \quad \text{and} \quad \eta(r) \sim C_2 r^{m-1} e^{-r^2/(2\sigma^2)}$$

by (4.13) and (4.17) for some positive constants C_1, C_2. Therefore $\nu(r)^2 \eta(r) \sim C_1^2 C_2 r^{-m-1} e^{r^2/(2\sigma^2)}$ and by l'Hôpital's rule $\int_0^x \nu(r)^2 \eta(r)\, dr \sim C_3 x^{-m-2} e^{x^2/(2\sigma^2)}$ for some positive constant C_3. Since $v(r; x) = \nu(r)/\nu(x)$,

$$\begin{aligned} \int_0^x v(r; x)^2 \eta(r)\, dr &\sim C_1^{-2} x^{2m} e^{-x^2/\sigma^2} C_3 x^{-m-2} e^{x^2/(2\sigma^2)} \\ &\sim C_1^{-2} C_3 x^{m-2} e^{-x^2/(2\sigma^2)}, \end{aligned}$$

which is somewhat stronger than the assertion of this lemma. □

Acknowledgments. This research was initiated by a suggestion of Professor M. Métivier while one of the authors (D. M.) visited the Ecole Polytechnique in Paris, France in 1984. We thank Professor S. R. S. Varadhan for suggesting the variational approach taken in this article. We also wish to express our thanks to Professor M. B. Marcus for the references to the Gaussian results stated for the purpose of comparison in the Introduction; to the referee for several valuable suggestions, in particular an interpretation of (2.12) via the induced Dirichlet form, essentially as in (2.15); and finally to Professor R. Wong for the reference to Olver's method used in Section 4.

REFERENCES

AUBIN, J.-P. (1979). *Applied Functional Analysis*. Wiley, New York.

BORELL, C. (1975). The Brunn–Minkowski inequality in Gauss space. *Invent. Math.* **30** 205–216.

DAWSON, D. A. (1972). Stochastic evolution equations. *Math. Biosci.* **15** 287–316.

DAWSON, D. A. (1975). Stochastic evolution equations and related measure processes. *J. Multivariate Anal.* **5** 1–55.

DONSKER, M. D. and VARADHAN, S. R. S. (1976). Asymptotic evaluation of certain Markov process expectations for large time–III. *Comm. Pure Appl. Math.* **29** 389–461.

DYNKIN, E. B. (1965). *Markov Processes* **2**. Springer, Berlin.

GILBARG, D. and TRUDINGER, N. (1983). *Elliptic Partial Differential Equations of Second Order*. Springer, Berlin.

ICHIHARA, K. (1978). Some global properties of symmetric diffusion processes. *Publ. Res. Inst. Math. Sci.* **14** 441–486.

ISCOE, I. and MCDONALD, D. (1987). Induced Dirichlet forms and capacitary inequalities. Unpublished.

ISCOE, I., MARCUS, M. B., MCDONALD, D., TALAGRAND, M. and ZINN, J. (1988). Continuity of l^2-valued Ornstein–Uhlenbeck processes. Unpublished.

KOTELENEZ, P. (1984a). Continuity properties of Hilbert space valued martingales. *Stochastic Process. Appl.* **17** 115–125.

KOTELENEZ, P. (1984b). A submartingale type inequality. *Stochastics* **8** 139–152.

NEWELL, G. F. (1962). Asymptotic extreme value distribution for one-dimensional diffusion processes. *J. Math. Mech.* **2** 481–496.

OLVER, F. W. J. (1974). Error bounds for stationary phase approximations. *SIAM J. Math. Anal.* **5** 19–29.

RUDIN, W. (1966). *Real and Complex Analysis*. McGraw-Hill, New York.

WONG, R. (1980). Error bounds for asymptotic expansions of integrals. *SIAM Rev.* **22** 401–435.

YOR, M. (1974). Existence et unicité de diffusions à valeurs dans un espace de Hilbert. *Ann. Inst. H. Poincaré Sect. B (N. S.)* **10** 55–88.

DEPARTMENT OF MATHEMATICS
AND STATISTICS
CASE WESTERN RESERVE UNIVERSITY
CLEVELAND, OHIO 44106

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF OTTAWA
585 KING EDWARD
OTTAWA, ONTARIO
CANADA K1N 6N5