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LARGE DEVIATIONS FOR SYSTEMS OF NONINTERACTING RECURRENT PARTICLES

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We consider noninteracting systems of infinite particles each of which follows an irreducible, null recurrent Markov process and prove a large deviation principle for the empirical density. The expected occupation time (up to time N) of this Markov process, named as $h(N)$, plays an essential role in our result. We impose on $h(N)$ a regularly varying property as $N \to \infty$, which a large class of transition probabilities does satisfy. Some features of our result are: (a) The large deviation tails decay like $\exp[-Nh^{-1}(N)I(\cdot)]$, more slowly than the known $\exp[-NI(\cdot)]$ type of decay in transient situations. (b) Our rate function $I(\lambda(\cdot))$ equals infinity unless $\lambda(\cdot)$ is an invariant distribution. (c) Our rate function is explicit and is rather insensitive to the underlying Markov process. For instance, if we randomized the time steps of a Markov chain by exponential waiting time of mean 1, the resultant system obeys exactly the same large deviation principle.

Introduction. In a noninteracting infinite particle system (abbreviated as a system) we assume that each single particle independently follows a Markov process. When the Markov process is transient, it is known that the large deviation tails decay exponentially like $\exp[-NI(\cdot)]$ (see, e.g., [1, 4, 6]). When it is recurrent (two examples were studied in [1]), however, the tails have $\exp[-Nh^{-1}(N)I(\cdot)]$ decay with $h(N) \to \infty$ as $N \to \infty$.

Our goal is to establish the scaling constant $h(N)$ and understand the large deviation rates $I(\cdot)$ in terms of the transition probability. We assume each particle follows an irreducible time homogeneous Markov process, which is null recurrent and satisfies a regularly varying property [for the definition, see (RV) in the following paragraph]. Some examples that we have in mind are:

(1) Discrete time chain with countable state space, e.g., random walks on $\mathbb{Z}^d$ ($d = 1, 2$) with mean zero and finite second moments: Let $\pi_{y-x}^{(m)} = \pi_{x,y}^{(m)}$ be the m-step transition probability. Note the regularity varying property (see, e.g., [7])

$$\sum_{m=1}^{N} \pi_{xy}^{(m)} \sim \begin{cases} \text{const.} \log N & \text{if } d = 2, \\ \text{const.}\, N^{1/2} & \text{if } d = 1. \end{cases}$$

(2) Continuous time chain with countable state space, e.g., Poisson random walks which are constructed using exponential waiting times of mean 1

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instead of rigid time steps. Its transition probability and resolvent are

$$p_\pi(t,x,y) = \sum_{m=0}^{\infty} \frac{e^{-t}t^m}{m!}\pi_{xy}^{(m)},$$

$$\int_0^\infty e^{-st}p_\pi(t,x,y)\,dt = \sum_{m=0}^{\infty} (s+1)^{-(m+1)}\pi_{xy}^{(m)}.$$

(3) Continuous time parameter and continuous state space, e.g., recurrent Bessel process which is a diffusion generated by $L \equiv D^2 + (\theta/x)D$, where $-1 < \theta \le 1$. Its transition probability and resolvent are, asymptotically,

$$p(t,x,y) \sim \text{const. } y^\theta t^{-(\theta+1)/2} \quad \text{as } t \to \infty,$$

$$\int_0^\infty e^{-st}p(t,x,y)\,dt \sim \begin{cases} \text{const. } y^\theta s^{(\theta-1)/2} & \text{when } -1 < \theta < 1, \\ \text{const. } y \ln\dfrac{1}{s} & \text{when } \theta = 1. \end{cases}$$

Note that $\alpha(x) = x^\theta$ is an invariant density, i.e., $\int_0^\infty \alpha(x)p(t,x,y)\,dx = \alpha(y)$.

(4) Discrete time and continuous state space, e.g., the examples in (3) at discrete time.

Since any attempt for a unified proof seems to involve obscure notation, we decide to give detailed proofs for the discrete time chain with countable state space and demonstrate, in the end, how the result can be generalized. We now focus on a discrete time chain with countable state space. Let X be the state space and $\pi_{xy}^{(m)}$ be the m-step transition probability of the Markov chain $X(m)$.

We assume a regularity varying property:

(RV) There exist $0 \le \beta < 1$, a slowly varying function $l(N)$ as $N \to \infty$, $x, y \in X$, and $c > 0$ such that

$$\lim_{N\to\infty} \sum_{m=1}^{N} \pi_{xy}^{(m)}/N^\beta l(N) = c.$$

A large class of transition probabilities satisfies this property. Among them are recurrent random walks [see (1)]. Let $\pi_{xy} \equiv \pi_{xy}^{(1)}$ and let $\alpha(x)$ be an invariant distribution, i.e., $\sum_{x\in X}\alpha(x)\pi_{xy} = \alpha(y)$ with $\alpha(x) \ge 0$. Some consequences of our assumptions are: In view of the null recurrence, this invariant distribution $\alpha(x)$ exists uniquely up to multiplicative constants and is an infinite distribution, i.e., $\sum_{x\in X}\alpha(x) = \infty$. Also $\alpha(x)$ is positive for all $x \in X$ due to the irreducibility. Doeblin's ratio theorem now implies that

$$\lim_{N\to\infty} \frac{\sum_1^N \pi_{xy}^{(m)}}{\sum_1^N \pi_{ab}^{(m)}} = \frac{\alpha(y)}{\alpha(b)} \quad \text{for all } a, b, x \text{ and } y \in X.$$

Thus, the condition (RV) is equivalent to:

(RV1) There exist $0 \le \beta < 1$ and slowly varying $l(\cdot)$ at ∞ such that

$$\lim_{N\to\infty} \frac{\sum_1^N \pi_{xy}^{(m)}}{N^\beta l(N)} = \alpha(y) \quad \text{for all } x, y \in X.$$

From Abelian and Karamata's Tauberian theorem, we obtain another equivalent condition:

(RV2) Let $h(t) = t^\beta l(t)\Gamma(\beta + 1)$ with β and $l(\cdot)$ as in (RV1).

$$\lim_{s\to 0} \frac{\sum_{m=0}^\infty e^{-sm}\pi_{xy}^{(m)}}{h(s^{-1})} = \alpha(y) \quad \text{for all } x, y \in X.$$

To state and prove our result we need some notation.

Let P_x (E_x resp.) be the probability distribution (expectation resp.) and let $(\pi u)(x) = \sum_{y\in X}\pi_{xy}u(y)$. For a λ: $X \to [0, \infty)$, let the number of particles at site x be Poisson distributed with mean $\lambda(x)$ and let the distribution be independent for distinct sites. We write $\mu(\lambda)$ for this distribution and $n(\cdot)$: $X \to \{0\} \cup N$ for a configuration. We denote by $\hat{P}_{\mu(\lambda)}$ ($\hat{E}_{\mu(\lambda)}$) the probability distribution (expectation) of a system with initial distribution $\mu(\lambda)$. Also let $n(m, x)$ be the number of particles at site x at time m and write n for $\{n(m, x)$: $m \ge 0$, $x \in X\}$. It is elementary to check that $\hat{P}_{\mu(\lambda)}$ is an ergodic Markov process. As in [4] and [6], we define the empirical density

$$\text{(ED)} \qquad D_{N,n}(x) \equiv N^{-1}\sum_{m=1}^N n(m, x).$$

$D_{N,n}$ is regarded as an element of $\underline{M} = \{\lambda$: $X \to R^+\}$ on which we impose the topology $\underline{\underline{M}}$ induced by projections $\lambda \to \lambda(x)$. This paper defines quantities $h(N)$ in terms of π_{xy} and characterizes $I(\cdot)$ such that $(\hat{P}_{\mu(\alpha)} \circ D_N^{-1}, \underline{\underline{M}})$ obeys the large deviation principle with scaling constants $Nh^{-1}(N)$ and rate function $I(\cdot)$, i.e.,

$$\text{(UB)} \qquad \limsup_{N\to\infty} N^{-1}h(N)\log \hat{P}_{\mu(\alpha)}\{D_{N,n} \in C\} \le -\inf_{\lambda\in C} I(\lambda)$$

for any closed set C,

$$\text{(LB)} \qquad \liminf_{N\to\infty} N^{-1}h(N)\log \hat{P}_{\mu(\alpha)}\{D_{N,n} \in G\} \ge -\inf_{\lambda\in G} I(\lambda)$$

for any open set G and $\{\lambda(\cdot)$: $I(\lambda) \le l\}$ is compact in the $\underline{\underline{M}}$ topology for any $0 \le l < \infty$. The last property will become obvious in view of our explicit formula for rates.

Note that $D_{N,n}(x)$ converges to $\alpha(x)$ for a.e. n wrt $\hat{P}_{\mu(\alpha)}$ and our interest is the rate of convergence. Our scaling constant $h(N)$ is found using a cumulant generating function approach (see, e.g., [5]) in which $h(N)$ is characterized by the existence, for small ε, of

$$\lim_{N\to\infty} h(N)N^{-1}\log \hat{E}_{\mu(\alpha)}\left\{\exp\left[\varepsilon h^{-1}(N)ND_{N,n}(x)\right]\right\}.$$

By an expectation formula for Poisson variables, the last is

$$\lim_{N\to\infty} h(N)N^{-1}\sum_{y\in X}\alpha(y)\left[E_y\left\{\exp\left[\varepsilon h^{-1}(N)ND_{N,n}(x)\right]\right\}-1\right]$$

$$=\lim_{N\to\infty}\left[\varepsilon\alpha(x)+\varepsilon^2h^{-1}(N)N^{-1}\sum\alpha(y)E_y\left\{\frac{1}{2!}\left[ND_{N,n}(x)\right]^2\right\}+o(\varepsilon^2)\right].$$

The second cumulant (order ε^2 part) now pinpoints $h(N)$ as

$$h(N)\sim N^{-1}\sum\alpha(y)E_y\left\{\frac{1}{2!}N^2D_{N,n}^2(x)\right\}\sim N^{-1}\sum_{y\in X}\alpha(y)\sum_{m_1+m_2\le N}\pi_{yx}^{(m_1)}\pi_{xx}^{(m_2)}$$

$$=\alpha(x)N^{-1}\sum_{m_1=1}^{N}\sum_{m_2=1}^{N-m_1}\pi_{xx}^{(m_2)}=\alpha(x)N^{-1}\sum_{m=1}^{N}\sum_{m_2=1}^{m}\pi_{xx}^{(m_2)}\sim\alpha(x)\sum_{m_2=1}^{N}\pi_{xx}^{(m_2)}.$$

Note this heuristic argument argument works as well for the transient cases where $\sum_{m=1}^{\infty}\pi_{xx}^{(m)}<\infty$.

We estimate the cumulant generating function in Theorem 1 and use it to get an upper bound in terms of the Mittag–Leffler distributions in Theorem 2. In Theorem 3 this upper bound is proved to be a lower bound and hence the true rates. The rate functionals are found to be finite only on an extremely thin set (Lemma 2). Theorem 4 is a generalization of Theorems 1–3 to Markov processes of continuous time parameter and/or continuous state space. In Theorem 5 we prove that the large deviation rates remain unchanged when the rigid time steps are replaced by exponential waiting times of mean 1 to form a new Markov chain. As a consequence, a system of random walks in dim 1 or 2 share the same rates with the corresponding Poisson systems of random walks. The latter is studied in [1].

1. Cumulant generating function of $D_{N,n}$ (Theorem 1). We first introduce the Mittag–Leffler generating function f_β and the scaling constant h,

$$f_\beta(c)=\sum_{k=0}^{\infty}\frac{c^k}{\Gamma(k\beta+1)}\quad\text{for } 1>\beta>0 \text{ and } c\in\mathbb{R}.$$

REMARK. $f_{1/2}(c)$ is the generating function of the truncated normal density $\pi^{-1/2}e^{-x^2/4}\,dx$, $x\ge 0$, and, by a simple calculation,

$$f_0(c)=\begin{cases}(1-c)^{-1}, & c<1,\\ +\infty, & c\ge 1,\end{cases}$$

which is associated with the exponential distribution $e^{-x}\,dx$, $x\ge 0$.

To simplify many expressions which we shall use, let

$$u(W,N,x)\equiv E_x\left\{\exp\sum_{1}^{N}W(X(m))\right\}, \tag{5}$$

and note the property of u,

$$u(W, m, x) = \sum_{y \in X} e^{V(y)} \pi_{xy} u(W, m-1, y) \quad \text{and} \quad u(W, 0, x) = 1. \tag{6}$$

The generating function of $D_{N,n}$ now takes a shorter form,

$$\begin{aligned} &\hat{E}_{\mu(\alpha)}\left\{\exp N \sum W(x) D_{N,n}(x)\right\} \\ &= \hat{E}_{\mu(\alpha)}\left\{\prod_{x \in X}\left[E_x\left\{\exp \sum_{m=1}^{N} W(X(m))\right\}\right]^{n(x)}\right\} \\ &= \hat{E}_{\mu(\alpha)}\left\{\prod_{x \in X} u(W, N, x)^{n(x)}\right\} \\ &= \exp\left\{\sum\left[u(W, N, x) - 1\right]\alpha(x)\right\}. \end{aligned} \tag{7}$$

We now state Theorem 1.

THEOREM 1. *If $V(x)$ is supported on a finite set and $\overline{V} = \Sigma \alpha(x) V(x)$, then*
(CG) $\lim_{N \to \infty} N^{-1} h(N) \log \hat{E}_{\mu(\alpha)}\{\exp[N h^{-1}(N) \Sigma_{x \in X} V(x) D_{N,n}(x)]\} =$
$\overline{V} \int_0^1 f_\beta(a^\beta \overline{V})\, da$, *where β is the exponent appearing in $h(t)$.*

PROOF. In view of (7) we shall prove that

$$\lim_{N \to \infty} N^{-1} h(N) \sum_x \left[u(h^{-1}(N) V, N, x) - 1\right] \alpha(x) = \overline{V} \int_0^1 f_\beta(a^\beta \overline{V})\, da. \tag{8}$$

We need a preliminary lemma.

LEMMA 1. *If $\lim_{n \to \infty}(m(N))/N = a > 0$, then*

$$\lim_{N \to \infty} u(h^{-1}(N) V, m(N), x) = f_\beta(a^\beta \overline{V}). \tag{9}$$

If W: $X \to R$ is supported on a finite set F, then

$$\sum \alpha(x)[(\pi u)(W, k, x) - u(W, k, x)] = 0. \tag{10}$$

PROOF OF LEMMA 1. Note that, if $m(N) = N$, (9) should be interpreted as convergence of $h^{-1}(N) \Sigma_1^N V(X(m))$ to Mittag-Leffler's distribution as $N \to \infty$. An application of Karamata's Tauberian theorem, as pointed out in [2], provides a very simple proof of (9) which we now give.

Let H_s be the resolvent

$$(H_s V)(x) = \sum_{m=0}^{\infty} e^{-Sm} \pi_{xy}^{(m)} V(y).$$

and note, from (RV2), that $(H_s V)(x) \sim \overline{V} h(s^{-1})$ as $s \to 0$. Let

$$v_k(N, x) \equiv \frac{1}{k!} E_x\left\{\left[\sum_{m=1}^{N} V(X(m))\right]^k\right\}$$

which satisfies

$$\sum_{\substack{m_1+\cdots+m_k\le N\\ m_i\ge 0}} \pi_{xy_1}^{(m_1)}V(y_1)\cdots \pi_{y_{k-1}y_k}^{(m_k)}V(y_k)$$
$$\ge v_k(N,x)$$
$$\ge \sum_{\substack{m_1+\cdots+m_k\le N\\ m_i\ge 1}} \pi_{xy_1}^{(m_1)}V(y_1)\cdots \pi_{y_{k-1}y_k}^{(m_k)}V(y_k).$$

Applying Laplace's transform we have

$$\sum_{N=0}^{\infty} e^{-sN}v_k(N,x) \sim [(H_sV)\cdots(H_sV)](x) \sim \left(\bar{V}h(s^{-1})\right)^k \quad \text{as } s\to 0,$$

where $[\cdot]$ is the composition of k "H_sV". Karamata's Tauberian theorem now implies that

$$v_k(N,x) \sim \frac{h^k(N)\bar{V}^k}{\Gamma(k\beta+1)} \quad \text{as } N\to\infty.$$

This proves (9) in the case when $m(N)=N$. The full result of (9) follows easily from the fact that

$$h^{-1}(N)V = h^{-1}(m)\left[h(m)h^{-1}(N)V\right] \sim h^{-1}(m)(a^{\beta}V) \quad \text{as } m\to\infty.$$

Because (10) can be restated as

$$\sum_{x\in X}\alpha(x)\pi(u-1)(x) = \sum_{y\in X}\alpha(y)[u(y)-1],$$

which is an interchange of summation formula, we need only show that $\sum\alpha(y)|u(W,k,y)-1|<\infty$, which justifies the interchange and is proved as: Using (5) we have

$$|u(W,k,y)-1| \le e^{k|W|}\sum_{j=1}^{k}\sum_{z\in F}\pi_{y,z}^{(j)},$$

where $|W| = \max_{x\in X}|W(x)|$ and, therefore,

$$\sum_{y}\alpha(y)|u(W,k,y)-1| \le e^{k|W|}\sum_{y}\alpha(y)\sum_{j=1}^{k}\sum_{z\in F}\pi_{y,z}^{(j)}$$
$$\le e^{k|W|}k\sum_{z\in F}\alpha(z) < \infty. \qquad \square$$

We are now ready to prove Theorem 1. First note that, using (6) and (10),

$$\sum_{x}\alpha(x)h(N)u\left(h^{-1}(N)V,m,x\right) - u\left(h^{-1}(N)V,m-1,x\right)\Big]$$
$$= \sum_{x}\alpha(x)h(N)\sum_{y\in F}\left[e^{h^{-1}(N)V(y)}-1\right]\pi_{xy}u\left(h^{-1}(N)V,m-1,y\right)$$
$$= \sum_{y\in F}h(N)\left[e^{h^{-1}(N)V(y)}-1\right]\alpha(y)u\left(h^{-1}(N)V,m-1,y\right).$$

Using (9) and $h(N)[e^{h^{-1}(N)V(y)} - 1] \to V(y)$ and letting $mN^{-1} \to a$ as $N \to \infty$, it follows that

$$\lim_{\substack{N\to\infty \\ m\sim aN}} h(N)\sum_x \alpha(x)\left[u\left(h^{-1}(N)V, m, x\right) - u\left(h^{-1}(N)V, m-1, x\right)\right] \tag{11}$$
$$= \overline{V} f_\beta\left(a^\beta \overline{V}\right).$$

As a final step we have that

$$N^{-1}h(N)\sum_x \alpha(x)\left[u\left(h^{-1}(N)V, N, x\right) - 1\right]$$
$$= N^{-1}\sum_{m=1}^{N}\left\{\sum_x \alpha(x)h(N)\left[u\left(h^{-1}(N)V, m, x\right) - u\left(h^{-1}(N)V, m-1, x\right)\right]\right\},$$

which, by (11), tends to $\overline{V}\int_0^1 f_\beta(a^\beta\overline{V})\,da$ as $N \to \infty$. Theorem 1 is completely proved. □

2. Upper bound (Theorem 2). Let us define I_β: $\underline{M} \to R^+ \cup \{\infty\}$:

$$I_\beta(\lambda) = \sup_{V:\ \text{finite support}}\left[\sum V(x)\lambda(x) - \overline{V}\int_0^1 f_\beta\left(a^\beta\overline{V}\right)da\right]. \tag{12}$$

Some properties of $I_\beta(\cdot)$ are:

LEMMA 2.

$$I_\beta(\lambda) = \infty \quad \textit{if } \lambda(\cdot) \textit{ is not a multiple of } \alpha(\cdot). \tag{13}$$

$$\textit{If } \beta \neq 0, \textit{ then } \quad I_\beta(b\alpha) = \sup_{c\in R}\left\{bc - c\int_0^1 f_\beta\left(a^\beta c\right)da\right\}. \tag{14}$$

$$I_0(b\alpha) = \begin{cases}(\sqrt{b} - 1)^2, & b \geq 0,\\ +\infty, & b < 0.\end{cases} \tag{15}$$

PROOF. If λ is not a multiple of α, then there exist x_1 and x_2 such that $\lambda(x_1)\alpha(x_2) - \lambda(x_2)\alpha(x_1) > 0$. We denote by $\{V_k\}$ a sequence of functions defined by

$$V_k(x) = \begin{cases} V(x_1) + k\alpha(x_2), & \text{for } x = x_1,\\ V(x_2) - k\alpha(x_1), & \text{for } x = x_2, \text{ for a fixed } V,\\ V(x), & \text{otherwise.}\end{cases}$$

Noting that $\overline{V}_k = V$ and that

$$\sum_{x\in X}\lambda(x)V_k(x) - \overline{V}_k\int_0^1 f_\beta\left(a^\beta\overline{V}_k\right)da$$
$$= k\left[\lambda(x_1)\alpha(x_2) - \lambda(x_2)\alpha(x_1)\right] + \sum\lambda(x)V(x) - \overline{V}\int_0^1 f_\beta\left(a^\beta\overline{V}\right)da,$$

and letting k tend to ∞, (13) is proved.

If $\lambda(x) = b\alpha(x)$, the functional of V to be maximized is a function of $\bar{V} \equiv \sum_{x \in X} V(x)\alpha(x)$. This fact yields (14). (15) can be derived by a simple calculation. □

We now prove that $I_\beta(\lambda)$ is an upper bound.

THEOREM 2. *If C is a closed subset of $\underline{M}$ and $h(t) = t^\beta l(t)\Gamma(\beta + 1)$ as in* (RV2), *then*

$$\text{(UB)} \qquad \limsup_{N \to \infty} N^{-1}h(N)\log \hat{P}_{\mu(\alpha)}\{D_{N,n} \in C\} \le -\inf_{\lambda \in C} I_\beta(\lambda).$$

PROOF. Important for the upper bound estimate are:

(16) If $I(\lambda) = +\infty$ and $l > 0$ is arbitrary, then there exists N_λ, a neighborhood of λ, such that

$$\limsup_{N \to \infty} N^{-1}h(N)\log \hat{P}_{\mu(\alpha)}\{D_{N,n} \in N_\lambda\} \le -l.$$

(17) If $I(\lambda) < \infty$ and $\varepsilon > 0$ is arbitrary, then there exists N_λ, a neighborhood of λ, such that

$$\limsup_{N \to \infty} N^{-1}h(N)\log \hat{P}_{\mu(\alpha)}\{D_{N,n} \in N_\lambda\} \le -I(\lambda) + \varepsilon.$$

We now prove (16) and (17) for $\beta > 0$. The case when $\beta = 0$ requires obvious modification and is omitted. If $I(\lambda) = \infty$, then there exists x, $y \in X$ and $\delta > 0$ such that $\lambda(x_1)\alpha(x_2) - \lambda(x_2)\alpha(x_1) > \delta > 0$. Let $N_\lambda \equiv \{\phi\colon\ \phi(x_1)\alpha(x_2) - \phi(x_2)\alpha(x_1) > \delta\}$. Using V: $V(x_1) = -\theta\alpha(X_2)$, $V(x_2) = \theta\alpha(x_1)$ and $V(x) = 0$ otherwise, in the formula (CG), it follows from Chebychev's inequality that if θ is sufficiently large,

$$\limsup_{N \to \infty} N^{-1}h(N)\log \hat{P}_{\mu(\alpha)}\{D_{N,n} \in N_\lambda\} \le \bar{V} - \theta\delta = 0 - \theta\delta < -l.$$

This is (16).

To prove (17), we assume $\lambda(\cdot) = b\alpha(\cdot) > \alpha(\cdot)$. The case when $\lambda(\cdot) < \alpha(\cdot)$ can be similarly proved and the case when $\lambda(\cdot) = \alpha(\cdot)$ is trivial. Let $N_\lambda \equiv \{\phi\colon\ \phi(x_1) > (b - \delta)\}$. Using $V(x) = \theta\chi_{x_1}(x)$ in the formula (CG), it again follows from Chebychev's inequality that

$$\limsup_{N \to \infty} N^{-1}h(N)\log \hat{P}_{\mu(\alpha)}\{D_{N,n} \in N_\lambda\}$$

$$\le \inf_{\theta > 0}\left[\theta\alpha(x_1)\int_0^1 f_\beta(a^\beta\theta\alpha(x))\,da - \theta(b - \delta)\right],$$

which, due to the lower semicontinuity of $I_\beta(\cdot)$, is less than $-I(\lambda) + \varepsilon$ when δ is sufficiently small. This completes the proof of (17). By using (16), (17) and a standard method developed in [3], the inequality (UB) holds for compact sets and it also holds for closed sets if there exists a sequence of compact subsets

$\{K_r\}$ such that

$$\limsup_{r\to\infty}\limsup_{N\to\infty} N^{-1}h(N)\log \hat{P}_{\mu(\alpha)}\{D_{N,n}\in K_r^c\} = -\infty. \tag{18}$$

We construct $\{K_r\}$ as follows. Choosing $\xi(x)>0$ for all $x\in X$ such that $\sum_{x\in X}\xi(x)=1$ and

$$\sum_{x\in x}\xi(x)\alpha(x)\int_0^1 f_\beta\big(a^\beta\alpha(x)\big)\,da<\infty,$$

we let K_r be

$$\Big\{\lambda(\cdot)\in \underline{\underline{M}}:\ \sum_x \xi(x)\lambda(x)\le r\Big\}.$$

It is clear that K_r is compact in $\underline{\underline{M}}$. From the convexity of the cumulant generating function we have that

$$\begin{aligned}
&\limsup_{N\to\infty} N^{-1}h(N)\log \hat{E}_{\mu(\alpha)}\Big\{\exp\Big[Nh^{-1}(N)\sum\xi(x)D_{N,n}(x)\Big]\Big\}\\
&\quad\le \limsup_{N\to\infty}\sum_{x\in X}\xi(x)N^{-1}h(N)\log \hat{E}_{\mu(\alpha)}\Big\{\exp\Big[Nh^{-1}(N)D_{N,n}(x)\Big]\Big\}\\
&\quad\le \sum_{x\in X}\xi(x)\alpha(x)\int_0^1 f_\beta\big(a^\beta\alpha(x)\big)\,da<\infty.
\end{aligned}$$

This, by Chebychev's inequality, implies the desired property (18). When $\beta=0$, the modification needed is obvious. □

3. Lower bounds (Theorem 3).

THEOREM 3. *If G is an open subset of $\underline{M}$ and $h(t)=t^\beta l(t)\Gamma(\beta+1)$ is as in* (RV2), *then*

$$\liminf_{N\to\infty} N^{-1}h(N)\log \hat{P}_{\mu(\alpha)}\{D_{N,n}\in G\}\ge -\inf_{\lambda\in G} I_\beta(\lambda). \tag{LB}$$

PROOF. In view of the topology $\underline{M}$, we need only prove, for $N_\lambda=\{\eta: |\eta(x_l)-\lambda(x_l)|<\varepsilon, \text{ for } 1\le l\le k\}$, with some $\varepsilon>0$ and some $k\in N$, that

$$\liminf_{N\to\infty} N^{-1}h(N)\log \hat{P}_{\mu(\alpha)}\{D_{N,n}\in N_\lambda\}\ge -I_\beta(\lambda).$$

Moreover it suffices to consider $N_{b\alpha(\cdot)}$ because $I_\beta(\lambda)=+\infty$ when $\lambda(\cdot)$ is not a constant multiple of $\alpha(\cdot)$. Let $\boldsymbol{\lambda}$ be the vector $(\lambda(x_1),\dots,\lambda(x_k))$ and $\mathbf{N}_\lambda$ be a neighborhood of $\boldsymbol{\lambda}$. We therefore need only prove that

$$\liminf_{N\to\infty} N^{-1}h(N)\log\big(\hat{P}_{\mu(\alpha)}\circ \mathbf{D}_N^{-1}\big)\{\mathbf{N}_{b\alpha}\}\ge -I_\beta(b\alpha). \tag{19}$$

For this finite dimensional situation we shall use a general standard result, which we state as a lemma (see, e.g., [5] for a proof).

LEMMA 3. *Let $\{\nu_N\}$ be a family of probability measures on R^k, $a(N)\to\infty$ be a sequence and let $M_N(\theta)\equiv\int\exp[a(N)\boldsymbol{\theta}\cdot\mathbf{Y}]\,d\nu_N(\mathbf{Y})\le\infty$ for $\theta\in R^k$.*

Assume also that $\lim_{N\to\infty}a^{-1}(N)\log M_N(\boldsymbol{\theta})=\phi(\boldsymbol{\theta})\le\infty$ exists for $\boldsymbol{\theta}\in R^k$ and that $\phi(\boldsymbol{\theta})$ is continuous differentiable and closed in $\{\theta: \phi(\boldsymbol{\theta})<\infty\}$. If

$\sigma \in R^k$ is such that $(\nabla\phi)(\theta_*) = \sigma$ for some θ_*, then

$$\liminf_{N\to\infty} a^{-1}(N)\log \nu_N\{n_\sigma\} \geq -[\sigma\cdot\theta_* - \phi(\theta_*)],$$

where n_σ is a neighborhood of σ.

Application of the preceding lemma in our situation:

It is clear that ν_N, $a(N)$ and $\phi(\theta)$ will be $(\hat{P}_{\mu(\alpha)}\circ \mathbf{D}_N^{-1})$, $Nh^{-1}(N)$ and $(\theta\cdot\alpha)\int_0^1 f_\beta(a^\beta\theta\cdot\alpha)\,da$, respectively.

Also we have a key fact, $\{(\nabla\phi)(\theta): \theta\in R^k\} = \{b\alpha: b>0\}$, which is easy to check.

It then follows from the preceding lemma that there exists θ_* such that

$$\begin{aligned}\liminf_{N\to\infty} N^{-1}h(N)\log \hat{P}_{\mu(\alpha)}\{D_N \in \mathbf{N}_{b\alpha}\} &\geq -[b\alpha\cdot\theta_* - \phi(\theta_*)]\\ &\geq -\sup_\theta[b\alpha\cdot\theta - \phi(\theta)]\\ &= -\sup_c\left[bc - c\int_0^1 f_\beta(a^\beta c)\,da\right]\\ &= -I_\beta(b\alpha),\end{aligned}$$

which is (19). The proof is complete. □

4. Generalization to other types of Markov processes (Theorem 4). We now generalize Theorems 1–3 to Markov processes with continuous time parameter and/or continuous state space. A compact subset of the state space is obviously the correct substitute for a finite set and a Poisson field is that of independent Poisson variables. The following notational modification should, of course, be made: If K is compact, then

$$(\overline{\text{RV1}})\qquad \int_1^t p(\tau,x,K)\,d\tau \sim \alpha(K)t^\beta l(t)\quad \text{as } t\to\infty,$$

$$(\overline{\text{RV2}})\qquad \int_1^\infty e^{-st}p(t,x,K)\,dt \sim \alpha(K)h(s^{-1})\quad \text{as } t\to\infty.$$

Note again that $h(t) = t^\beta l(t)\Gamma(\beta+1)$.

$$(\overline{\text{ED}})\qquad D_{t,\omega}(A) \equiv t^{-1}\int_0^t \sum_i \chi_A(\omega_i(s))\,ds\quad \text{for measurable subsets } A,$$

where $\omega_i(s)$ is the trajectory of the ith particle. The Bessel processes, for example, satisfy ($\overline{\text{RV1}}$) and ($\overline{\text{RV2}}$) [see (3)]. We can then go through the same kind of arguments to establish

THEOREM 4. *If the Markov process of a single particle is null recurrent and satisfies a regularly varying property (expressed in appropriate notation), then we have ($t\in R^+$ or N):*

$$(\overline{\text{CG}})\qquad \begin{aligned}&\lim_{t\to\infty} t^{-1}h(t)\log \hat{E}_{\mu(\alpha)}\left\{\exp\left[th^{-1}(t)\int V(x)D_{t,w}(dx)\right]\right\}\\ &\quad = \overline{V}\int_0^1 f_\beta(a^\beta\overline{V})\,da,\end{aligned}$$

where V is compact supported, $\overline{V} \equiv \int V(x)\alpha(dx)$ *and* β *is the exponent appearing in* $h(t)$.

$$(\overline{\mathrm{UB}}) \qquad \limsup_{t\to\infty} t^{-1}h(t)\log \hat{P}_{\mu(\alpha)}\{D_{t,w} \in C\} \le -\inf_{\lambda\in C} I_\beta(\lambda),$$

where C is open in the topology induced by continuous functions with compact support and $I_\beta(\lambda(\cdot)) < \infty$ *is as in* (14) *and* (15).

$$(\overline{\mathrm{LB}}) \qquad \liminf_{t\to\infty} t^{-1}h(t)\log \hat{P}_{\mu(\alpha)}\{D_{t,w} \in G\} \ge -\inf_{\lambda\in G} I_\beta(\lambda),$$

where G is open and β, I_β *are as in* ($\overline{\mathrm{UB}}$).

5. Large deviations of a system modified by time change (Theorem 5). It should be noted that our large deviation result involves only an invariant distribution $\alpha(\cdot)$ and an index β related to the occupation time (see Lemma 2),

$$\int_1^t p(\tau, x, K)\, d\tau \sim \alpha(K)t^\beta l(t).$$

This radically differs from the results (see, e.g., [4] and [6]) in transient cases where the kernel $G(x, K) = \int_0^\infty p(t, x, K)\, dt$ is finite and plays an essential role in rate functions. This also shows that our large deviation result is rather insensitive to the underlying evaluation of each particle. We now explain this point. For a discrete time Markov process, let us replace, as explained in (2), the rigid time steps by exponential waiting times of mean 1. The resultant process has the transition probability $p_\pi(t, x, y) \equiv \sum_{m=0}^\infty e^{-t}t^m/m!$. We then have

THEOREM 5. *If* π_{xy} *satisfies* (RV2), *then* $p_\pi(t, x, y)$ *satisfies* ($\overline{\mathrm{RV2}}$) *with the same* $h(\cdot)$ *function. Thus, the system associated with* $p_\pi(t, x, y)$ *shares the same large deviation rates with that associated with* $\pi(x, y)$.

PROOF. The proof is a direct calculation,

$$\begin{aligned}
\int_0^t e^{-st}p_\pi(t, x, y)\, dt &= \int_0^\infty e^{-st} \sum_{m=0}^\infty \frac{e^{-t}t^m}{m!}\pi_{xy}^{(m)}\, dt \\
&= \sum_{m=0}^\infty \left(\int_0^\infty \frac{e^{-st}e^{-t}t^m}{m!} \right) \pi_{xy}^{(m)} \\
&= \sum_{m=0}^\infty (1+s)^{-(m+1)}\pi_{xy}^{(m)} \\
&\sim (1+s)^{-1}\alpha(y)h\big(\log^{-1}(1+s)\big) \\
&\sim \alpha(y)h(s^{-1}) \\
&\sim \sum_{m=0}^\infty e^{-sm}\pi_{xy}^{(m)} \quad \text{as } s \to 0,
\end{aligned}$$

where the interchange of summation and integration is justified by the Fubini theorem. □

Cox and Griffeath [1] have studied the large deviation for occupation time of Poisson systems of the independent random walk. As a consequence of Theorem 5, the same results hold for systems of independent random walks on Z^1 (or Z^2).

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