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MAXIMIZING $E\max_{1\le k\le n} S_k^+ / ES_n^+$: A PROPHET INEQUALITY FOR SUMS OF I.I.D. MEAN ZERO VARIATES[1]

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Let $X, X_1, X_2, \dots$ be i.i.d. mean zero random variables. Put $S_k = X_1 + \cdots + X_k$. We prove that for every $n \ge 1$, $E\max_{1\le k\le n} S_n^+ \le (2 - n^{-1})ES_n^+$. This result is nearly sharp, since if

$$P(X = 1) = P(X = -1) = \tfrac{1}{2},$$

then $E\max_{1\le k\le n} S_k^+ = (2 - n^{-1/2}\gamma_n^+)ES_n^+$, where $\lim_{n\to\infty}\gamma_n^+ = \sqrt{\pi/2}$.

Let $X, X_1, X_2, \dots$ be i.i.d. mean zero random variables and put $S_n = X_1 + \cdots + X_n$. Doob [(1953), Theorem 5.1, Chapter VII] proved that

$$E\max_{1\le k\le n}|S_k| \le c^* E|S_n|, \tag{1}$$

where $c^* \le 8$. This was improved to $c^* \le 3$ in Klass (1988). Echoing a 1987 conjecture of Harrison (private communication), we conjecture that if

$$\begin{aligned} C_n^* &\equiv \sup\left\{E\max_{1\le k\le n}|S_k|/E|S_n|:\ EX = 0 \quad\text{and}\quad 0 < E|X| < \infty\right\}, \\ C^* &\equiv \limsup_{n\to\infty} C_n^*, \end{aligned} \tag{2}$$

then

$$\begin{aligned} C^* &= \lim_{n\to\infty} C_n^*, \\ C^* &= E\sup_{0\le t\le 1}|B(t)|/E|B(1)| = \pi/2, \end{aligned} \tag{3}$$

where $B(\cdot)$ is a standard Brownian motion. Moreover, we conjecture that $C^* = \sup_{n\ge 1} C_n^*$.

Unable to solve this problem, we consider a related one. Define

$$C_n^+ \equiv \sup\left\{E\max_{1\le k\le n} S_k^+/ES_n^+:\ EX = 0 \quad\text{and}\quad 0 < E|X| < \infty\right\} \tag{4}$$

and

$$C^+ \equiv \limsup_{n\to\infty} C_n^+. \tag{5}$$

How large are C_n^+ and C^+? We prove that

$$C^+ = 2 \tag{6}$$

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and

$$2 - \gamma_n^+ n^{-1/2} \le C_n^+ \le 2 - n^{-1}, \tag{7}$$

where γ_n^+ is given by the simultaneous solution of (16)–(19) and

$$\lim_{n \to \infty} \gamma_n^+ = \sqrt{\pi/2}\,. \tag{8}$$

Clearly (6) follows from (7) and (8), as do the facts that $\lim_{n \to \infty} C_n^+$ exists and $C^+ = \sup_{n \ge 1} C_n^+$. We conjecture that the lower bound in the left-hand side of (7) is actually an equality. Note further that since $E \max_{1 \le k \le n} |S_n| \le E \max_{1 \le k \le n} S_k^+ + E \max_{1 \le k \le n} S_k^-$ and $ES_n^+ = ES_n^-$, we also have $C_n^* \le C_n^+ \le 2 - n^{-1}$ and $C^* \le C^+ = 2$.

In a classical queueing model, the waiting time for the nth customer (between when he arrives and when service begins) has the same distribution as the maximum of a related partial sum process. Hence the foregoing results have some application to queueing theory; specifically, to $GI/G/1$ queues with traffic intensity $\rho = \lambda/\mu$ equal to 1.

Observe that $ES_n^+ = \sup_{t_n \in T_n} ES_{t_n}^+$, where T_n is the collection of all stopping times t_n which halt by time n. Therefore, C_n^+ also represents the largest proportional expected advantage achievable by a prophet (a prophet is one who has exact knowledge of what the sequence $S_1^+, \dots, S_n^+$ will be and so can stop at the first random time $\tau \le n$ such that $S_\tau^+ = \max_{1 \le k \le n} S_k^+$) over a mere mortal (a mortal is one who is constrained to the use of random times t_n which are stopping times which halt by time n and so do not look into the future). Hence the results in (6) and (7) may be interpreted as so-called prophet inequalities. Viewed from this perspective (with $\sup_{t_n \in T_n} ES_{t_n}^+$ replacing ES_n^+), the evaluation of C_n^+ and C^+ in case $EX \ne 0$ is also of interest. When $EX < 0$ and $E(X^+)^2 < \infty$, $E \sup_{1 \le k < \infty} S_k^+$ is finite and so C_∞^+ can then be defined. Darling, Liggett and Taylor (1972) proved that $C_\infty^+ = e$, whence C^+ for this case is not 2 but is in fact at least e. Consequently, how $E \max_{1 \le k \le n}(S_k^+ + ky)^+ / \sup_{t \in T_n} E(S_t^+ + ty)^+$ can and does vary as n and y vary is a mystery yet to be fathomed. For a list of references on prophet inequalities, consult Hill (1986).

The principal result which we seek is:

Theorem. *Let $X_1, X_2, \dots$ be i.i.d. mean zero random variables. Let $S_k = X_1 + \cdots + X_k$. Then*

$$E \max_{1 \le k \le n} S_n^+ \le (2 - n^{-1}) ES_n^+. \tag{9}$$

Let $M_0^+ = 0$ and $M_k^+ = \max_{1 \le k \le n} S_j^+$. Inequality (9) depends critically on establishing that

$$ES_n^+ \ge E \sum_{k=1}^{n} (M_k^+ - M_{k-1}^+) I(S_n \ge S_k). \tag{10}$$

I had originally intended to show how a careful scrutiny of previous approaches could be used to evolve (10). However, the referee has suggested a shorter, more elegant derivation, based on a stronger statement. I will therefore dispense with the somewhat elaborate motivation of its discovery.

Notice that if (10) holds for all mean zero X-distributions, one might well conjecture that it holds for the random variables themselves—without expectations. Thus, it should hold for *real numbers*. The next lemma (due to the referee) verifies that this is indeed the case.

LEMMA. *Let* $x_1, \dots, x_n$ *be any real numbers. Put* $s_0 = 0$, $s_k = x_1 + \cdots x_k$ *and* $m_k^+ = \max_{0 \le j \le k} s_j$. *Then*

$$s_n^+ \ge \sum_{k=1}^{n} (m_k^+ - m_{k-1}^+) I(s_n \ge s_k). \tag{11}$$

PROOF. Let $\tau =$ last $0 \le k \le n$: $s_k = m_k^+ \le s_n^+$. Then

$$\begin{aligned}\sum_{k=1}^{n} (m_k^+ - m_{k-1}^+) I(s_n \ge s_k) &= \sum_{k=1}^{n} (m_k^+ - m_{k-1}^+) I(s_n \ge m_k^+ = s_k)\\ &= \sum_{k=1}^{n} (m_k^+ - m_{k-1}^+) I(\tau \ge k)\\ &= I(\tau \ge 1) \sum_{k=1}^{\tau} (m_k^+ - m_{k-1}^+)\\ &= m_\tau^+\\ &\le s_n^+ \quad \text{(by construction).}\end{aligned}$$

□

REMARK. Observe that equality obtains in (11) if each x_j is an integer not exceeding 1 (i.e., if each $x_j \in \{1, 0, -1, -2, \dots\}$).

In Chung (1974), page 287 it is shown that (regardless of whether $EX = 0$ or not)

$$EM_k^+ = \sum_{j=1}^{k} \frac{ES_j^+}{j}. \tag{12}$$

Therefore, combining (11) and (12),

$$ES_n^+ \ge \sum_{k=1}^{n} \frac{ES_k^+}{k} P_{n-k}^+, \tag{13}$$

where

$$P_j^+ = P(S_j \ge 0), \qquad P_0^+ = 1. \tag{14}$$

Note that (13) is an *equality* if X takes values in $\{1, 0, -1, -2, \dots\}$. If we had to approximate P_{n-k}^+ we would have reached an impasse. However, if we replace X

by $-X$ and put $P_{n-k}^{-} = P(-S_{n-k} \geq 0)$, then (13) also gives

$$E(-S_n)^+ \geq \sum_{j=1}^{k} \frac{E(-S_k)^+}{k} P_{n-k}^{-}. \tag{15}$$

Since $ES_k^+ = E(-S_k)^+$ for all mean zero variables, the coefficients of P_{n-k}^+ and P_{n-k}^- are identical. Adding (13) and (15),

$$\begin{aligned} 2ES_n^+ &\geq \sum_{k=1}^{n} \frac{ES_k^+}{k}(P_{n-k}^+ + P_{n-k}^-) \\ &\geq \frac{ES_n^+}{n} + \sum_{k=1}^{n} \frac{ES_k^+}{k} \quad \left[\text{since } P_{n-k}^+ + P_{n-k}^- = 1 + P(S_{n-k} = 0), \text{ which is 2 if } n - k = 0\right] \\ &= \frac{ES_n^+}{n} + EM_n^+ \quad [\text{by}(12)]. \end{aligned}$$

Consequently (9) holds.

We now show by example that (9) is best possible asymptotically (in the sense that $C^+ = 2$). To do so, we establish the left-hand side of (7) together with (8).

EXAMPLE. Let

$$X = \begin{cases} 1 & \text{wp } \frac{1}{2}, \\ -1 & \text{wp } \frac{1}{2}. \end{cases}$$

Then

$$P(S_{2k} = 0) = \binom{2k}{k} 2^{-2k} \sim (\pi k)^{-1/2} \quad \text{as } k \to \infty \tag{16}$$

and

$$ES_k^+ = 2^{-1} + 2^{-1} \sum_{j=1}^{[(k-1)/2]} P(S_{2j} = 0) \sim (k/2\pi)^{1/2} \quad \text{as } k \to \infty. \tag{17}$$

[To verify the equality in (17), note that

$$\begin{aligned} ES_k^+ &= E(S_{k-1} + X_k)I(S_{k-1} \geq 1) + EX_k^+ I(S_{k-1} = 0) \\ &= ES_{k-1}^+ + 2^{-1}P(S_{k-1} = 0).] \end{aligned}$$

Combining (13) and (15) we also obtain

$$\begin{aligned} 2ES_n^+ &= \sum_{k=1}^{n} \frac{ES_k^+}{k}(1 + P(S_{n-k} = 0)) \\ &= EM_n^+ + \frac{ES_n^+}{n} + \sum_{j=1}^{[(n-1)/2]} P(S_{2j} = 0) \frac{ES_{n-2j}^+}{n - 2j} \end{aligned}$$

and so

$$EM_n^+ = (2 - n^{-1}) - \sum_{j=1}^{[(n-1)/2]} \frac{P(S_{2j} = 0)ES_{n-2j}^+}{(n-2j)ES_n^+} ES_n^+. \tag{18}$$

Inserting the formulas in (16) and (17) into (18), an explicit formula for γ_n^+ can be obtained, where

$$EM_n^+ = \left(2 - n^{-1/2}\gamma_n^+\right)ES_n^+. \tag{19}$$

We will not record it here. Instead, we will content ourselves with proving that $\gamma_n^+ \to \sqrt{\pi/2}$, whence (8) and the left-hand side of (7) hold. Notice that

$$\gamma_n^+ \sim n^{1/2} \sum_{j=1}^{[(n-1)/2]} \frac{P(S_{2j} = 0)ES_{n-2j}^+}{(n-2j)ES_n^+} \sim \sum_{j=1}^{[(n-1)/2]} (\pi j)^{-1/2}(n-2j)^{-1/2}$$

$$= n^{-1} \sum_{j=1}^{[(n-1)/2]} \left(\frac{\pi j}{n}\right)^{-1/2}\left(1 - \frac{2j}{n}\right)^{-1/2}$$

$$\sim \int_0^{1/2} \frac{\pi^{-1/2}dx}{\sqrt{x(1-2x)}} \quad \text{(by the definition of the Riemann integral, together with its existence in this case)} \tag{20}$$

$$= (2\pi)^{-1/2}\int_0^1 \frac{dy}{\sqrt{y(1-y)}}$$

$$= (2\pi)^{-1/2}\frac{\left(\Gamma(\frac{1}{2})\right)^2}{\Gamma(1)} = \sqrt{\frac{\pi}{2}}.$$

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