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CENTRAL LIMIT THEOREMS FOR THE LOCAL TIMES OF CERTAIN MARKOV PROCESSES AND THE SQUARES OF GAUSSIAN PROCESSES[1]

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Let $\{X_t, t \geq 0\}$ be an R^d-valued, symmetric, right Markov process with stationary transition density. Let $\{\hat{X}_t, t \geq 0\}$ denote the version of X_t "killed" at an exponential random time, independent of X_t. Associated with $\hat{X}_t$ is a Green's function $g(x, y)$, which we assume satisfies $0 < g(x, x) < \infty$ for all x and a local time $\{L_x, x \in R^d\}$. It follows from an isomorphism theorem of Dynkin that L_x has continuous sample paths whenever $\{G(x), x \in R^d\}$, a Gaussian process with covariance $g(x, y)$, does. In this paper we use Dynkin's theorem to show that L_x satisfies the central limit theorem in the space of continuous functions on R^d if and only if $G(x)$ has continuous sample paths. This result strengthens a result of Adler and Epstein on the construction of the free field by means of a central limit theorem involving the local time, in the case when the local time is a point indexed process. In order to apply Dynkin's theorem the following result is obtained: The square of a continuous Gaussian process satisfies the central limit theorem in the space of continuous functions.

1. Introduction. The relationship between a Gaussian random field (the free field) and the local time of a symmetric Markov process has been observed by many authors starting with Symanzik [13]. It has been studied by Dynkin in great detail; see [6] and the references therein. Wolpert [15], treating Brownian motion in R^2, and later Adler and Epstein [1], treating a wide class of R^d-valued Markov processes, have pointed out that a central limit theorem (CLT) for the local times of a Markov process gives a "physical" explanation of how the associated Gaussian random field comes about. Whereas in the theory that most of these papers deal with the local time exists only as a random distribution, we shall restrict ourselves to the still interesting case in which the local time is defined as a point indexed process with finite variance. We shall then use an isomorphism theorem of Dynkin [6] to show that a stronger CLT holds than the one obtained in [15] and [1]. Our result is valid without any conditions beyond the most elementary ones that are necessary for the local time to be defined and the CLT to make sense.

Let $\{X_t, t \geq 0\}$ be an R^d-valued, symmetric, right Markov process with stationary symmetric transition density $p_t(x, y) = p_t(y, x)$. Let ξ be an exponential random variable with mean 1, independent of X, which we treat as a

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death time for X, and let Δ be the "cemetery" state for X so that the "killed version" of X is given by the process

$$\hat{X}_t = \begin{cases} X_t, & t < \xi, \\ \Delta, & t \geq \xi. \end{cases}$$

The killed process $\hat{X}_t$ is still a Markov process, with transition density $e^{-t}p_t(x,y)$, $x, y \in R^d$, and Green's function

$$g(x,y) = \int_0^\infty e^{-t} p_t(x,y)\, dt. \tag{1.1}$$

We shall assume throughout that $0 < g(x,x) < \infty$ for all x.

For each $x_0 \in R^d$ we define the probability P_{x_0} on the space of paths of X, augmented by Δ, possessing the finite-dimensional distributions

$$\begin{aligned} &P_{x_0}\left(\hat{X}_{t_1} \in B_1, \dots, \hat{X}_{t_k} \in B_k\right) \\ &\quad = \frac{1}{g(x_0,x_0)} \int_{B_1} \cdots \int_{B_k} e^{-t_k} p_{t_1}(x_0,x_1) p_{t_2-t_1}(x_1,x_2) \dots p_{t_k-t_{k-1}}(x_{k-1},x_k) \\ &\qquad \times g(x_k,x_0)\, dx_k \dots dx_1, \end{aligned} \tag{1.2}$$

for $0 < t_1 < \cdots < t_k$, and Borel sets $B_1, \dots, B_k$. As pointed out in [5], Section 3, properly formulated $P_{x_0}(\cdot)$ can be interpreted as $P(\cdot \mid \hat{X}_0 = x_0,\ \hat{X}_{\xi^-} = x_0)$, so that P_{x_0} describes a process starting and finishing (after an exponential killing time) at x_0. [Note that at time ξ the path takes the value Δ. Therefore, $P_{x_0}(\hat{X}_{t_1} \in R^d, \dots, \hat{X}_{t_k} \in R^d) = \mathrm{Prob}(\xi > t_k \mid \hat{X}_0 = x_0,\ \hat{X}_{\xi^-} = x_0)$.]

We shall be interested in the local time process $\{L_x,\ x \in R^d\}$ of the killed version $\hat{X}_t$ of X_t, which can be formally expressed as

$$L_x = \int_0^\xi \delta_x(\hat{X}(t))\, dt, \tag{1.3}$$

where δ_x is the delta function centered at x. This definition can be made rigorous by taking L_x as the Radon-Nikodym derivative of the occupation measure

$$\mu(A) = \int_0^\infty I_A(\hat{X}(t))\, dt,$$

where we follow the convention that for all Borel sets $A \in R^d$, $I_A(\hat{X}_t) = 0$ for $t \geq \xi$. Another approach is to approximate δ_x by some density function and then pass to the limit as is done in [6]. In this paper we will mainly be concerned with the case when $\{L_x,\ x \in R^d\}$ has continuous sample paths almost surely.

This is the version of Dynkin's theorem which we will use. Let $\{G(x),\ x \in R^d\}$ be a mean-zero Gaussian process with covariance $g(x,y)$ as given in (1.1). Let (Ω, π) denote the probability space of this process. The measure π is determined by $\{g(x,y),\ (x,y) \in R^d \times R^d\}$. Let $L = \{L_x,\ x \in R^d\}$ denote the local time process defined in (1.3). This process is defined on a probability space (Ω_1, P_{x_0}), where P_{x_0} is given in (1.2). We denote by E_π and $E_{P_{x_0}}$

expectation with respect to the measures π and P_{x_0}, respectively. The isomorphism theorem is as follows.

THEOREM 1.1 (Dynkin). *Let F be any positive functional on the space of functions from R^d to R and let $\eta_x = G^2(x)/2$. Then*

$$E_\pi\left(F(\eta)\frac{G^2(x_0)}{g(x_0,x_0)}\right) = E_\pi E_{P_{x_0}}(F(\eta + L)). \tag{1.4}$$

Equivalently, the process $\eta_{\cdot}(\omega)$ with the measure $[G^2(x_0)/g(x_0,x_0)]\pi(d\omega)$ and $\eta_{\cdot}(\omega) + L_{\cdot}(\omega_1)$ with the measure $\pi(d\omega)P_{x_0}(d\omega_1)$ are equal in distribution.

Since the local time exists as long as $0 < g(x,x) < \infty$, it is an immediate consequence of this theorem that it has a continuous version whenever $\{G(x), x \in R^d\}$ has a version with continuous sample paths. The continuity of Gaussian processes is well understood (see, e.g., [9], [7] and [14]) and is completely determined by the Green's function $g(x,y)$. Thus one can obtain conditions for the continuity of L in terms of $g(x,y)$. Of course, we must add that in the case when $\{X(t),\ t \in R^d\}$ is Brownian motion or a Lévy process, $g(x,x)$ is finite only when $d = 1$.

Suppose that L has a version with continuous sample paths which we will again denote by $L = \{L_x,\ x \in R^d\}$. Let $\{L_i\}_{i=1}^{\infty} = \{L_{x,i},\ x \in R^d\}_{i=1}^{\infty}$ be i.i.d. copies of L. Let $\{\varepsilon_i\}_{i=1}^{\infty}$ be a Rademacher sequence, [i.e., a sequence of i.i.d. random variables satisfying $P(\varepsilon_i = 1) = P(\varepsilon_i = -1) = 1/2$], independent of $\{L_i\}_{i=1}^{\infty}$. Consider the normed sums

$$S_n = S_n(x) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\varepsilon_i L_{x,i}, \qquad x \in R^d, \tag{1.5a}$$

and

$$S_n' = S_n'(x) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(L_{x,i} - EL_x), \qquad x \in R^d. \tag{1.5b}$$

Let $K \subset R^d$ be compact. We say that L satisfies the CLT on $C(K)$, the space of continuous functions on K, if the finite-dimensional distributions of S_n or equivalently of S_n' converge weakly and if the measures induced by S_n or equivalently by S_n' on $C(K)$ are tight. If L satisfies the CLT on $C(K)$ for all compact $K \subset R^d$ we say that L satisfies the CLT on $C(R^d)$.

One can check by Theorem 1.1, or from the definition of local time, that

$$ES_n(x)S_n(y) = \frac{2g(x_0,x)g(x_0,y)g(x,y)}{g(x_0,x_0)} \tag{1.6}$$

and similarly that

$$(1.7)\quad ES_n'(x)S_n'(y) = \frac{2g(x_0,x)g(x_0,y)g(x,y)}{g(x_0,x_0)} - \frac{g^2(x_0,x)g^2(x_0,y)}{g^2(x_0,x_0)}.$$

Since $\sup_{x,y\in K} g(x,y) < \infty$ for all compact sets $K \subset R^d$ we see by the standard CLT on finite-dimensional spaces that the finite-dimensional distributions of S_n and S_n' converge weakly to those of the Gaussian processes

$$Z(x) = \frac{\sqrt{2}\,g(x_0,x)G(x)}{\sqrt{g(x_0,x_0)}}, \qquad x \in R^d,$$

and

$$Z'(x) = \frac{g(x_0,x)}{\sqrt{g(x_0,x_0)}}\left(\sqrt{2}\left(G(x) - \frac{g(x_0,x)}{g(x_0,x_0)}G(x_0)\right) + \frac{g(x_0,x)}{g(x_0,x_0)}G(x_0)\right), \qquad x \in R^d.$$

Now suppose that L satisfies the CLT on $C(R^d)$. This implies, by definition, that the Gaussian processes $\{Z(x),\ x \in R^d\}$ and $\{Z'(x),\ x \in R^d\}$ have continuous sample paths. Therefore, an obvious necessary condition for L to satisfy the CLT on $C(R^d)$ is that $\{G(x),\ x \in R^d\}$ is a continuous Gaussian process. The main result of this paper is that this condition is also sufficient.

THEOREM 1.2. *Let $L = \{L_x,\ x \in R^d\}$ be the local time for the killed, symmetric, right Markov process $\hat{X}_t$, as described above, with Green's function $g(x,y)$. Then L satisfies the CLT on $C(R^d)$ if and only if $\{G(x),\ x \in R^d\}$, the mean-zero Gaussian process with covariance $g(x,y)$, has continuous sample paths.*

The fact that the measures induced by S_n, (S_n') are tight on $C(K)$ for all compact sets $K \subset R^d$ whenever $\{G(x),\ x \in R^d\}$ has continuous sample paths gives a more meaningful construction of the free field [i.e., of $\{G(x),\ x \in R^d\}$] by the local time of $\hat{X}_t$ than to say that S_n, (S_n') satisfies the CLT for $x \in X$, where X is a finite subset of R^d.

It is also noteworthy that the local time process L always satisfies the CLT on $C(R^d)$ as long as there exists a continuous limiting Gaussian process to which it can converge. This is reminiscent of the results of Barlow and Hawkes ([4] and [3]), which imply, in particular, that the local time of a symmetry Lévy process has jointly continuous sample paths if and only if the Green's function of the Lévy process is the covariance of a continuous stationary Gaussian process.

It suffices, for the proof of Theorem 1.2, to show that $\{\eta.(\omega),\ [G^2(x_0)/g(x_0,x_0)]\pi\}$ satisfies the CLT whenever $\{G(x),\ x \in R^d\}$ is continuous. But, as we shall see, this follows easily from the fact that $\{\eta.(\omega), \pi\}$ satisfies the CLT. To be more precise, the main ingredient in the proof of Theorem 1.2 is the following theorem.

THEOREM 1.3. *Let* $\{X(t),\ t \in T\}$, *where* T *is some index set, be a real centered continuous Gaussian process. Then the continuous stochastic process* $\{X^2(t),\ t \in T\}$ *satisfies the CLT in* $C(T)$ *the space of continuous functions on* T.

This theorem is contained in Corollary 2.6 which is an immediate corollary of Theorem 2.3. Theorem 2.3 and some more general results on the CLT for products of Gaussian processes are obtained in Section 2. To prove Theorem 1.2, we relate the CLT for the local time to the CLT for the square of the associated Gaussian process by (2.32), which is an immediate consequence of Theorem 1.1. The main work in this paper is to obtain Theorem 1.3 and its direct consequence Corollary 2.8.

2. A central limit theorem for products of Gaussian processes. Let $\{G(t),\ t \in T\}$ be a real-valued centered continuous Gaussian process. In considering this process we will always take the topology on T to be given by the pseudodistance d_G, where

$$d_G(s,t) = \left(E|G(s) - G(t)|^2\right)^{1/2}, \qquad \forall\, s,t \in T. \tag{2.1}$$

When (T, d_G) is compact we will sometimes consider G as a random variable with values in $(C(T, d_G), \|\cdot\|_\infty)$, the Banach space of continuous functions on (T, d_G), where $\|\cdot\|_\infty$ denotes the supremum norm.

DEFINITION 2.1. Let (T, d) be a compact metric space and $C(T, d) \equiv (C(T, d), \|\cdot\|_\infty)$. Let $Z = \{Z(t),\ t \in T\}$ be a stochastic process such that $Z \in C(T, d)$. We say that Z satisfies the central limit theorem in $C(T, d)$ [written $Z \in \mathrm{CLT}(C(T, d))$] if

$$n^{-1/2} \sum_{j=1}^{n} (Z_j - EZ) \tag{2.2}$$

converges in distribution in $C(T, d)$, where $\{Z_j\}_{j=1}^\infty$ are i.i.d. copies of Z.

Note that if (2.2) does converge in distribution in $C(T, d)$ then the limit is, necessarily, a centered $C(T, d)$-valued Gaussian random variable G with

$$EG(s)G(t) = \mathrm{cov}(Z(s), Z(t)), \qquad \forall\, s,t \in T. \tag{2.3}$$

The metric d that appears in the expression $Z \in C(T, d)$ is not unique. One can always take

$$d = d(s,t) = \left(E|Z(s) - Z(t)|^2\right)^{1/2},$$

which, necessarily, must exist by (2.3) and which is equal to $d_G(s,t)$, defined in (2.1), the corresponding metric of the limiting Gaussian process.

We will use several equivalent formulations for $Z \in \mathrm{CLT}(C(T,d))$.

PROPOSITION 2.2 (see, e.g., [8], Theorem 2.14). *Let $Z = \{Z(t),\ t \in T\} \in C(T,d)$ and let $\{Z_j\}_{j=1}^{\infty}$ be i.i.d. copies of Z. Let $\{\varepsilon_j\}_{j=1}^{\infty}$ be a Rademacher sequence and $\{g_j\}_{j=1}^{\infty}$ a sequence of i.i.d. normal random variables with mean 0 and variance 1, such that the three sequences are independent of each other. The following are equivalent*:

(i) $Z \in \mathrm{CLT}(C(T,d))$;
(ii) $n^{-1/2}\sum_{j=1}^{n}\varepsilon_j Z_j$ *converges in distribution in* $C(T,d)$;
(iii) $n^{-1/2}\sum_{j=1}^{n} g_j Z_j$ *converges in distribution in* $C(T,d)$;
(iv) (T,d) *is totally bounded and*

$$\lim_{\delta\to 0}\sup_n n^{-1/2}E\left\|\sum_{j=1}^{n}\varepsilon_j Z_j\right\|_\delta = 0,$$

where for real-valued functions $f(t)$ and $g(t)$, $t \in (T,d)$, we define

$$\|g\|_\delta = \sup_{\{s,t\in T:\, d(s,t)\le\delta\}} |g(s) - g(t)|;$$

(v) (T,d) *is totally bounded and*

$$\lim_{\delta\to 0}\sup_n n^{-1/2}E\left\|\sum_{j=1}^{n} g_j Z_j\right\|_\delta = 0.$$

The next theorem is the main result of this section.

THEOREM 2.3. *Let $\{(X(u),Y(v)),\ (u,v) \in U\times V\}$ be an R^2-valued centered continuous Gaussian process. Assume that $(U\times V, d)$ is compact, where*

$$d((u_1,v_1),(u_2,v_2)) = \left(E|X(u_1)Y(v_1) - X(u_2)Y(v_2)|^2\right)^{1/2}.$$

Then $XY \subseteq \mathrm{CLT}(C(U\times V,d))$.

Note that to simplify the notation we write XY for $X\otimes Y$, where $X\otimes Y(u,v) = X(u)Y(v)$ since the noncommutivity of the tensor product does not affect anything that follows.

The following lemmas, which will be used in the proof of Theorem 2.3, are interesting in their own right. The first one is a form of "decoupling" lemma. Because we are dealing with Gaussian random variables the "decoupling" is quite simple. The general topic of decoupling random variables was initiated in [11]; see also [16].

LEMMA 2.4. *Let $(X,Y) = \{(X(u),Y(v)),\ (u,v) \in U\times V\}$ be an R^2-valued centered bounded Gaussian process. Let $\{(X_j,Y_j)\}_{j=1}^{n}$, $\{(X_j',Y_j')\}_{j=1}^{n}$ be* *independent copies of* (X,Y) *and let* $\{\varepsilon_j\}_{j=1}^n$ *be a Rademacher sequence such that the three sequences are independent of each other. Then*

$$(2.4)\qquad E\left\| n^{-1/2}\sum_{j=1}^n \varepsilon_j X_j Y_j\right\|_\infty \le \|EX^2\|_\infty^{1/2}\|EY^2\|_\infty^{1/2} + 2E\left\| n^{-1/2}\sum_{j=1}^n \varepsilon_j X_j Y_j'\right\|_\infty .$$

PROOF. Note that

$$(2.5)\qquad (X,Y') =_{\mathscr{D}} \left(\frac{X+X'}{\sqrt{2}}, \frac{Y-Y'}{\sqrt{2}}\right),$$

where " $=_{\mathscr{D}}$ " denotes equality in distribution. This follows from the fact that

$$E[(X(u)+X'(u))(Y(v)-Y'(v))] = 0, \qquad \forall (u,v)\in U\times V.$$

Using (2.5), we see that

$$(2.6)\qquad \begin{aligned} E&\left\| n^{-1/2}\sum_{j=1}^n \varepsilon_j X_j Y_j'\right\|_\infty \\ &= \frac{1}{2}E\left\| n^{-1/2}\sum_{j=1}^n \varepsilon_j (X_j+X_j')(Y_j-Y_j')\right\|_\infty \\ &= \frac{1}{2}E_{\varepsilon,(X,Y)}E_{(X',Y')}\left\| n^{-1/2}\sum_{j=1}^n \varepsilon_j (X_jY_j + X_j'Y_j - X_jY_j' - X_j'Y_j')\right\|_\infty \\ &\ge \frac{1}{2}E\left\| n^{-1/2}\sum_{j=1}^n \varepsilon_j (X_jY_j - EXY)\right\|_\infty , \end{aligned}$$

where $E_{\varepsilon,(X,Y)}, (E_{(X',Y')})$ denotes expectation with respect to the sequences $\{\varepsilon_j\}_{j=1}^n$ and $\{(X_j,Y_j)\}_{j=1}^n, (\{X_j',Y_j')\}_{j=1}^n)$, and we bring $E_{(X',Y')}$ inside $\|\cdot\|_\infty$ to obtain the final inequality in (2.6). The last term in (2.6)

$$(2.7)\qquad \ge \frac{1}{2}E\left\| n^{-1/2}\sum_{j=1}^n \varepsilon_j X_j Y_j\right\|_\infty - \frac{1}{2}\|EXY\|_\infty E\left| n^{-1/2}\sum_{j=1}^n \varepsilon_j\right|,$$

which gives us (2.4) by the Schwarz inequality. □

LEMMA 2.5. *Let* $U=\{U(s),\ s\in S\}$ *and* $V=\{V(t),\ t\in T\}$, S *and* T *countable, be independent stochastic processes. Let* $\{U_j\}_{j=1}^n$ *and* $\{V_j\}_{j=1}^n$ *be i.i.d. copies of* U *and* V *and let* $\{g_j\}_{j=1}^\infty$ *be a sequence of i.i.d. normal random variables with mean* 0 *and variance* 1, *such that the three sequences are independent of each other. Then*

$$(2.8)\qquad E\left\|\sum_{j=1}^n g_jU_jV_j\right\|_\infty \le 8\left(E\left\|\sum_{j=1}^n g_j\|V_j\|_\infty U_j\right\|_\infty + E\left\|\sum_{j=1}^n g_j\|U_j\|_\infty V_j\right\|_\infty\right).$$

If, in addition to the above, U and V are mean-zero Gaussian processes, then

$$E\left\| n^{-1/2} \sum_{j=1}^{n} g_j U_j V_j \right\|_\infty \le 16\left(E\|U\|_\infty^2 E\|V\|_\infty^2\right)^{1/2}. \tag{2.9}$$

PROOF. We will use the following comparison result of Fernique [7], Corollary 2.13, which is based on Slepian's lemma: Let X be a countable index set and $\{G_1(x),\ x \in X\}$ and $\{G_2(x),\ x \in X\}$ be mean-zero Gaussian processes such that

$$E|G_1(x) - G_1(y)|^2 \le E|G_2(x) - G_2(y)|^2, \qquad \forall\, x, y \in X, \tag{2.10}$$

and such that 0 is in the range of $G_1(\cdot, \omega)$ for almost all $\omega \in \Omega$. [$(G_1, \Omega, \mathscr{P})$ is the probability space supporting G_1]. Then

$$E \sup_{x \in X} |G_1(x)| \le 2E \sup_{x \in X} |G_2(x)|. \tag{2.11}$$

Now, fix $s_0 \in S$ and set

$$\overline{U}_j(s) = U_j(s) - U_j(s_0), \qquad \forall\, s \in S,\ j = 1, \dots, n,$$

and consider

$$H(s,t) = \sum_{j=1}^{n} g_j \overline{U}_j(s) V_j(t), \qquad \forall\, (s,t) \in S \times T. \tag{2.12}$$

Obviously, $H(s_0, t) = 0$ for all $t \in T$. As usual let E_g denote expectation with respect to the sequence $\{g_j\}_{j=1}^n$. We have

$$\begin{aligned} E_g\left|H(s_1,t_1) - H(s_2,t_2)\right|^2 &= \sum_{j=1}^{n} \left|\overline{U}_j(s_1)V_j(t_1) - \overline{U}_j(s_2)V_j(t_2)\right|^2 \\ &\le 2\left(\sum_{j=1}^{n} \overline{U}_j^2(s_1)\left|V_j(t_1) - V_j(t_2)\right|^2 \right. \\ &\qquad \left. + \sum_{j=1}^{n} V_j^2(t_2)\left|\overline{U}_j(s_1) - \overline{U}_j(s_2)\right|^2\right) \\ &\le 8\left(\sum_{j=1}^{n} \|U_j\|_\infty^2 |V_j(t_1) - V_j(t_2)|^2 \right. \\ &\qquad \left. + \sum_{j=1}^{n} \|V_j\|_\infty^2 |\overline{U}_j(s_1) - \overline{U}_j(s_2)|^2\right). \end{aligned} \tag{2.13}$$

Define

$$Y(s) = 2\sqrt{2} \sum_{j=1}^{n} g_j \|U_j\|_\infty V_j(s), \qquad \forall\, s \in S,$$

and

$$Z(t) = 2\sqrt{2} \sum_{j=1}^{n} g_j' \|V_j\|_\infty U_j(t), \qquad \forall\, t \in T,$$

where $\{g_j'\}_{j=1}^{\infty}$ is a sequence of normal random variables with mean 0 and variance 1, independent of everything else. Then by Fernique's comparison result with

$$G_1(s,t) = H(s,t), \qquad \forall\, (s,t) \in S \times T,$$

and

$$G_2(s,t) = Y(s) + Z(t), \qquad \forall\, (s,t) \in S \times T,$$

we see that

$$E_g \sup_{(s,t)\in S\times T} |H(s,t)| \le 2\Big(E_g \sup_{s\in S} |Y(s)| + E_g \sup_{t\in T} |Z(t)| \Big). \tag{2.14}$$

It follows from (2.14) that

$$E_g \left\| \sum_{j=1}^{n} g_j U_j V_j \right\|_\infty \le E_g \sup_{t\in T} \left| \sum_{j=1}^{n} g_j U_j(s_0) V_j(t) \right| + 2\Big(E_g \sup_{s\in S} |Y(s)| + E_g \sup_{t\in T} |Z(t)| \Big)$$

and since

$$E_g \sup_{t\in T} \left| \sum_{j=1}^{n} g_j U_j(s_0) V_j(t) \right| \le 2 E_g \sup_{t\in T} \left| \sum_{j=1}^{n} g_j \|U_j\|_\infty V_j(t) \right|$$

by the comparison result of Fernique, we get (2.8).

Now assume that U is a Gaussian process. Recall that $\{U_j\}_{j=1}^{\infty}$ is independent of $\{V_j\}_{j=1}^{\infty}$ and $\{g_j\}_{j=1}^{\infty}$. Let E_U denote expectation with respect to the probability space generated by $\{U_j\}_{j=1}^{\infty}$. Note that for fixed sequences $\{V_j\}_{j=1}^{\infty}$ and $\{g_j\}_{j=1}^{\infty}$,

$$E_U \left\| \sum_{j=1}^{n} g_j \|V_j\|_\infty U_j \right\|_\infty = \left(\sum_{j=1}^{n} g_j^2 \|V_j\|_\infty^2 \right)^{1/2} E\|U\|_\infty.$$

Therefore,

$$\begin{aligned} E \left\| \sum_{j=1}^{n} g_j \|V_j\|_\infty U_j \right\|_\infty &= E\left(\sum_{j=1}^{n} g_j^2 \|V_j\|_\infty^2 \right)^{1/2} E\|U\|_\infty \\ &\le n^{1/2} \big(E\|V\|_\infty^2 \big)^{1/2} E\|U\|_\infty \end{aligned} \tag{2.15}$$

and similarly for the last term in (2.8). This and the Schwarz inequality give (2.9). □

PROOF OF THEOREM 2.3. Let $\{\varepsilon_j\}_{j=1}^{\infty}$ be a sequence of Rademacher functions independent of all the other other variables. Then for any sequence of functions $\{\varphi_j(t)\}_{j=1}^{\infty}$,

$$
\begin{aligned}
E\left\|\sum_{j=1}^{n} g_j\varphi_j\right\|_{\infty} &= E_g E_{\varepsilon}\left\|\sum_{j=1}^{n} \varepsilon_j g_j\varphi_j\right\|_{\infty} \\
&\geq E\left\|\sum_{j=1}^{n} \varepsilon_j (E|g|)\varphi_j\right\|_{\infty} = \sqrt{2/\pi}\, E\left\|\sum_{j=1}^{n} \varepsilon_j\varphi_j\right\|_{\infty}.
\end{aligned} \tag{2.16}
$$

Therefore, by (2.16), (2.4) and (2.9) we have

$$
E\left\|n^{-1/2}\sum_{j=1}^{n} \varepsilon_j X_j Y_j\right\|_{\infty} \leq (1 + 16\sqrt{2\pi})\big(E\|X\|_{\infty}^2 E\|Y\|_{\infty}^2\big)^{1/2}. \tag{2.17}
$$

This shows that XY satisfies the "bounded CLT", i.e., that the left side of (2.17) is uniformly bounded in n. Note that, since XY is continuous and $(U \times V, d)$ is compact, it is enough to take the supremum of XY over a countable set as is required by the hypothesis of Lemma 2.5. This will always be the case in what follows. We will continue to use Lemma 2.5 in this way without further comment.

In order to show that $XY \in \mathrm{CLT}(C(U \times V, d))$ we will approximate XY by finite-dimensional random vectors in $C(U \times V, d)$. This we can do by using a Karhunen–Loève expansion for the R^2-valued Gaussian process $\{(X(u), Y(v)), (u, v) \in U \times V\}$ (see [2], Theorem 6.8). We write

$$
(X(u), Y(v)) = \sum_{k=1}^{\infty} g_k(\varphi_k(u), \psi_k(v)), \qquad (u, v) \in U \times V, \tag{2.18}
$$

where $\{\varphi_k(u)\}_{k=1}^{\infty}, (\{\psi_k(v)\}_{k=1}^{\infty})$ are continuous functions on $U, (V)$. Let

$$
X_N(u) = \sum_{k=N}^{\infty} g_k\varphi_k(u), \qquad u \in U,
$$

$$
Y_N(v) = \sum_{k=N}^{\infty} g_k\psi_k(v), \qquad v \in V.
$$

Note that $(X - X_N)(Y - Y_N)$ has finite-dimensional range in $C(U \times V, d)$ and hence satisfies the CLT by classical considerations. Now let $\{(X - X_N)_i\}_{i=1}^{\infty}$ and $\{(Y - Y_N)_i\}_{i=1}^{\infty}$ be i.i.d. copies of $(X - X_N)$ and $(Y - Y_N)$ and $\{X_{N,i}\}_{i=1}^{\infty}$ and $\{Y_{N,i}\}_{i=1}^{\infty}$ be i.i.d. copies of X_N and Y_N. Define $X_i = (X - X_N)_i + X_{N,i}$ and $Y_i = (Y - Y_N)_i + Y_{N,i}$, $i = 1, \dots, \infty$.

We have

$$
\begin{aligned}
&E\left\| m^{-1/2}\sum_{i=1}^{m}\varepsilon_i[X_iY_i-(X-X_N)_i(Y-Y_N)_i]\right\|_\infty \\
&\quad = E\left\| m^{-1/2}\sum_{i=1}^{m}\varepsilon_i[X_iY_i-(X_i-X_{N,i})(Y_i-Y_{N,i})]\right\|_\infty \\
&\quad \le E\left\| m^{-1/2}\sum_{i=1}^{m}\varepsilon_i X_{N,i}Y_i\right\|_\infty + E\left\| m^{-1/2}\sum_{i=1}^{m}\varepsilon_i Y_{N,i}X_i\right\|_\infty \qquad (2.19)\\
&\qquad + E\left\| m^{-1/2}\sum_{i=1}^{m}\varepsilon_i X_{N,i}Y_{N,i}\right\|_\infty \\
&\quad \le (1+16\sqrt{\pi})\left(E\|X_N\|_\infty^2 E\|Y\|_\infty^2\right)^{1/2} \\
&\qquad + (1+16\sqrt{\pi})\left[\left(E\|Y_N\|_\infty^2 E\|X\|_\infty^2\right)^{1/2} + \left(E\|X_N\|_\infty^2 E\|Y_N\|_\infty^2\right)^{1/2}\right],
\end{aligned}
$$

where at the last stage we use (2.17).

Since both $E\|X_N\|_\infty^2$ and $E\|Y_N\|_\infty^2$ go to 0 as $N \to \infty$ we see that the first expectation in (2.19) goes to 0 as $N \to \infty$ uniformly in m. It now follows from [12], Theorem 3.1, that XY satisfies the CLT. See also [10], Lemma 1.2, Chapter 4. (In "Proof of Lemma 1.2" in [10] replace X by Y_ε.)

COROLLARY 2.6. *Let $\{X(t),\ t \in T\}$ be a real-valued centered continuous Gaussian process. Then $X^n \in \mathrm{CLT}(C(T,d))$ for all integers $n \ge 1$.*

PROOF. The case $n = 2$ follows from Theorem 2.3. For $n > 2$ one can prove this by induction using the fact that

$$X'X^n =_{\mathscr{D}} \left(\frac{X-X'}{\sqrt{2}}\right)\left(\frac{X+X'}{\sqrt{2}}\right)^n$$

and proceeding as in the proof of Lemmas 2.4 and 2.5. □

REMARK 2.7. It seems clear that Theorem 2.3 can be extended in a similar fashion as Corollary 2.6. Thus, if $\{(X_1(u_1),\dots,X_m(u_m)),\ (u_1,\dots,u_m) \in U_1 \times \cdots \times U_m\}$ is an R^m-valued centered continuous Gaussian process, it ought to follow that $X_1^{n_1}X_2^{n_2}\cdots X_m^{n_m} \in \mathrm{CLT}(C(U_1\times\cdots\times U_m, d))$ for d appropriately defined. However, we have not tried to write out a proof of this statement and will leave it to the interested reader.

In order to use Theorem 1.1 to prove Theorem 1.2 we need a CLT for a stochastic process that is the square of a Gaussian process with a change of measure. Let $\{X(t),\ t \in T\}$ be a real-valued centered continuous Gaussian process as defined in the beginning of this section. Consider the stochastic process $\{X^2(t),\ t \in T\}$ and let π denote the measure induced on $C(T,d)$ by this process. For $t_0 \in T$ we define the measure π_{t_0} on $C(T, d)$ by

$$d\pi_{t_0} = \frac{X^2(t_0)}{E_\pi X^2(t_0)}\, d\pi \tag{2.20}$$

and let E_π denote expectation with respect to the measure π. [To avoid trivialities, we will assume that $E_\pi X^2(t_0) \neq 0$.] We define the (obviously continuous) stochastic process $Z^2 = \{Z^2(t),\ t \in T\}$ to be the process given by the measure π_{t_0} on $C(T)$. It is clear that for all measurable functions f on $C(T)$ we have

$$E_{\pi_{t_0}} f(Z^2) = E_\pi \frac{X^2(t_0)}{E_\pi X^2(t_0)} f(X^2), \tag{2.21}$$

where $E_{\pi_{t_0}}$ denotes expectation with respect to the measure π_{t_0}.

The next result is a corollary of Theorem 2.3.

COROLLARY 2.8. *Let $Z^2 = \{Z^2(t),\ t \in T\}$ be the continuous stochastic process defined by* (2.20) *and* (2.21). *Then $Z^2 \in \mathrm{CLT}(C(T, d))$, where $d = d(s, t)$ can be taken to be any of the equivalent metrics*

$$\left(E_{\pi_{t_0}}|Z^j(s) - Z^j(t)|^2\right)^{1/2}, \qquad \left(E_\pi|X^j(s) - X^j(t)|^2\right)^{1/2}, \qquad j = 1, 2. \tag{2.22}$$

PROOF. Note that

$$\|fg\|_\delta \le \|f\|_\infty \|g\|_\delta + \|g\|_\infty \|f\|_\delta, \tag{2.23}$$

where $\|\ \|_\delta$ is defined in Proposition 2.2(iv). To show that $Z^2 \in \mathrm{CLT}(C(T, d))$, it is enough, by Proposition 2.2, to show that

$$\lim_{\delta \to \infty} \sup_n E_{\pi_{t_0}} E_\varepsilon \left\| n^{-1/2} \sum_{j=1}^n \varepsilon_j Z_j^2 \right\|_\delta = 0 \tag{2.24}$$

or equivalently, by (2.21), that

$$\lim_{\delta \to \infty} \sup_n E_\pi E_\varepsilon \left[\left\| n^{-1/2} \sum_{j=1}^n \varepsilon_j X_j^2 \right\|_\delta \prod_{j=1}^n \left(\frac{X_j^2(t_0)}{E_\pi X_j^2(t_0)} \right) \right] = 0. \tag{2.25}$$

Define

$$W(t) = X(t) - r(t) X(t_0),$$

where

$$r(t) = \frac{E_\pi X(t) X(t_0)}{E_\pi X^2(t_0)}.$$

We see that W is π independent of $X(t_0)$. As usual let $\{W_j\}_{j=1}^\infty$ and $\{X_j\}_{j=1}^\infty$ be i.i.d. copies of W and X, respectively, and set

$$\beta_j = \frac{X_j^2(t_0)}{E_\pi X_j^2(t_0)}, \qquad j = 1, \dots, n.$$

Then

$$\begin{aligned}
&E_\pi E_\varepsilon\left[\left\| n^{-1/2} \sum_{j=1}^n \varepsilon_j X_j^2 \right\|_\delta \prod_{j=1}^n \beta_j\right] \\
&\quad = E_\pi E_\varepsilon\left[\left\| n^{-1/2} \sum_{j=1}^n \varepsilon_j \big(W_j + rX_j(t_0)\big)^2 \right\|_\delta \prod_{j=1}^n \beta_j\right] \\
&\quad \le E_\pi E_\varepsilon\left[\left\| n^{-1/2} \sum_{j=1}^n \varepsilon_j W_j^2 \right\|_\delta\right] + 2E_\pi E_\varepsilon\left[\left\| rn^{-1/2} \sum_{j=1}^n \varepsilon_j X_j(t_0) W_j \right\|_\delta \prod_{j=1}^n \beta_j\right] \qquad (2.26)\\
&\qquad + E_\pi E_\varepsilon\left[\left\| r^2 n^{-1/2} \sum_{j=1}^n \varepsilon_j X_j^2(t_0) \right\|_\delta \prod_{j=1}^n \beta_j\right] \\
&\quad = \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

Note that W is a Gaussian process and so by Theorem 2.3, W^2 satisfies the CLT. The fact that

$$\lim_{\delta \to 0} \sup_n (\mathrm{I}) = 0 \tag{2.27}$$

follows from Proposition 2.2(iv).

Furthermore, by the same argument we used prior to (2.15),

$$\begin{aligned}
\mathrm{II} &= 2E_{\varepsilon, X(t_0)} E_W\left[\left\| rn^{-1/2} \sum_{j=1}^n \varepsilon_j X_j(t_0) W_j \right\|_\delta \prod_{j=1}^n \beta_j\right] \\
&\le 2E_{\varepsilon, X(t_0)} E_W\left[\left\| r\left(\frac{1}{n} \sum_{j=1}^n X_j^2(t_0)\right)^{1/2} W \right\|_\delta \prod_{j=1}^n \beta_j\right] \\
&\le 2E_{\pi_{t_0}}\left(\frac{1}{n} \sum_{j=1}^n X_j^2(t_0)\right)^{1/2} E_\pi \|rW\|_\delta \qquad (2.28)\\
&\le 2\big(\|r\|_\infty E_\pi \|W\|_\delta + \|r\|_\delta E_\pi \|W\|_\infty\big)\left[E_{\pi_{t_0}}\left(\frac{1}{n} \sum_{j=1}^n X_j^2(t_0)\right)\right]^{1/2},
\end{aligned}$$

where

$$E_{\pi_{t_0}}\left(\frac{1}{n}\sum_{j=1}^{n} X_j^2(t_0)\right) = E_{\pi}\left[\left(\frac{1}{n}\sum_{j=1}^{n} X_j^2(t_0)\right)\prod_{j=1}^{n}\beta_j\right] = \frac{E_\pi X^4(t_0)}{E_\pi X^2(t_0)} = 3E_\pi X^2(t_0). \tag{2.29}$$

Since $\lim_{\delta\to 0} E_\pi \|W\|_\delta = 0$ and $\lim_{\delta\to 0}\|r\|_\delta = 0$, we see from (2.28) and (2.29) that

$$\lim_{\delta\to 0}\sup_n (\mathrm{II}) = 0. \tag{2.30}$$

Finally, since $\{X_j^2(t_0)\}_{j=1}^\infty$ is a sequence of real-valued random variables, we have

$$\begin{aligned}\mathrm{III} &\le \|r\|_\delta^2 E_{\pi_{t_0}} E_\varepsilon\left(n^{-1/2}\sum_{j=1}^n \varepsilon_j X_j^2(t_0)\right) \le \|r\|_\delta^2\left(E_{\pi_{t_0}} X_j^4(t_0)\right)^{1/2}\\ &= \|r\|_\delta^2\left(\frac{E_\pi X^6(t_0)}{E_\pi X^2(t_0)}\right)^{1/2}\\ &= \sqrt{15}\,\|r\|_\delta^2 E_\pi X^2(t_0)\end{aligned}$$

and so

$$\lim_{\delta\to 0}\sup_n (\mathrm{III}) = 0. \tag{2.31}$$

Combining (2.27), (2.30) and (2.31), we get (2.24). This completes the proof of Corollary 2.8. □

PROOF OF THEOREM 1.2. Let $\{L_j\}_{j=1}^\infty$ and $\{\eta_j\}_{j=1}^\infty$ be i.i.d. copies of L and η as given in Theorem 1.1 and let $\{\varepsilon_j\}_{j=1}^\infty$ be a Rademacher sequence independent of L and η. It follows from Theorem 1.1 and the triangle inequality that

$$\begin{aligned}&\lim_{\delta\to\infty}\sup_n E\left[\left\|n^{-1/2}\sum_{j=1}^n \varepsilon_j L_j\right\|_\delta\right]\\ &\quad\le \lim_{\delta\to\infty}\sup_n E_\pi E_\varepsilon\left[\left\|n^{-1/2}\sum_{j=1}^n \varepsilon_j\eta_j\right\|_\delta\left|\prod_{j=1}^n\left(\frac{\eta_j(t_0)}{E_\pi\eta_j(t_0)}\right)\right|\right]\\ &\qquad+\lim_{\delta\to\infty}\sup_n E_\pi E_\varepsilon\left[\left\|n^{-1/2}\sum_{j=1}^n\varepsilon_j\eta_j\right\|_\delta\right].\end{aligned} \tag{2.32}$$

We showed in the proof of Corollary 2.8 that the first term on the right of the inequality in (2.32) is equal to 0. That the last term in (2.32) is equal to 0 follows from Proposition 2.2(iv). Thus the first term in (2.32) equals 0 and, again by Proposition 2.2, this implies that L satisfies the CLT on all compact sets of R^d (and hence, by definition, on R^d). □

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