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ERGODICITY FOR SPIN SYSTEMS WITH STIRRINGS[1]

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We study a class of particle systems which includes finite-range spin systems and combinations of those systems with stirring processes. We give sufficient conditions for ergodicity of the processes. The method is based on a graphical representation of the system and construction of a "generalized dual process."

1. Introduction. A spin-flip system is a Markov process in the state space $\mathbf{X} = \{-1, 1\}^{\mathbb{Z}^d}$. At each site of $\mathbb{Z}^d$ is located a spin taking the values $+1$ or -1. Each site, after an exponentially distributed random time, is updated by a rule that depends on the configuration in the site's neighborhood. Such processes have been used to study the Gibbs states associated with some potential. Reversible spin-flip systems having as invariant measure a Gibbs state related to some potential are called stochastic Ising models.

A particle system is "ergodic" if (a) there exists a unique invariant measure and (b) starting from any initial measure, the process converges to that invariant measure. Ergodicity for an attractive Ising model is equivalent to absence of phase transition for the related Gibbs state. Indeed, the study of Gibbs states allowed the proof of ergodicity of attractive short-range one-dimensional Ising models and the absence of ergodicity for nearest-neighbor two-dimensional Ising models, two of the most important results on the stochastic Ising model.

A criterion for the ergodicity of spin systems has been given by Dobrushin [4]. Dobrushin established that $M < \varepsilon$ is a sufficient condition for the ergodicity of a spin-flip system, where M represents the maximal influence of other sites on the spin-flip rate at any site, and ε is in some sense the minimum spin-flip rate at any site. This approach was also studied by Gray and Griffeath [11], Sullivan [19] and Holley and Stroock [14, 15]. A review can be found in Liggett [17]. Recent related results are obtained by Aizenman and Holley [1].

In this paper we exploit a "generalized graphical representation" of the systems to obtain some results for spin systems and combinations of spin systems with stirring processes. In Theorem 2.5, we show that $M \leq 2\varepsilon$ is a sufficient condition for the ergodicity of the one-dimensional nearest-neighbor Ising model. The result is interesting because no attractiveness conditions are imposed. In the attractive case, Gray [7] proved that any one-dimensional

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nearest-neighbor Ising model is ergodic. We note in Fact 4.9 and Remark 4.10, that Gray's result can be easily extended to "anti-attractive" systems.

Our approach applies to spin-flip stirring processes [2]. These processes are also called "Glauber–Kawasaky" dynamics or "reaction–diffusion processes." They combine the spin-flip dynamics described above with "stirring dynamics." These processes arise when one wants to derive hydrodynamical equations of the reaction–diffusion type [2]. The stirring dynamics can be described informally by saying that each pair of sites $x, y \in \mathbb{Z}^d$, after an exponentially distributed random time with parameter $p(x, y)$, exchanges their spins. The function $p(x, y)$ is assumed symmetric; i.e., $p(x, y) = p(y, x)$. This process is also known as "symmetric simple exclusion" [17]. In Theorem 2.1 we give a sufficient condition for the ergodicity of spin-flip stirring processes that depends only on the flip rates. Our condition is an inequality:

$$(K - m)(r - 1) \leq 2m, \tag{1.1}$$

where K (respectively, m) is the maximum (minimum) spin-flip rate and r is the (maximum) number of sites determining the spin-flip rate at any given site. In the absence of stirring dynamics, our condition implies Dobrushin's condition $M < \varepsilon$.

The method also applies when various spin-flip processes are combined. We consider spin-flip systems with generators L_i and respective K_i, m_i and r_i, and study the process with generator $L = \sum_i L_i$. We show in Theorem 2.2 that $\sum_i (K_i - m_i)(r_i - 1) \leq 2\sum_i m_i$ is a sufficient condition for ergodicity even if (1.1) does not hold for some of the processes. Moreover, we prove that the addition of a "voter model" generator to L does not affect the condition for ergodicity.

Finally, we consider the unique invariant measure of a spin-flip stirring process in the regime $(K - m)(r - 1) < 2m$ and prove that, when the rate of stirring increases to ∞, this invariant measure approaches a product measure.

The proofs of our theorems make use of a "generalized dual process." The method consists of constructing a graphical realization of the process and then studying a reverse-time process as is done in the usual duality theory. The difference is that our generalized dual processes are not Markovian. We overcome this difficulty by dominating the dual structure with Markov processes such as branching processes and one-dimensional random walks. For reviews of graphical methods and duality see Griffeath [12], Durrett [5] and Liggett [17]; generalized duality for attractive systems can be found in Gray [8]. The idea of generalized duality that we exploit here appeared first in De Masi, Ferrari and Lebowitz [2]. The comparison of the dual with subcritical branching processes to get a criterion for ergodicity was used by Holley and Stroock [16].

Our technique can also be applied to discrete-time processes (probabilistic automata). This topic will be discussed in [6]. Using different techniques, Gray [9] proves ergodicity for certain discrete-time majority-vote models in one dimension.

In the next section we introduce our processes and state the theorems. In Section 3 we construct the generalized graphical representation and prove Theorems 2.1 and 2.2. In Section 4 we prove one-dimensional results, while in Section 5 we study the behavior of the invariant measures when the rate of stirring goes to ∞.

2. Definitions and results. A spin system is a Markov process on the state space $\mathbf{X} = \{-1, 1\}^{\mathbb{Z}^d}$ with pregenerator defined on cylinder functions, given by (the subscript "g" below is for Glauber dynamics)

$$L_g f(\sigma) = \sum_{x \in \mathbb{Z}^d} c(x, \sigma)[f(\sigma^x) - f(\sigma)], \tag{2.1}$$

where the configuration $\sigma^x \in \mathbf{X}$ is given by

$$\sigma^x(z) = \begin{cases} \sigma(z) & \text{if } z \neq x, \\ -\sigma(z) & \text{if } z = x, \end{cases} \tag{2.2}$$

and the rates $c(x, \sigma)$ are nonnegative functions depending on σ only through a finite set $R_x \subset \mathbb{Z}^d$ of sites depending on x: i.e., $\sigma(y) = \xi(y)$ for all $y \in R_x$ implies $c(x, \sigma) = c(x, \xi)$. Furthermore, to guarantee the existence of the process, we assume [17]

$$M := \sup_{x \in \mathbb{Z}^d} \sum_{y \neq x} \sup_{\sigma} |c(x, \sigma) - c(x, \sigma^y)| < \infty. \tag{2.3}$$

A "stirring process" (or symmetric simple exclusion process) is a Markov process on $\mathbf{X}$ with pregenerator (the subscript "s" below is for stirring)

$$L_s f(\sigma) = \sum_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} p(x, y)[f(\sigma^{xy}) - f(\sigma)], \tag{2.4}$$

where σ^{xy} is defined by

$$\sigma^{xy}(z) = \begin{cases} \sigma(z) & \text{if } z \neq x, y, \\ \sigma(x) & \text{if } z = y, \\ \sigma(y) & \text{if } z = x. \end{cases} \tag{2.5}$$

The function $p(x, y)$ is assumed symmetric; i.e., $p(x, y) = p(y, x)$ for all $x, y \in \mathbb{Z}^d$ and uniformly integrable in x; i.e.,

$$\sup_x \sum_y p(x, y) < \infty. \tag{2.6}$$

Now fix a nonnegative constant a and define the generator

$$L = L_g + aL_s. \tag{2.7}$$

The conditions imposed on p and c [in (2.3) and (2.6)] are sufficient to guarantee the existence of a Markov process $\sigma_t \in \mathbf{X}$ such that for all nonnegative a, the semigroup $S(t)$ with generator L satisfies $S(t)f(\sigma) = E_\sigma f(\sigma_t)$, where f is a continuous function and E_σ is the expectation with respect to the process σ_t when the initial configuration is σ. The standard reference for existence problems is Liggett [17]. A process with generator L given by (2.7) is called a "*spin-flip stirring process*."

A process is "ergodic" if (a) there exists a unique invariant measure μ for the process and (b) for any measure ν on $\mathbf{X}$, $\lim_{t\to\infty} \nu S(t) = \mu$. The limit is understood as the weak limit of the sequence of measures $\nu S(t)$. This is equivalent to $\nu S(t) f \to \mu f$, when $t \to \infty$, for any cylinder function f. A process is "exponentially ergodic" if for any cylinder function f, there exist positive constants $\alpha_1 = \alpha_1(f), \alpha_2$ such that for any initial measure ν, $|\nu S(t) f - \mu f| < \alpha_1 e^{-\alpha_2 t}$. Let f_A be a cylinder function depending on the finite set of coordinates A. In the theorems below we consider $\alpha_1(f_A) = C|A| \, \|f_A\|$, where C is a positive constant depending on the rates of the process, $|A|$ is the number of elements of A and $\|f\| = \sup_\eta f(\eta)$. For a spin-flip process as defined in (2.1), define

$$
\begin{aligned}
m &= \inf\{c(x,\sigma) \colon x \in \mathbb{Z}, \sigma \in \mathbf{X}\},\\
K &= \sup\{c(x,\sigma) \colon x \in \mathbb{Z}, \sigma \in \mathbf{X}\}, \qquad (2.8)\\
r &= \max_{x \in \mathbb{Z}} |R_x|,
\end{aligned}
$$

where $|R|$ is the number of elements of the set R. Informally, m represents the minimal rate of spin flip, K the maximal rate and r the maximum number of sites on which the flip rate depends.

THEOREM 2.1. *Let σ_t be a process with generator L given by* (2.7). *Assume $m > 0$. If L_g satisfies the condition*

$$(r-1)(K-m) < 2m, \qquad (2.9)$$

then the process σ_t is exponentially ergodic. If equality holds in (2.9), *then the process is ergodic.*

REMARK 2.10. In order to compare the condition that Theorem 2.1 gives for usual spin systems ($a = 0$) and Dobrushin's criterion, define

$$\varepsilon = \inf_{x,\sigma} [c(x,\sigma) + c(x,\sigma^x)]. \qquad (2.11)$$

The value M, defined in (2.3) represents, intuitively, the effect of other sites on the spin-flip rate of a given site and ε is, in some sense, the minimum flip rate. Dobrushin's criterion says that $M < \varepsilon$ is sufficient for ergodicity of the process [17]. The condition $(r-1)(K-m) < 2m$ implies $M < \varepsilon$ because $2m \le \varepsilon$ and $M \le (K-m)(r-1)$.

Our next result is a generalization of Theorem 2.1. Let L_v be a spin-flip generator whose rates are defined by

$$c(x,\sigma) = \sum_y \{v(x,y)[1 + \sigma(x)\sigma(y)] + \bar{v}(x,y)[1 - \sigma(x)\sigma(y)]\},$$

where the functions v and $\bar{v}$ are uniformly summable. A process with such rates is a combination of the "voter model" and the "anti-voter model" [17].

THEOREM 2.2. *Let $L_g = \sum_{i=1}^{l} L_i$, where L_i are spin-flip generators as defined in* (2.1), *with corresponding K_i, m_i and r_i, as defined in* (2.8). *Assume $\sum_i m_i > 0$. If*

$$\sum_i (r_i - 1)(K_i - m_i) < 2\sum_i m_i, \tag{2.12}$$

then the process with generator $L_g + aL_s + bL_v$ is exponentially ergodic. If equality holds in (2.12), *then the process is ergodic.*

REMARK 2.13. Sometimes it is possible to decompose a spin-flip generator with rates depending on r sites, into a sum of generators with rates depending on fewer sites. When this is possible, the above condition for ergodicity may be weaker than the one given by Theorem 2.1, and even than Dobrushin's condition $M < \varepsilon$. Next we present an example along these lines.

EXAMPLE 2.14 (One-dimensional nearest-neighbor Glauber-stirring dynamics [2]). Let μ be the Gibbs state with nearest-neighbor interaction, defined by

$$\begin{aligned} &\mu(\sigma(x) = \xi(x), x \in F | \sigma(x) = \xi(x), x \notin F) \\ &\quad = Z^{-1}(\xi)\exp\Big\{\beta \sum_{x,y} \xi(x)\xi(y)\Big\}, \end{aligned} \tag{2.15}$$

where β is a parameter (the "inverse temperature") and the sum runs over the set $\{(x,y) \in \mathbb{Z}^2 \colon x \in F \subset \mathbb{Z}, y \in \mathbb{Z}, |x-y| = 1\}$. The normalizing constant $Z(\xi)$ makes μ a probability. Let L_g be a spin-flip generator with rates satisfying the following condition:

$$\frac{c(x,\sigma)}{c(x,\sigma^x)} = \frac{\exp\{\beta[\sigma^x(x-1)\sigma^x(x) + \sigma^x(x)\sigma^x(x+1)]\}}{\exp\{\beta[\sigma(x-1)\sigma(x) + \sigma(x)\sigma(x+1)]\}}. \tag{2.16}$$

Then μ is reversible for the spin-flip process with generator L_g. This process is called a Glauber dynamics or a stochastic Ising model. Define

$$\begin{aligned} A_1(x) &= \{\sigma \in \mathbf{X} \colon \sigma(x-1) \neq \sigma(x) \neq \sigma(x+1)\}, \\ A_2(x) &= \{\sigma \in \mathbf{X} \colon \sigma(x-1) \neq \sigma(x+1)\}, \\ A_3(x) &= \{\sigma \in \mathbf{X} \colon \sigma(x-1) = \sigma(x) = \sigma(x+1)\}. \end{aligned} \tag{2.17}$$

For each fixed x, $\{A_i(x)\}_i$ is a partition of $\mathbf{X}$ and $c(x,\sigma)$ is constant in $A_i(x)$, $i = 1,2,3$. Hence the generator L_g can be rewritten as follows:

$$L_g f(\sigma) = \sum_{x \in \mathbb{Z}} \sum_{i=1}^{3} \lambda_i 1\{\sigma \in A_i(x)\}[f(\sigma^x) - f(\sigma)], \tag{2.18}$$

where $1\{\cdot\}$ is the indicator function of the set $\{\cdot\}$ and the constants λ_i are given by

$$\lambda_i = c(x,\sigma) \quad \text{for some } \sigma \in A_i(x). \tag{2.19}$$

Notice that (2.16) only imposes that $\lambda_1/\lambda_3 = e^{4\beta}$, leaving free λ_2. This is justified because λ_2 is the rate for jumping between configurations with the same "energy." Fixing the total rate $\lambda := \lambda_1 + \lambda_2 + \lambda_3$, we have a one-parameter family of processes L_g with reversible measure μ. Sometimes one can use properties of one of the processes to derive results concerning the others. See Chapter 3 of [17] for examples in this direction.

COROLLARY 2.3. *Let σ_t be a spin-flip stirring process with generator $L = L_g + aL_s$, where L_g is given by* (2.18). *Assume* $\min\{\lambda_1, \lambda_2, \lambda_3\} > 0$. *If*

$$2\max\{|\lambda_2 - \lambda_3|, |\lambda_1 - \lambda_2|\} < \min\{2\lambda_2, \lambda_1 + \lambda_3\}, \tag{2.20}$$

then σ_t is exponentially ergodic. If equality holds in (2.20), *then the process is ergodic.*

REMARK 2.21. Notice that (2.20) is equivalent, in this context, to Dobrushin's condition $M < \varepsilon$.

Corollary 2.3 gives a partial answer to a question raised in [2]. In that paper, a process with generator $L_g + aL_s$ was considered, where the stirring part L_s is the nearest-neighbor simple exclusion process [$p(x, x \pm 1) = \frac{1}{2}$ and $p(x,y) = 0$ if $y \neq x \pm 1$ in (2.4)] and L_g is given by (2.18) with rates

$$\lambda_1 = (1+\gamma)^2, \qquad \lambda_2 = 1 - \gamma^2, \qquad \lambda_3 = (1-\gamma)^2, \tag{2.22}$$

where $\gamma = \tanh\beta$, $\beta > 0$. In this case, Corollary 2.3 implies that a sufficient condition for ergodicity of the process is $\gamma \le \frac{1}{2}$. On the other hand, this process is related to a reaction–diffusion equation via the hydrodynamical limit. In order to define the latter, let $S_a(t)$ be the semigroup corresponding to the generator $L_g + aL_s$ and let $\{\nu^a\}$ be a family of product measures with density $\nu^a(\eta([ra^{1/2}])) = u_0(r)$, $r \in \mathbb{R}$, where $[\cdot]$ is the integer part and u_0 is a smooth function. Define now $u_a(r,t) := \nu^a S_a(t)\sigma([ra^{1/2}])$. In [2] it was proved that the hydrodynamical limit $\lim_{a\to\infty} u_a(r,t)$ exists and equals $u(r,t)$, the solution of the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + F(u), \qquad u(r,0) = u_0(r),$$

where $F(u) = \nu_u L_g \sigma(0) = -2(1-2\gamma)u - 2\gamma^2 u^3$. Here ν_u is a product measure with magnetization $\nu_u\sigma(0) = u$. The above reaction–diffusion equation admits a unique stationary solution for $\gamma \le \frac{1}{2}$ and three solutions for $\gamma > \frac{1}{2}$. The question posed in the introduction of [2] was whether the existence of more than one stationary state can be seen at a microscopic level ($a < \infty$). Our result is unsatisfactory because it establishes the ergodicity of the process in the region $\gamma \le \frac{1}{2}$, where there exists only one stationary solution for the macroscopic equation.

For one-dimensional spin systems (without the stirring part), it is possible to get better results, as shown by the following example.

EXAMPLE 2.23 (One-dimensional nearest-neighbors spin systems). Consider the previous example with $a = 0$ (a spin system). The nearest-neighbor condition yields a factor of 2: Condition (2.24) is equivalent to $M < 2\varepsilon$ in this context.

THEOREM 2.4. *Let σ_t be a one-dimensional spin system with generator given by* (2.18). *Assume $\min\{\lambda_1, \lambda_2, \lambda_3\} > 0$. Then, if*

$$\max\{|\lambda_2 - \lambda_3|, |\lambda_1 - \lambda_2|\} < \min\{2\lambda_2, \lambda_1 + \lambda_3\}, \tag{2.24}$$

the process is exponentially ergodic. Equality in (2.24) *implies that the process is ergodic.*

3. Generalized dual processes.

Consider the spin-flip rates $c(x, \sigma)$ of (2.1), depending on the finite set R_x. Write

$$c(x, \sigma) = \sum_{j \in J} \lambda_j(x) 1\{\sigma \in A_j(x)\}, \tag{3.1}$$

where $\lambda_{j+1}(x) \geq \lambda_j(x) > 0$ and $A_j(x)$ are cylinder sets of $\mathbf{X}$ depending on the set $R_x \subset \mathbb{Z}^d$, such that $c(x, \sigma) = \lambda_j(x)$ for all $\sigma \in A_j(x)$. For each x the family $\{A_j(x)\}_j$ is a partition of $\mathbf{X}$. The set of labels $J \subset \mathbb{N}$ is finite. The generator L_g is written as

$$\begin{aligned} L_g f(\sigma) &= \sum_{x \in \mathbb{Z}^d} \sum_{j \in J} \lambda_j(x) 1\{\sigma \in A_j(x)\}[f(\sigma^x) - f(\sigma)] \\ &= \sum_{x \in \mathbb{Z}^d} \Bigg(\sum_{j \in J} \bar{\lambda}_j(x) \sum_{l \geq j} 1\{\sigma \in A_l(x)\}[f(\sigma^x) - f(\sigma)] \\ &\qquad + \lambda_0(x) \sum_{l \geq 0} 1\{\sigma \in A_l(x)\}[f(\sigma^x) - f(\sigma)] \Bigg), \end{aligned} \tag{3.2}$$

where $\bar{\lambda}_j(x) = \lambda_j(x) - \lambda_{j-1}(x)$, $j \geq 1$. Observe that since $\sum_{l \geq 0} 1\{\sigma \in A_j(x)\} = 1$, the last line of (3.2) can be written as a noise:

$$2\lambda_0\left[\tfrac{1}{2}\left[f(\sigma^{x,-1}) - f(\sigma)\right] + \tfrac{1}{2}\left[f(\sigma^{x,+1}) - f(\sigma)\right]\right], \tag{3.3a}$$

where the configurations $\sigma^{x,+1}$ and $\sigma^{x,-1}$ are defined by

$$\sigma^{x,\pm 1}(z) = \begin{cases} \pm 1 & \text{if } z = x, \\ \sigma(z) & \text{otherwise.} \end{cases} \tag{3.3b}$$

Next we construct the graphical representation of our process. This is basically the construction given in [2], with the addition of Bernoulli random variables related to the noise (3.3).

Graphical construction of the process with generator $L = L_g + aL_s$.

1. The Glauber dynamics. With each coordinate x, associate Poisson point processes (Ppp's) with rates $\bar{\lambda}(x) = \max_j\{\bar{\lambda}_j(x)\}$ and $2\lambda_0(x)$, respectively. A realization of a Ppp is an increasing sequence of times. At each of these times we say that there is a mark of the corresponding Ppp. Each mark corresponding to the process with rate $\bar{\lambda}(x)$ is marked j, $j \geq 0$, with probability $(\bar{\lambda}_j(x) - \bar{\lambda}_{j-1}(x))/\bar{\lambda}(x)$. Each mark of the Ppp with rate $2\lambda_0(x)$ is marked δ. Notice that, the j-marks of site x are distributed according to a Ppp of rate $\lambda_j(x) - \lambda_{j-1}(x)$. Analogously, the δ-marks of site x form a Ppp of rate $2\lambda_0(x)$.

2. The stirring dynamics. At each pair of sites (x, y) associate a Ppp with parameter $ap(x, y)$. At each mark of this process put a double arrow linking sites x and y.

All these marked Ppp are mutually independent. Call $(\Omega, \mathbf{F}, P)_g$ [respectively, $(\Omega, \mathbf{F}, P)_s$] the probability space induced by the spin-flip (stirring) family of marked Ppp. Consider also a family of independent variables $\{B_{x,n}: x \in \mathbb{Z}^d,\ n \geq 1\}$, with Bernoulli distribution

$$P(B_{x,n} = 1) = P(B_{x,n} = -1) = \tfrac{1}{2}. \tag{3.4}$$

Let $(\Omega, \mathbf{F}, P)_0$ be the probability space associated to those random variables. Call $(\Omega, \mathbf{F}, P)$ the direct product of the three probability spaces just defined. Discard the null event corresponding to the occurrence of two marks simultaneously at any given time.

Given a configuration $\omega \in \Omega$ of marked Ppp, construct σ_t $(= \sigma_{t,\omega})$ as follows: Suppose that the configuration at time T^- is σ_{T^-} and a mark of ω is present at site x, at time T. There are three possibilities.

1(a). A j-mark. In this case, if the configuration σ_{T^-} belongs to at least one of the sets $A_l(x)$, $l \geq j$, then flip the spin at x, so that $\sigma_T = (\sigma_{T^-})^x$. Otherwise nothing happens.

1(b). A δ-mark. Assuming that this is the nth δ-mark involved with site x, then $\sigma_T(x) = B_{x,n}$. In other words, the spin at x at time T is changed to 1 with probability $\frac{1}{2}$ and to -1 with probability $\frac{1}{2}$, independent of everything.

2. A double arrow linking x and y. In this case the contents of sites x and y are interchanged; i.e., $\sigma_T = (\sigma_{T^-})^{x,y}$.

Let $0 < T_1 \leq \cdots \leq T_{n-1} < t$ be the successive marks involving site x in the time interval $[0, t]$. Let $T_0 = 0$ and $T_n = t$. Then we define $\sigma_s(x) = \sigma_{T_i}(x)$, for $s \in [T_i, T_{i+1})$. It can be proven that this is well-defined by approximating the infinite volume process by processes constructed in finite boxes $\Lambda_N \uparrow \mathbb{Z}^d$. Furthermore, it is easy to see that the process constructed as above has generator L.

Construction of the generalized dual process. Suppose that, for a given time interval $[0, t]$, we have a realization of the marked Ppp described above. We reverse the time direction calling $\hat{s} = t - s$ and for any finite set $D \subset \mathbb{Z}^d$

we construct a space–time branching structure contained in $\mathbb{Z}^d \times [\hat{0}, \hat{t}]$ with base $(D, \hat{0})$, and top $(D_{\hat{t}}, \hat{t})$. Now write t for $\hat{t}$, but remember that we are going back in time. We proceed by induction. Suppose that the spatial projection of the structure at time s is D_s. Let T be the first Poisson mark after s involving some site of D_s. There are the following possibilities.

1(a). A j-mark involving site $x \in D_{T^-}$. In this case, the point (x, T) is marked j and the set D_T will be $D_{T^-} \cup R_x$.

1(b). A δ-mark involving site $x \in D_{T^-}$. In this case, the point (x, T) is marked δ and the set D_T will be $D_{T^-} \setminus \{x\}$.

2(a). A double arrow involving site $x \in D_{T^-}$ and $y \notin D_{T^-}$. In this case, the points (x, T) and (y, T) are marked with s and the set D_T will be $D_{T^-} \cup \{y\} \setminus \{x\}$.

2(b). A double arrow involving sites $x, y \in D_{T^-}$. In this case points (x, T) and (y, T) are marked with s and $D_T = D_{T^-}$.

According to this construction, for each finite set D and time t, we are defining a map from the probability space $(\Omega, \mathbf{F}, P)$ into the space of all possible marked branching structures on $\mathbb{Z}^d \times [0, t]$,

$$(3.5\text{a}) \qquad \hat{D}^D_{[0,t]}\colon (\omega, D, t) \mapsto (N, (t_k, x_k, y_k, j_k, D_k), k = 1, \dots, N),$$

where N is the number of marks in the interval $[0, t)$, t_k is the time of occurrence of the kth mark, x_k is the site involved with the kth mark, y_k is the other site involved with the kth mark, if this mark is an s-mark (if not $y_k \equiv x_k$), j_k is the type of the kth mark (δ, s or j, $1 \le j \le |J|$) and finally, D_k is the set of sites in the spatial projection of the structure between times t_k and t_{k+1}. The duality equation is

$$(3.5\text{b}) \qquad \sigma_t^{\sigma}(D) = H\Big(\hat{D}^D_{[0,t]}, B, \sigma\Big), \quad P \text{ a.s.},$$

where $B := \{B_{x,n}\}$ is the sequence of independent Bernoulli random variables with distribution (3.4). These are the variables that one has to use to compute $\sigma_T(x)$ when a δ-mark appears at x at time T. We do not have a formula for H but it is computable for each realization ω, because it is easy to know the value of σ_t on D once we know $\hat{D}^D_{[0,t]}$, the independent random variables $B_{x,i}$ and σ_0. The central idea of this dual construction is this: When the dual process meets a δ-mark at site x, at time $\hat{T}$, it is not necessary to go further in time to know the value of $\sigma_T(x)$, because it is determined at that point by an independent Bernoulli random variable with parameter $1/2$. The idea of representing the noise with Bernoulli random variables was used by Griffeath [12] to prove ergodicity of cancelative processes.

Proof of Theorem 2.1. The proof follows from the construction of the dual structure. The main observation is the following: If the spatial projection of the dual structure started at time $t = \hat{0}$ is empty at time $0 = \hat{t}$, i.e., $D_t^D = \varnothing$, then $\sigma_t(D)$ does not depend on $\sigma_0 = \sigma$. This implies that a sufficient condition for the exponential ergodicity of the process is that, for all finite D, there exist positive constants c, α_2 such that

$$P(D_t^D \neq \varnothing) \leq c|D|e^{-t\alpha_2}. \tag{3.6}$$

We observe now that $|D_t^D|$ can be coupled to a usual branching process $R_t^{|D|} \in \mathbb{N}$, such that $|D_t^D| \leq R_t^{|D|}$ for all t with probability 1. In this branching process, at rate $\lambda + 2m$, where $\lambda = \sup_x \overline{\lambda}(x) = K - m$, each branch dies and is replaced by either r new branches with probability $\lambda/(\lambda + 2m)$ or 0 new branches with probability $2m/(\lambda + 2m)$. The initial state of the branching process is $R_0^{|D|} \equiv |D|$. A sufficient condition for (3.6) is that the average number of branches created at each branching be less than 1 (cf., for example, [13]). This happens when

$$\frac{\lambda r}{\lambda + 2m} < 1, \tag{3.7}$$

which is equivalent to (2.9). Equality in (3.7) also implies $\lim_{t\to\infty} P(D_t^D = \varnothing) = 1$, which implies ergodicity. This completes the proof of Theorem 2.1. □

PROOF OF THEOREM 2.2. (a) Consider first $b = 0$. For each generator L_i we construct $(\Omega, \mathbf{F}, P)_i$ as in the proof of Theorem 2.1 and define

$$(\Omega, \mathbf{F}, P) := (\Omega, \mathbf{F}, P)_s \times \prod_i (\Omega, \mathbf{F}, P)_{g,i} \times \prod_i (\Omega, \mathbf{F}, P)_{0,i}.$$

A typical $\omega \in \Omega$ has marks of type δ_i, (i, j), s, etc. The construction of the process and its generalized dual follows the lines of the previous construction. The branching process $R_t^{|D|}$ dominating $|D_t^D|$ is, in this case, the following: At rate $\theta := \sum_i(K_i - m_i) + \sum_i 2m_i$, each branch dies and, with probability $(K_i - m_i)/\theta$, creates r_i new branches. The condition for this process to die exponentially fast is

$$\frac{\sum r_i(K_i - m_i)}{\sum r_i[(K_i - m_i) + 2m_i]} < 1,$$

while equality implies that the branching process will die with probability 1.

(b) Now assume $b \neq 0$. The process with generator L_v is a linear combination of the voter and the anti-voter model. Each of these processes admits a (coalescing) graphical representation. Take, for instance, the voter model. At each ordered pair of sites (x, y) associate a Ppp with parameter $v(x, y)$. At each mark of this Ppp put an oriented arrow going from y to x. At site x at time T, if an arrow starting at y and ending at x is present, then site x adopts the spin of site y; i.e., $\sigma_T = (\sigma_{T^-})^{x\leftarrow y}$, where

$$\sigma^{x\leftarrow y}(z) = \begin{cases} \sigma(z) & \text{if } z \neq x, \\ \sigma(y) & \text{if } z = x. \end{cases}$$

If there is an arrow from x to y, then site y adopts the spin of site x; i.e., $\sigma_T = (\sigma_{T^-})^{y\leftarrow x}$.

Duality. When an arrow starting at y and ending at $x \in D_{T^-}$ is present, mark points (x, T) and (y, T) with v. D_{T^-} will be $D_{T^-} \setminus \{x\} \cup \{y\}$.

For the anti-voter model do the same, substituting $\bar{v}$ for v and using the following definition of $\sigma^{x \leftarrow y}$:

$$\sigma^{x \leftarrow y}(z) = \begin{cases} \sigma(z) & \text{if } z \neq x, \\ -\sigma(y) & \text{if } z = x. \end{cases}$$

Now it is clear that both the voter and the anti-voter model marks cannot increase the number of elements of the dominating branching process R_t. This completes the proof of Theorem 2.2. □

4. The one-dimensional case. In this section we prove Corollary 2.3 and Theorem 2.4.

PROOF OF THEOREM 2.4. We divide the set of parameters $\{\lambda_1 \geq 0,\ \lambda_2 \geq 0,\ \lambda_3 \geq 0\colon \min_i \lambda_i > 0\}$ into five regions.

CASE 1. $0 < \lambda_2 - \lambda_3 \leq \lambda_1 - \lambda_2$. Rewrite the generator of the one-dimensional process of (2.18) as

$$\begin{aligned} L_g f(\sigma) = \sum_{x \in \mathbb{Z}} \big(&\lambda_3[f(\sigma^x) - f(\sigma)] \\ &+ (\lambda_2 - \lambda_3)1\{\sigma \in A_2(x)\}[f(\sigma^x) - f(\sigma)] \\ &+ (\lambda_1 - \lambda_3)1\{\sigma \in A_1(x)\}[f(\sigma^x) - f(\sigma)]\big) \\ = 2&\lambda_3 L_n + (\lambda_2 - \lambda_3)L_v + (\lambda_1 + \lambda_3 - 2\lambda_2)L_3, \end{aligned} \tag{4.1}$$

where, using (3.3),

$$L_n f(\sigma) = \sum_{x \in \mathbb{Z}} \left(\tfrac{1}{2}[f(\sigma^{x,1}) - f(\sigma)] + \tfrac{1}{2}[f(\sigma^{x,-1}) - f(\sigma)]\right), \tag{4.2}$$

$$L_v f(\sigma) = \sum_{x \in \mathbb{Z}} (1\{\sigma \in A_2(x)\} + 2.1\{\sigma \in A_1(x)\})[f(\sigma^x) - f(\sigma)], \tag{4.3}$$

$$L_i f(\sigma) = \sum_{x \in \mathbb{Z}} 1\{\sigma \in A_i(x)\}[f(\sigma^x) - f(\sigma)], \qquad i = 1,2,3. \tag{4.4}$$

The assumption $0 \leq \lambda_1 + \lambda_3 - 2\lambda_2$ guarantees that the last member in (4.1) is the sum of three generators. The process with generator L_n is a noise: At rate 1, the spin at x chooses a new value between $+1$ and -1 with probability $\frac{1}{2}$ independent of everything. The process with generator L_v is a nearest-neighbor voter model: When the number of neighbors of x with different spin is 1, the spin-flip rate is 1 and when that number is 2, the rate is 2. Finally, the process with generator L_3 is a majority-vote model: at rate 1, the spin at x looks to its nearest neighbors; if both of them have the opposite spin, the spin at x flips aligning the three spins. If not, nothing happens. More details on these processes can be found in [17], [12] and [9].

The generalized dual is a process of traveling particles in $\mathbb{Z}$; behaving as coalescing random walks at rate $\lambda_2 - \lambda_3$; branching to empty nearest-neighbor sites at rate $\lambda_1 + \lambda_3 - 2\lambda_2$ and dying at rate $2\lambda_3$. The number of particles of this process is dominated by a nearest-neighbor random walk in $\mathbb{N}$ with rates $2[\lambda_1 + \lambda_3 - 2\lambda_2 + \lambda_2 - \lambda_3]$ and $2[2\lambda_3 + \lambda_2 - \lambda_3]$ for right and left jumps, respectively. The origin is an absorbing point. This process goes to the origin exponentially fast if

$$(\lambda_1 + \lambda_3 - 2\lambda_2) < 2\lambda_3. \tag{4.5}$$

This proves Theorem 2.4 in Case 1.

REMARKS 4.6. (1) Actually, (4.1) to (4.4) imply that, under the conditions of case 1, any nearest-neighbor one-dimensional Glauber dynamics has a dual process in the usual sense. Its dual will be a combination of the dual of the voter model (coalescing random walks), the dual of the noise process (4.2) (death of particles) and the dual of the majority-vote model. The last model was studied by Gray [9]. (2) Choosing $\lambda_2 = (\lambda_1 + \lambda_3)/2$, the one-dimensional attractive nearest-neighbor Glauber dynamics with generator L_g of (4.1) is just a voter model with noise. The exponential ergodicity for this model is immediate for any β, as the dual branching contains only δ-marks. In fact, this model (without stirrings) was the one studied by Glauber. However, notice that the hydrodynamical equation given by this model has a linear term instead of the cubic term obtained from (2.22); thus in this case there is only one stationary solution also for the macroscopic equation. (3) Taking $\lambda_2 = \lambda_3$ in (4.1), the Glauber dynamics is just a majority-vote model with noise.

CASE 2. $0 \le \lambda_1 - \lambda_2 \le \lambda_2 - \lambda_3$. Using (4.2) to (4.4), write

$$L_g = 2\lambda_3 L_n + \left(\frac{\lambda_1 - \lambda_3}{2}\right) L_\nu + \left(\lambda_2 - \lambda_3 - \frac{\lambda_1 - \lambda_3}{2}\right) L_2. \tag{4.7}$$

Analogously to the first case, we get exponential ergodicity when $\lambda_2 - \lambda_3 - (\lambda_1 - \lambda_3/2 < 2\lambda_3$. In this case, this is equivalent to $\lambda_1 - \lambda_2 \le \lambda_2 - \lambda_3 \le 2\lambda_3$, which is equivalent to (2.24).

CASE 3. $\lambda_2 \le \lambda_1$, $\lambda_2 \le \lambda_3$. In this case,

$$L_g = 2\lambda_2 L_n + (\lambda_1 - \lambda_2) L_1 + (\lambda_3 - \lambda_2) L_3 \tag{4.8}$$

and the condition is $\max\{\lambda_1 - \lambda_2, \lambda_3 - \lambda_2\} < 2\lambda_2$, which is equivalent to (2.24).

CASE 4. $0 \le \lambda_2 - \lambda_1 \le \lambda_3 - \lambda_2$. Let $A\sigma$ be the configuration given by $A\sigma(2k+1) = \sigma(2k+1)$, $A\sigma(2k) = -\sigma(2k)$, $k \in \mathbb{Z}$. Let $\xi_t = A\sigma_t$. This case follows from Case 1, by checking that the rates of ξ_t satisfy the conditions of Case 1 and the following easily proven fact.

FACT 4.9. The process σ_t is ergodic iff ξ_t is.

CASE 5. $0 \le \lambda_3 - \lambda_2 \le \lambda_2 - \lambda_1$. This is similar to Case 4, using Case 2 and Fact 4.9.

We have proven exponential ergodicity. When the strict inequalities are replaced by equality, it suffices to observe that the identities are sufficient for random walk to be eventually absorbed at the origin with probability 1. □

REMARK 4.10. Fact 4.9 can be used in greater dimensions. Define A by $A\sigma(x_1,\dots,x_d) = \sigma(x_1,\dots,x_d)$ if $\sum x_i$ is even and $A\sigma(x_1,\dots,x_d) = -\sigma(x_1,\dots,x_d)$, otherwise. Hence all results about ergodicity of attractive nearest-neighbor spin systems extend to anti-attractive systems. An "attractive" spin-flip system is a system whose rates satisfy (a) $c(x,\sigma) \ge c(x,\xi)$ if $\sigma(x) = \xi(x) = 1$ and $\sigma \ge \xi$ coordinatewise, and (b) $c(x,\sigma) \le c(x,\xi)$ if $\sigma(x) = \xi(x) = 0$ and $\sigma \le \xi$. A nearest-neighbor spin system ξ_t is called "anti-attractive," if the process $\sigma_t := A\xi_t$ is attractive.

PROOF OF COROLLARY 2.3. Divide the set of parameters into five regions, as we did in the proof of Theorem 2.5. Then apply Theorem 2.2 to Cases 1, 2 and 3. For Case 4 write

$$L_g = 2\lambda_1 L_n + (\lambda_2 - \lambda_1)L_v + (\lambda_1 + \lambda_3 - 2\lambda_2)L_1,$$

where $L_v := L_2 + 2L_3$ is an anti-voter model satisfying the conditions of the generator L_v of Theorem 2.2. Hence this case follows by applying that theorem. Case 5 is solved analogously by writing

$$L_g = 2\lambda_1 L_n + \left(\frac{\lambda_3 - \lambda_1}{2}\right)L_v + \left(\lambda_2 - \lambda_1 - \frac{\lambda_3 - \lambda_1}{2}\right)L_2. \qquad \square$$

5. Asymptotic behavior of the invariant measures. In the following theorem we study the asymptotic behavior of the unique invariant measure in the exponential ergodic regime as the rate of stirring tends to ∞. Let ν_u be the product measure with magnetization u, defined by

$$\nu_u\{\sigma(x) = 1 : x \in A\} = \left(\frac{1+u}{2}\right)^{|A|}. \tag{5.1}$$

THEOREM 5.1. *Let σ_t be a spin-flip stirring process with generator $L_a = L_g + aL_s$ with L_g satisfying the conditions* (2.9). *Assume that $c(x,\sigma) = c(x,-\sigma)$, where $(-\sigma)(z) = -\sigma(z)$ for all z. Let μ_a be the unique invariant measure for the process and ν_0 the product measure with average magnetization* 0. *Then, as $a \to \infty$, μ_a converges weakly to ν_0.*

PROOF. It is sufficient to prove that, if $f_D(\sigma) = 1\{\sigma(x) = 1,\ x \in D\}$, then $\lim_{a\to\infty} \mu_a f_D = (\frac{1}{2})^{|D|}$. To avoid heavy use of notation, we assume $|D| = 2$, $D = \{x,y\}$, the proof being essentially the same for the other D as we will see below. The function f is defined by $f(\sigma) = 1\{\sigma(x) = 1, \sigma(y) = 1\}$. Since μ_a is invariant, $\mu_a f = \mu_a S(t) f$, and by the duality equation (3.5),

$$\begin{aligned}\mu_a S(t) f &= E\left[H\left(\hat{D}^{\{x,y\}}_{[0,t]}\right)\right]\\ &= E\left[H\left(\hat{D}^{\{x,y\}}_{[0,t]}\right), C_{[0,t]}\right] + E\left[H\left(\hat{D}^{\{x,y\}}_{[0,t]}\right), C^c_{[0,t]}\right],\end{aligned} \tag{5.2}$$

where $C_{[0,t]} = \{\hat{D}^{\{x\}}_{[0,t]} \cap \hat{D}^{\{y\}}_{[0,t]} = \varnothing\}$ is the event in which the two structures thought of as subsets of $\mathbb{Z}^d \times [0,t]$, do not have points in common. $C^c_{[0,t]}$ is the complementary set. We prove the theorem by showing that there exist positive constants α_1 and α_2 such that

$$\lim_{t\to\infty} P\left[C_{[0,t]}, \hat{D}^{\{x\}}_{[0,t]} = \varnothing, \hat{D}^{\{y\}}_{[0,t]} = \varnothing\right] \ge 1 - \alpha_1 a^{-(d/2)\alpha_2} \tag{5.3}$$

and that, for all initial σ, as $t \to \infty$,

$$E\left[H\left(\hat{D}^{\{x,y\}}_{[0,t]}, \sigma\right), \hat{D}^{\{x\}}_{[0,t]} \cap \hat{D}^{\{y\}}_{[0,t]} = \varnothing\right] \to \tfrac{1}{4}. \tag{5.4}$$

PROOF OF (5.3). Since the branching process R^2_t dominating $|D_t|$ is subcritical, it dies in a finite time with probability 1. This means that for almost all ω, there exists a time $T(\omega)$ such that $D^{\{x,y\}}_t = \varnothing$ for all $t \ge T$. Moreover, the number of branchings in the time interval $[0,t]$ is bounded for almost all ω and has finite expectation. Now, $\hat{D}^{\{x\}}_{[0,t]} \cap \hat{D}^{\{y\}}_{[0,t]} \neq \varnothing$ only when a branching mark appears that involves two sites occupied by $D^{\{x,y\}}_t$. We fix now the first branching mark. Thus we condition on the event "the first branching mark happens in the interval $T + dt$." Since the Ppp defining the graphic representation are mutually independent, this event is independent of the event "at least two sites involved in this mark are occupied." Hence the probability that at least two sites involved in the first branching mark are occupied is of order $1/a^{d/2}$ ([2] and [3]). Since the number of sites involved in each mark is uniformly bounded by r [defined in (2.8)], the probability of an intersection in the first mark is of order $r/a^{d/2}$. Let $N^{|D|}$ be the number of new branches created in $[0,t]$, when the initial state of the branching process is $|D|$. Since the process is subcritical, $E(N^{|D|}) < \infty$ and the probability of an intersection in at least one of the branching marks is bounded by $CE(N^{|D|})/a^{d/2}$, where C is a constant depending on the rates $c(\cdot,\cdot)$ and $p(\cdot,\cdot)$. This proves also the analog of (5.3) when $|D| > 2$.

PROOF OF (5.4). Notice that on the event $C_{[0,t]} \cap \{D^{\{x\}}_{[0,t]} = \varnothing, D^{\{y\}}_{[0,t]} = \varnothing\}$, the random variables $\sigma_t(x)$ and $\sigma_t(y)$ are independent [and equal to $H(\hat{D}^{\{x\}}_{[0,t]})$ and $H(\hat{D}^{\{y\}}_{[0,t]})$, respectively]. The left-hand side of (5.4) is then equal to

$$\begin{aligned}&E\left[H\left(\hat{D}^{\{x\}}_{[0,t]}\right)H\left(\hat{D}^{\{y\}}_{[0,t]}\right), C_{[0,t]}, D^{\{x\}}_t = \varnothing, D^{\{y\}}_t = \varnothing\right]\\ &\quad + E\left[H\left(\hat{D}^{\{x,y\}}_{[0,t]}\right), C_{[0,t]}, \left(D^{\{x\}}_t = \varnothing, D^{\{y\}}_t = \varnothing\right)^c\right].\end{aligned} \tag{5.5}$$

Now, since the branching process is subcritical, as $t \to \infty$, $P(D^{\{x\}}_t = \varnothing, D^{\{y\}}_t = \varnothing$

eventually) = 1. In this way, by dominated convergence, the second term in (5.5) goes to 0 as $t \to \infty$. Finally, by symmetry of the noise with respect to 1 and -1, the first term of (5.5) converges to $\frac{1}{4}$ when $t \to \infty$. □

REMARK. Notice that this proof works only in the exponentially ergodic case.

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Note added in proof. After finishing the paper, I learned that Claudia Neuhauser [18] proved the following: If the process with generator L_g is exponentially ergodic, then there exists $a > 0$ such that the process with generator $L_g + aL_e$ is also exponentially ergodic. This has a small intersection with the results in this paper.

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