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MOMENT INEQUALITIES FOR FUNCTIONALS OF THE BROWNIAN CONVEX HULL

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We briefly show an extension of inequalities of Burkholder and Gundy for linear Brownian motion to certain monotone functionals of the d-dimensional Brownian convex hull. Our results belong to a class of results that imply that Brownian hulls are much like the one-dimensional maximal process.

1. Introduction. Let $\{B(t);\, t \geq 0\}$ be a standard linear Brownian motion. Burkholder [3] shows that if τ is a stopping time (with respect to the natural filtration of B), for all $r > 0$ there are constants c_1, c_2 satisfying

$$c_1 E\{\tau\}^{r/2} \leq E\Big\{\max_{s \leq \tau} B(s)\Big\}^r \leq c_2 E\{\tau\}^{r/2}, \tag{1}$$

which, at least informally, states that the Brownian scaling property, in some sense, carries over to stopping times. The multidimensional version of this result holds, with B replaced by $|B|$, in (1).

In the next section we show that in higher dimensions an analogous result holds for the set-valued stochastic process that is defined to be the convex hull of the range of B. Section 3 closes with some concluding remarks and a brief description of some recent work on the Brownian convex hull.

2. The main result. Throughout this paper, we define $\mathscr{C}^d$ to be the collection of all convex subsets of the d-dimensional Euclidean space, $\mathbb{R}^d$, that contain the origin in their interior. Endow $\mathscr{C}^d$ with the Hausdorff metric, H. In other words, for $A, B \in \mathscr{C}^d$,

$$H(A, B) \equiv \max\Big\{\sup_{x \in A} \inf_{y \in B} |x - y|,\ \sup_{y \in B} \inf_{x \in A} |x - y|\Big\}.$$

The following lemma is easy.

Lemma 2.1. *$(\mathscr{C}^d, H)$ is a separable metric space.*

Notice that $(\mathscr{C}^d, H)$ is incomplete for all d. For example, in $d = 1$ take $A_n = [-1, 1/n] \in \mathscr{C}^1$ and notice that $A_n \to A = [-1, 0) \notin \mathscr{C}^1$.

Before we state the inequalities, we need to define the class of functionals that we shall be looking at.

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DEFINITION 2.2. We say $\varphi\colon \mathscr{C}^d \to \mathbb{R}^1_+$ is *increasing* if whenever $A, B \in \mathscr{C}^d$, and $A \subseteq B$, then $\varphi(A) \le \varphi(B)$. For $\alpha > 0$, we say that such a φ is *α-scaling*, if for all $C \in \mathscr{C}^d$, and for all $r > 0$, $\varphi(rC) = r^\alpha \varphi(C)$. Let Φ denote the class of all increasing functionals that are α-scaling for some $\alpha > 0$.

As examples of elements of Φ one has the volume, surface area and the diameter functionals. Actually, all of the so-called *mixed volumes* are known to be in Φ (see Eggelston [7]). Other nontrivial and interesting examples of $\varphi \in \Phi$ are, for example, $\varphi(C)$ being the largest surface area or volume obtained by looking at k-sided polytopes inscribing C, or $\varphi(C)$ being the smallest surface area or volume obtained by considering k-sided polytopes circumscribing C.

The following is an immediate consequence of the definitions. As well as giving us an idea of the elements of Φ, it also settles all related measurability problems.

LEMMA 2.3. *If $\varphi \in \Phi$, then φ is continuous.*

PROOF. There exists an $\alpha > 0$, such that φ is α-scaling. Therefore, if $A_n \in \mathscr{C}^d$, and $A \in \mathscr{C}^d$ are such that $\lim_{n\to\infty} H(A_n, A) = 0$, it follows that

$$\forall\, \varepsilon > 0\ \exists\, N \ni \forall\, n \ge N\colon (1-\varepsilon)A \subseteq A_n \subseteq (1+\varepsilon)A.$$

Notice that we are using the fact that elements of $\mathscr{C}^d$ contain 0 in their interior. Applying φ to the above, by scaling and monotonicity,

$$\forall\, \varepsilon > 0\ \exists\, N \ni \forall\, n \ge N\colon (1-\varepsilon)^\alpha \varphi(A) \le \varphi(A_n) \le (1+\varepsilon)^\alpha \varphi(A),$$

which is the result. □

We shall introduce some notation and then state and prove the main result. Let $\{X(t);\, t \ge 0\}$ be a d-dimensional Brownian motion. Define $\{C(t);\, t \ge 0\}$ to be the associated convex hull process, that is, $C(t)$ is the convex hull of the set of points $X([0,t]) \equiv \{x \in \mathbb{R}^d | \exists\ s \le t \ni X(s) = x\}$. Let $\mathscr{F}_t$ be the natural filtration of X and define $U_d \equiv \{x \in \mathbb{R}^d\colon |x| \le 1\}$ to be the d-dimensional unit disk.

PROPOSITION 2.4. *Let $\varphi \in \Phi$. If α is the scaling index of φ, then there exist constants c_1 and c_2, such that for all $\mathscr{F}_t$-stopping times, τ,*

$$c_1 E\tau^{\alpha/2} \le E\{\varphi(C(\tau))\} \le c_2 E\tau^{\alpha/2}.$$

PROOF OF THE UPPER BOUND. Let $M(t) = \sup_{s \le t} |X(s)|$. Then the most generous estimate yields the desired upper bound, viz.,

$$\forall\, t > 0, \qquad C(t) \subseteq M(t) U_d$$

implying that

$$\varphi(C(\tau)) \le \varphi(M(\tau)U_d) = M(\tau)^{\alpha}\varphi(U_d).$$

Taking expectations,

$$E\varphi(C(\tau)) \le EM(\tau)^{\alpha}\varphi(U_d) \le \kappa E\tau^{\alpha/2}\varphi(U_d),$$

by an application of the ordinary Burkholder–Gundy inequality; see Burkholder [3]. Letting $c_2 = \kappa\varphi(U_d)$, we get the desired upper bound. □

PROOF OF THE LOWER BOUND. Here, we shall use the good-lambda inequalities. For this, simply notice the following sequence of inequalities:

$$\begin{aligned}
&\Pr\{\tau^{\alpha/2} \ge 2r, \varphi(C(\tau)) \le \delta r\} \\
&\quad\le \Pr\Big\{\tau \ge r^{2/\alpha}, \varphi\Big(C\big((2r)^{2/\alpha}\big)\Big) \le \delta r\Big\} \\
(2)\qquad &\quad= E\Big\{1_{\{\tau \ge r^{2/\alpha}\}} \Pr\Big\{\varphi\Big(C\big((2r)^{2/\alpha}\big)\Big) \le \delta r | \mathscr{F}(r^{2/\alpha})\Big\}\Big\} \\
&\quad\le E\Big\{1_{\{\tau \ge r^{2/\alpha}\}} \Pr_{X(r^{2/\alpha})}\big[\varphi\big(C((2^{2/\alpha}-1)r^{2/\alpha})\big) \le \delta r\big]\Big\} \\
&\quad= \Pr\{\tau^{\alpha/2} \ge r\}\Pr\Big\{\varphi(C(1)) \le \delta(2^{2/\alpha}-1)^{-\alpha/2}\Big\}.
\end{aligned}$$

Here we have used, in inequality (2), some standard Markov process notation. Hence, as $\delta \to 0$, we have shown that

$$\Pr\{\tau^{\alpha/2} \ge 2r, \varphi(C(\tau)) \le \delta r | \tau^{\alpha/2} \ge r\} = o(\delta).$$

The justification for the above sequence of equalities/inequalities is easy: They follow from the Markov property, the independent increments property and some basic geometry. The details are left to the interested reader. At this point, the good-lambda inequality implies the lower bound. For this and more, see Burkholder [3]. □

3. Remarks.

1. Since $C(t) \in \mathscr{C}^d$ for all t at once, almost surely, Lemma 2.3 is more than sufficient. However, we do not know of a proof or a disproof for the lemma if $\mathscr{C}^d$ is replaced by *all* compact convex subsets of $\mathbb{R}^d$. In this case, the lemma does go through if we further assume that elements of Φ are location invariant as well. It would be interesting to see if this assumption can be dispensed with altogether.
2. For results on the growth rates for the convex hull of multidimensional Brownian motion, see the interesting results of Lévy [9] and Evans [8].
3. Our application of Burkholder's good-lambda inequality is patterned after those in Bass [1] and Davis [5]. What makes things slightly different here is the geometric argument required in the proof of the lower bound.

4. Using Cauchy's formula, Takács [10] gives a beautifully simple calculation for the expected perimeter length of the convex hull of planar Brownian motion: $E|\partial C(t)| = (8\pi t)^{1/2}$. El Bachir [6] has a simple proof, using polar coordinates, for the analogous result for the area, that is, he has proved that if $d = 2$, then $E|C(t)| = \pi t/2$.
5. For results on the smoothness of the boundary of the convex hull of planar Brownian motion, see Cranston, Hsu and March [4]. Further information on this subject has appeared in the recent article of Burdzy and San Martin [2]. Mountford (unpublished) uses estimates of [2] to give the exact result.
6. In the notation of Proposition 1, if $\varphi \in \Phi$, then for all $p > 0$, $\varphi^p \in \Phi$, and hence the term "moment inequalities" in the title.

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