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MULTIPLE POINTS OF SAMPLE PATHS OF MARKOV PROCESSES[1]

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We show that certain d-dimensional Markov processes $X(t)$, $t \geq 0$, have the property that if E is a closed subset of R_+ with sufficiently large Hausdorff dimension, then $X(E)$ has k-multiple points. This is applied directly to diffusions driven by stochastic differential equations and Lévy processes with positive lower indices, solving problems posed by J. P. Kahane and S. J. Taylor.

1. Introduction. Let $X(t)$ be a d-dimensional Brownian motion and let $E \subset \mathbb{R}_+$ be a closed set. For which E does $X(E) \triangleq \{X(t): t \in E\}$ have k-multiple points, that is, does there exist x such that $x = X(t_1) = \cdots = X(t_k)$ for at least k different $t_j \in E$? This problem was posed by Kahane [7] and Taylor [15]. In [7], Kahane treated symmetric stable Lévy processes and obtained some zero-probability and positive-probability results. The probability-1 result in the case $k = 2$ was then proved by Testard [16]. In this paper, we study this problem (both positive-probability and probability-1) for two large classes of Markov processes, namely, diffusions driven by stochastic differential equations and Lévy processes with positive lower indices. We first prove, in Theorem 2.1, a result for a general time-homogeneous Markov process with the transition density functions satisfying certain explicit conditions. The existence of the k-multiple points of $X(E)$ is then ensured by the positivity of $H^{\theta_1}(E)$, the Hausdorff θ_1-measure of E, where $0 < \theta < \theta_1 \leq 1$ and θ is a certain suitable number. Theorem 2.1 is then applied directly to obtain Theorems 3.1 and 3.2, which are, respectively, for diffusions and Lévy processes. Finally, Theorem 3.3 is concerned with processes with stable components.

The novelty of our results is to extend the known cases when the E's are intervals (i.e., we may take $\theta_1 = 1$) to the cases when the E's are certain "fractals." (See the remarks following Theorems 2.1 and 3.1–3.3.) The basic idea of our study is to consider the k-parameter random field

$$(1.1) \qquad Z(t_1, \ldots, t_k) \triangleq \big(X(t_2) - X(t_1), \ldots, X(t_k) - X(t_{k-1})\big)$$

and to consider the construction of a certain nonzero positive measure supported on $\operatorname{supp} \sigma \cap Z^{-1}(\mathbf{0}) = \{\mathbf{t} \in \operatorname{supp} \sigma | Z(\mathbf{t}) = \mathbf{0}\}$, where $t = (t_1, \ldots, t_k)$, $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^{d(k-1)}$ and σ is a certain Borel measure. For E an interval and σ

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Lebesgue measure, this idea has already appeared in the work of Wolpert [17], Geman, Horowitz and Rosen [4], Rosen [13], Le Gall, Rosen and Shieh [9] and others. However, the calculations and methods involved in this paper are quite different from those in the previously mentioned literature.

In the case where E is an interval, Rogers [12] proved a certain positive-probability result for general Markov processes in a complete metric space; the probability-1 result was then considered by Shieh [14].

2. Multiple points of general Markov processes. In this section, $X(t)$ will be a time-homogeneous Markov process in $\mathbb{R}^d$, of which the sample paths $X(\cdot, \omega)$ are right-continuous and have left limits everywhere (the "cadlag" paths). We assume the existence of transition density functions $p(t, x, y)$, $t > 0$ and $x, y \in \mathbb{R}^d$.

THEOREM 2.1. *Assume that the following hold:*

$$p(t, x, y) \text{ is jointly measurable in } (t, x, y); \tag{2.1}$$

$$p(t, x, y) \text{ is positive and continuous in } (x, y) \text{ for each } t > 0; \tag{2.2}$$

there exists some θ, $0 < \theta < 1$, *such that*

$$\begin{aligned} \sup_x \left(\int_{\mathbb{R}^d} p^k(t, x, y)\, dy \right)^{1/k} &= O(t^{-\theta}), \\ \sup_y \left(\int_{\mathbb{R}^d} p^k(t, x, y)\, dx \right)^{1/k} &= O(t^{-\theta}); \end{aligned} \tag{2.3}$$

$$M_\delta \triangleq \sup_{t \geq \delta} \sup_{x, y} p(t, x, y) < \infty, \quad \text{for all } \delta > 0, \tag{2.4}$$

and

$$\beta(t) \triangleq \inf_x p(t, x, x) > 0, \quad \text{for all } t > 0. \tag{2.5}$$

Let $E \subset \mathbb{R}_+$ *be a compact set such that*

$$H^{\theta_1}(E) > 0, \quad \text{for some } \theta_1 \colon \theta < \theta_1 \leq 1,$$

where $H^{\theta_1}(E)$ *denotes the Hausdorff* θ_1*-measure of* E *(see, e.g.,* [8], *Chapter* 10). *Then the following hold*:

(i) $P\{X(E)$ *has k-multiple points*$\} > 0$;
(ii) $P\{X(E')$ *has k-multiple points*$\} = 1$, *where*

$$E' = \bigcup_{n=0}^{\infty} (E + \tau n), \qquad \text{diam } E < \tau < \infty.$$

REMARK. In the case where E is an interval and $X(t)$ has continuous sample paths, a version of Theorem 2.1 is considered by Shieh [14] in which (2.3) is weakened.

PROOF OF THEOREM 2.1. Let $[a_j, b_j]$ be k compact intervals in $\mathbb{R}_+$ such that $0 < a_j < b_j < a_{j+1} < b_{j+1}$. We consider the random field $Z(\mathbf{t})$ defined by (1.1), where $\mathbf{t} = (t_1, \dots, t_k) \in \prod_{j=1}^k [a_j, b_j]$. Note that Z is $\mathbb{R}^{d(k-1)}$-dimensional. Let $\sigma(\mathbf{t})$ be a finite positive Borel measure on $\prod_{j=1}^k [a_j, b_j]$. Assuming that

$$(2.6) \qquad I_\theta(\sigma) = \iint \frac{d\sigma(\mathbf{s})\, d\sigma(\mathbf{t})}{\prod_{j=1}^k |t_j - s_j|^\theta} < \infty,$$

we shall construct a.s. a positive measure $\mu_k^\sigma(\cdot, \omega)$ which can be expressed formally by

$$(2.7) \qquad \begin{aligned} \mu_k^\sigma(\Lambda) &= \int_\Lambda \delta(Z(\mathbf{t}))\, d\sigma(\mathbf{t}) \\ &= \int \cdots \int_\Lambda \int_{\mathbb{R}^d} \prod_{j=1}^k \delta_x(X(t_j))\, dx\, d\sigma(t_1 \cdots t_k). \end{aligned}$$

It is intuitive that μ_k^σ should be the limit of

$$(2.8) \qquad \mu_{k,\varepsilon}^\sigma(\Lambda) = \int_\Lambda \phi_\varepsilon(Z(\mathbf{t}))\, d\sigma(\mathbf{t}),$$

where ϕ_ε is a certain approximation to the identity on $\mathbb{R}^{d(k-1)}$ and supp $\phi_\varepsilon \subset \{|x| \le \varepsilon\}$. Marcus [10] explicitly uses (2.7) and (2.8), with σ being Lebesgue measure, to study the level sets of real stochastic processes, and his idea can be traced back to the early work of Kahane [8]. We shall prove below that some key displays in [10] (Theorem 1 and its proof) are indeed valid in our present case and hence we can use Marcus' construction (adapted to our case). We denote the density of $Z(\mathbf{t})$ at $\mathbf{x}$ by $q(\mathbf{t}; \mathbf{x})$, $\mathbf{x} = (x_1, \dots, x_{k-1}) \in \mathbb{R}^{d(k-1)}$, and we also denote the bivariate density of $(Z(\mathbf{s}), Z(\mathbf{t}))$ at $(\mathbf{x}, \mathbf{y})$ by $r(\mathbf{s}, \mathbf{t}; \mathbf{x}, \mathbf{y})$. Using the Markov property, we see that the explicit expressions for q and r are given as follows (cf. (2.1) and (2.2) in [14]): Let $X(0) = x_0$, $\sum_{i=1}^j x_i = u_j$, $\sum_{i=1}^j y_i = v_j$, $\sum_{i=1}^0 \cdot \triangleq 0$. For $\mathbf{t} = (t_1, \dots, t_k)$ and $\mathbf{x} = (x_1, \dots, x_{k-1})$, we have

$$(2.9) \quad q(\mathbf{t}; \mathbf{x}) = \int_{\mathbb{R}^d} \left\{ p(t_1, x_0, z) \prod_{j=2}^k p(t_j - t_{j-1}, z + u_{j-2}, z + u_{j-1}) \right\} dz.$$

For $\mathbf{s} = (s_1, \dots, s_k)$ and $\mathbf{y} = (y_1, \dots, y_{k-1})$, assuming $s_j < t_j$ for all j: $1 \le j \le l$

while $t_j < s_j$ for all j: $l < j \le k$, we have

$$\begin{aligned} r(\mathbf{s},\mathbf{t};\mathbf{x},\mathbf{y}) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Bigg\{ & p(s_1,x_0,z_1)p(t_1-s_1,z_1,z_2) \\ & \times \prod_{j=2}^{l}\big[p(s_j-t_{j-1},z_2+v_{j-2},z_1+u_{j-1}) \\ & \qquad \times p(t_j-s_j,z_1+u_{j-1},z_2+v_{j-1})\big] \\ & \times p(t_{l+1}-t_l,z_2+v_{l-1},z_2+v_l) \\ & \times p(s_{l+1}-t_{l+1},z_2+v_l,z_1+u_l) \\ & \times \prod_{j=l+2}^{k}\big[p(t_j-s_{j-1},z_1+u_{j-2},z_2+v_{j-1}) \\ & \qquad \times p(s_j-t_j,z_2+v_{j-1},z_1+u_{j-1})\big]\Bigg\}\,dz_1\,dz_2. \end{aligned} \tag{2.10}$$

Note that we have expressions similar to (2.10) when $\mathbf{s}$ and $\mathbf{t}$ have other ordering relations.

By (2.1) and (2.2), q and r are continuous in the spatial variables and $q(\mathbf{t};\mathbf{x}) > 0$ everywhere. We prove that $r(\mathbf{s},\mathbf{t};\mathbf{x},\mathbf{y})$ is uniformly dominated by a function $g(\mathbf{s},\mathbf{t})$ which is integrable with respect to $d\sigma(\mathbf{s})\,d\sigma(\mathbf{t})$. Let $\delta = \min_j(a_{j+1}-b_j)$ and let M_δ be defined by (2.4). By (2.6), $d\sigma(\mathbf{s})\,d\sigma(\mathbf{t})$ carries no mass on each hyperplane $t_j = s_j$. We split $(\mathbf{s},\mathbf{t})$ into 2^k cases. On any one of these, say $s_j < t_j$ for all $1 \le j \le l$ while $t_j < s_j$ for all $l < j \le k$, by (2.10) we have

$$\begin{aligned} & r(\mathbf{s},\mathbf{t};\mathbf{x},\mathbf{y}) \\ & \le M_\delta^{k-1}\iint_{\mathbb{R}^d\times\mathbb{R}^d}\Bigg\{p(s_1,x_0,z_1)\prod_{j=1}^{l}p(t_j-s_j,z_1+u_{j-1},z_2+v_{j-1}) \\ & \qquad \times \prod_{j=l+1}^{k}p(s_j-t_j,z_2+v_{j-1},z_1+u_{j-1})\Bigg\}\,dz_1\,dz_2. \end{aligned} \tag{2.11}$$

Since

$$\int_{\mathbb{R}^d}p(s_1,x_0,z_1)\,dz_1 = 1, \quad \text{for all } s_1 > 0,$$

we may apply the generalized Hölder inequality:

$$\begin{aligned} & \int_{\mathbb{R}^d}\prod_{j=1}^{l}p_j(t_j-s_j,z_1+\cdot,z_2+\cdot)\prod_{j=l+1}^{k}p_j(s_j-t_j,z_2+\cdot,z_1+\cdot)\,dz_2 \\ & \quad \le \prod_{j=1}^{l}\sup_x\|p_j(t_j-s_j,x,\cdot)\|_{L^k(\mathbb{R}^d)}\prod_{j=l+1}^{k}\sup_y\|p_j(s_j-t_j,\cdot,y)\|_{L^k(\mathbb{R}^d)}, \end{aligned}$$

and then we conclude, by (2.3) and (2.11), that

$$\sup_{\mathbf{x},\mathbf{y}} r(\mathbf{s},\mathbf{t};\mathbf{x},\mathbf{y}) \le g(\mathbf{s},\mathbf{t}) \triangleq \text{Const.} \prod_{j=1}^{k} |t_j - s_j|^{-\theta}. \tag{2.12}$$

By (2.6), $g(\mathbf{s},\mathbf{t})$ is integrable with respect to $d\sigma(\mathbf{s})\,d\sigma(\mathbf{t})$. Let

$$\mathscr{H} = \left\{ J \subset \prod_{j=1}^{k} [a_j, b_j] : J = \prod_{j=1}^{k} [c_j, d_j],\ c_j \text{ and } d_j \text{ are rationals} \right\}.$$

We can then apply the arguments in Marcus ([10], page 281), with Lebesgue measure there now being replaced by $d\sigma$, to conclude that there exists a.s. a measure μ_k^σ, suggested by (2.7), such that, for all $J \in \mathscr{H}$,

$$\mu_k^\sigma(J,\omega) = \lim_{n\to\infty} \mu_{k,\varepsilon_n}^\sigma(J,\omega), \tag{2.13}$$

where $\mu_{k,\varepsilon}^\sigma$ is defined by (2.8) and $\varepsilon_n = \varepsilon_n(\omega)$ is a certain sequence decreasing to zero and independent of J. Provided we assume the path-continuity of $X(\cdot,\omega)$, then it is immediate that μ_k^σ is supported on $\operatorname{supp}\sigma \cap Z^{-1}(\mathbf{0})$ (cf. [10], page 282, line 5). However, in our case this assertion is not obviously true since our $Z(\cdot)$ may have discontinuities, arising from those of $X(\cdot)$. Provided that

$$I_{\theta_2} = \iint \frac{d\sigma(\mathbf{s})\,d\sigma(\mathbf{t})}{\prod_{j=1}^{k}|t_j - s_j|^{\theta_2}} < \infty, \tag{2.14}$$

for some θ_2: $\theta < \theta_2$, then we can use the following arguments, similar to those in Le Gall, Rosen and Shieh ([9], page 507), to show that μ_k^σ is indeed supported on $\operatorname{supp}\sigma \in Z^{-1}(\mathbf{0})$. We observe that, in the case of (2.14),

$$\iint \frac{g(\mathbf{s},\mathbf{t})\,d\sigma(\mathbf{s})\,d\sigma(\mathbf{t})}{\prod_{j=1}^{k}|t_j - s_j|^{\theta_2-\theta}} < \infty.$$

Thus, in view of the arguments in [10], [(2.6)–(2.8)], we have

$$\iint \frac{d\mu_k^\sigma(\mathbf{s})\,d\mu_k^\sigma(\mathbf{t})}{\prod_{j=1}^{k}|t_j - s_j|^{\theta_2-\theta_1}} < \infty \quad \text{a.s.} \tag{2.15}$$

Now we show that, for each $\varepsilon_0 > 0$,

$$\Lambda_{\varepsilon_0} \triangleq \{\mathbf{t}: |Z(\mathbf{t})| > \varepsilon_0\}$$

is of μ_k^σ-measure zero, from which μ_k^σ is indeed supported on $\operatorname{supp}\sigma \cap Z^{-1}(\mathbf{0})$. Since $X(\cdot,\omega)$ is cadlag, we may express $\Lambda_{\varepsilon_0} = S_1 \cup S_2$, where S_1 is contained in a countable union of hyperplanes resulting from the discontinuities $X(t_j) \neq X(t_j-)$ and S_2 is open in $\mathbb{R}_+^k$. We have $\mu_k^\sigma(S_1) = 0$, because from (2.15) $d\mu_k^\sigma$ carries no mass on each hyperplane $\{\mathbf{t}: t_j = t_0$ for some j and some $t_0\}$. The set S_2, being open, itself can be expressed as a countable union of members J in $\mathscr{H}$. On each such J, $|Z(\mathbf{t})|$ is bounded away from zero at least by ε_0. Thus, by (2.13) and the fact that $\operatorname{supp}\phi_\varepsilon \subset \{|x| \le \varepsilon\}$, $\mu_k^\sigma(J) = 0$ and hence $\mu_k^\sigma(S_2) = 0$.

Now we complete the proof of Theorem 2.1 as follows. Since $H^{\theta_1}(E) > 0$, by Frostman's lemma in [8] (page 130), there exists a probability measure ν supported on E such that $\nu[a, a + x] \le Cx^{\theta_1}$, where C is independent of a and x. Choose $E_1, \dots, E_k$ to be k disjoint compact subsets of E such that $\nu(E_j) > 0$ and that $E_j \subset [a_j, b_j]$ with $0 < a_j < b_j < a_{j+1} < b_{j+1}$. Let σ be the restriction of $\otimes^k \nu$ to $E_1 \times \cdots \times E_k$. Whenever $\theta < \theta_2 < \theta_1$, it is easy to see that both (2.6) and (2.14) hold. We have then:

$$\begin{aligned} &P\{X(E) \text{ has } k\text{-multiple points}\} \\ &\quad \ge P\{\exists\, \mathbf{t} = (t_1, \dots, t_k) \text{ such that } t_j \in E_j \text{ and that } X(t_1) = \cdots = X(t_k)\} \\ &\quad = P\{\exists\, \mathbf{t} \in \operatorname{supp} \sigma \cap Z^{-1}(\mathbf{0})\} \\ &\quad \ge P\{\mu_\sigma^k > 0\} \\ &\quad \ge \frac{\left(\int q(\mathbf{t}; \mathbf{0})\, d\sigma(\mathbf{t})\right)^2}{\int\int r(\mathbf{s}, \mathbf{t}; \mathbf{0}, \mathbf{0})\, d\sigma(\mathbf{s})\, d\sigma(\mathbf{t})}. \end{aligned}$$

For the last inequality, see (2.10) in [10]. Now, in view of (2.5), (2.6), (2.9) and (2.12), we have

$$\begin{aligned} &P\{X(E) \text{ has } k\text{-multiple points}\} \\ &\quad \ge \frac{\left(\int \cdots \int \prod_{j=1}^{k-1} \beta(t_{j+1} - t_j)\, d\nu(t_j)\right)^2}{\int\int \cdots \int\int \prod_{j=1}^{k} d\nu(s_j)\, d\nu(t_j) / |t_j - s_j|^\theta} \\ &\quad \triangleq \rho > 0; \end{aligned} \tag{2.16}$$

this proves the first assertion of Theorem 2.1. Next, $E \subset [0, \tau]$ whenever $\tau > \operatorname{diam} E$. By the Markov property, we have for all $n = 0, 1, 2, \dots,$

$$\begin{aligned} &P\{X(E + \tau n) \text{ has } k\text{-multiple points} | \sigma(X(u); 0 \le u \le \tau n)\} \\ &\quad = P^{X(\tau n)}\{X(E) \text{ has } k\text{-multiple points}\}, \end{aligned}$$

where P^z denotes the same probability measure as that corresponding to X but with $X(0) = z$. We observe that (2.16) is independent of the choice of the starting point of X. Thus, by the conditional Borel–Cantelli lemma, see for example Breiman ([2], page 96), we have

$$P\{X(E + \tau n) \text{ has } k\text{-multiple points i.o.}\} = 1;$$

this proves the second assertion of Theorem 2.1. □

3. Multiple points of diffusions and Lévy processes. In this section, we first assume that $X(t)$ is a d-dimensional diffusion driven by a stochastic differential equation

$$dX(t) = \sigma(X(t))\, dB(t) + b(X(t))\, dt,$$

where σ and b are bounded, uniformly Hölder continuous of certain order, and σ^2 is uniformly elliptic.

THEOREM 3.1. *Let $X(t)$ be a diffusion in $\mathbb{R}^d$ as given previously. Let $E \subset \mathbb{R}_+$ be a compact set. Let*

$$\theta = \frac{d(k-1)}{2k}, \tag{3.1}$$

and assume that

$$H^{\theta_1}(E) > 0, \quad \text{for some } \theta_1: 0 < \theta < \theta_1 \le 1.$$

Then the conclusions of Theorem 2.1 hold for $X(E)$ and $X(E')$.

REMARK. Since $\theta_1 \le 1$, Theorem 3.1 is applicable in the case $d = 3$ to $k = 2$ and in the case $d = 2$ to any $k \ge 2$. For three-dimensional Brownian motion with $k = 2$, we have $\theta = \frac{3}{4}$, which agrees with the result of Testard [16].

PROOF OF THEOREM 3.1. It is well known that $X(t)$ has continuous sample paths and that the transition density functions $p(t, x, y)$ are jointly measurable in (t, x, y), are continuous in (x, y) for each $t > 0$ and satisfy the following bound:

$$M_1 t^{-d/2} \exp\left(-\frac{\alpha_1 |y-x|^2}{t}\right) - M_2 t^{-(d/2)+\lambda} \exp\left(-\frac{\alpha_2 |y-x|^2}{t}\right)$$
$$\le p(t, x, y) \le M_3 t^{-d/2} \exp\left(-\frac{\alpha_3 |y-x|^2}{t}\right),$$

where M_1, M_2, M_3, λ, α_1, α_2, α_3 are some positive constants; see Dynkin ([3], Appendix, Section 6, Theorem 0.5). Therefore,

$$\sup_x \| p(t, x, \cdot) \|_{L^k(\mathbb{R}^d)} = O(t^{-\theta})$$

and

$$\sup_y \| p(t, \cdot\,, y) \|_{L^k(\mathbb{R}^d)} = O(t^{-\theta}),$$

where θ is given by (3.1). Thus, Theorem 2.1 is directly applicable. □

Next, we assume that $X(t)$ is a d-dimensional Lévy process. Recall that the Blumenthal–Getoor [1] lower index of X is defined to be

$$\beta'' = \sup\{\alpha \ge 0: |y|^{-\alpha} \operatorname{Re} \psi(y) \to \infty \text{ as } |y| \to \infty\},$$

where

$$E e^{iy \cdot X(t)} = e^{-t\psi(y)}.$$

Note that $0 \le \beta'' \le 2$ and that, in the case $\beta'' > 0$, the density function $p(t, x)$ of each $X(t)$, $t > 0$, exists and is bounded and continuous in x.

THEOREM 3.2. *Let $X(t)$ be a Lévy process in $\mathbb{R}^d$ as given previously. Assume moreover that $p(t,x) > 0$, for all t and x. Let $E \subset \mathbb{R}_+$ be a compact set. Let*

$$\theta = \frac{d(k-1)}{\beta'' k},$$

and assume that

$$H^{\theta_1}(E) > 0, \quad \textit{for some } \theta_1\colon \theta < \theta_1 \le 1.$$

Then the conclusions of Theorem 2.1 *hold for $X(E)$ and $X(E')$.*

REMARK. Since $\theta_1 \le 1$, we have

$$\beta'' > \frac{d(k-1)}{k}.$$

In the case where E is an interval, Theorem 3.2 was conjectured by Orey [11] and was proved by Le Gall, Rosen and Shieh ([9], Theorem 2).

PROOF OF THEOREM 3.2. Choose α such that

$$\theta = \frac{d(k-1)}{\beta'' k} < \theta_0 \triangleq \frac{d(k-1)}{\alpha k} < \theta_1 \le 1. \tag{3.2}$$

Since $\alpha < \beta''$, there exists some $K > 0$ such that $|\mathrm{Re}\, \psi(y)| \ge |y|^\alpha$, for all y: $|y| \ge K$. Observe that, by the Fourier inversion formula,

$$\begin{aligned} p^k(t,x) &= p^{k-1}(t,x) p(t,x) \\ &\le \left\{ \int_{\mathbb{R}^d} \exp(-t\, \mathrm{Re}\, \psi(y))\, dy \right\}^{k-1} p(t,x). \end{aligned}$$

Hence,

$$\left(\int_{\mathbb{R}^d} p^k(t,x)\, dx \right)^{1/k} = O(t^{-\theta_0}),$$

where θ_0 is given in (3.2), and therefore Theorem 2.1 is again directly applicable with θ there now replaced by θ_0. □

Finally, we mention a Lévy case which is not covered by Theorem 3.2. Let $X = (X_1, \dots, X_d)$ be a d-dimensional Lévy process such that the X_j are independent and each X_j is strictly stable in $\mathbb{R}^1$. Such an X is called a process with stable components. Let α_j be the index of X_j and $p_j(t, x_j)$, $t > 0$ and $x_j \in \mathbb{R}^1$, be the density function of $X_j(t)$. Since the density function of $X(t)$ is the product of $p_j(t, \cdot\,)$, we may use the same arguments as those in the proof of Theorem 3.2 to assert the following.

THEOREM 3.3. *Let X be a process in $\mathbb{R}^d$ with stable components as given previously. Assume that $p_j(t, x) > 0$ for all j, t and x. Then, the conclusions of Theorem 3.2 hold, with*

$$\theta = \left(\sum_{j=1}^{d} \frac{1}{\alpha_j} \right) \frac{k-1}{k}.$$

REMARK 1. Note that the lower index of X is $\min_j \alpha_j$. Thus, Theorem 3.3 is not contained in Theorem 3.2 unless $\alpha_j = \alpha$ for all j, in which case X itself is a stable process of index α.

REMARK 2. Since $0 < \alpha_j \leq 2$, Theorem 3.3 is applicable in the case $d = 3$ to $k = 2$ and in the case $d = 2$ to any k such that $k(\alpha_1 + \alpha_2 - \alpha_1\alpha_2) < \alpha_1 + \alpha_2$. In the case where E is an interval, these results were proved by Hendricks [5, 6].

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Note added in proof. After this paper had been submitted and accepted, the author received a thesis from F. Testard (June, 1987, Orsay) who considered our problem for certain Gaussian random fields.

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