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# A NONSTANDARD LAW OF THE ITERATED LOGARITHM FOR TRIMMED SUMS

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Let $X_i$, $i \geq 1$, be independent random variables with a common distribution in the domain of attraction of a strictly stable law, and for each $n \geq 1$ let $X_{1,n} \leq \cdots \leq X_{n,n}$ denote the order statistics of $X_1, \ldots, X_n$. In 1986, S. Csörgő, Horváth and Mason showed that for each sequence $k_n$, $n \geq 1$, of nonnegative integers with $k_n \to \infty$ and $k_n/n \to 0$ as $n \to \infty$, the trimmed sums $S_n(k_n) = X_{k_n+1,n} + \cdots + X_{n-k_n,n}$ converge in distribution to the standard normal distribution, when properly centered and normalized, despite the fact that the entire sums $X_1 + \cdots + X_n$ have a strictly stable limit, when properly centered and normalized. The asymptotic almost sure behavior of $S_n(k_n)$ strongly depends on the rate at which $k_n$ converges to $\infty$. The sequences $k_n \sim c \log\log n$ as $n \to \infty$ for $0 < c < \infty$ constitute a borderline case between a classical law of the iterated logarithm and a radically different behavior. This borderline case is investigated in detail for nonnegative summands $X_i$.

**1. Introduction and statements of results.** Let $X_i$, $i \geq 1$, be a sequence of independent and identically distributed (iid) real-valued random variables. Limit theorems for the partial sums $S_n = \sum_{i=1}^n X_i$, $n \geq 1$, constitute a central part of probability theory. As early as 1925, Lévy described the class of all possible limit distributions of $S_n$ after centering and normalization, that is, the class of all nondegenerate distributions $G$ for which

$$\sigma_n^{-1}(S_n - \mu_n) \to_{\mathscr{D}} G \quad \text{as } n \to \infty \tag{1.1}$$

holds for some sequences $\mu_n, \sigma_n$, $n \geq 1$, of real centering and norming constants with $\sigma_n > 0$ for every $n$. It consists of all normal laws and the strictly stable laws; cf., for example, Gnedenko and Kolmogorov (1954), Section 33. The latter are completely characterized by their exponent $\alpha \in (0, 2)$, a skewness parameter in $[-1, 1]$ and location and scale parameters; see, for example, Feller (1966), Section 17.4. Recall that for a given law $G$ its domain of attraction is defined to be the set of all distributions $F$ of the iid summands $X_i$ for which there exist two sequences $\mu_n, \sigma_n$, $n \geq 1$, of real constants with $\sigma_n > 0$ such that (1.1) holds. As usual, the fact that $F$ belongs to the domain of attraction of a strictly stable law with exponent $\alpha \in (0, 2)$ will be denoted by $F \in D(\alpha)$ in the sequel.

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Since Lévy's work, it is part of the folklore that for a strictly stable limit $G$ in (1.1), the sums $S_n$ are dominated by a few of their largest and smallest summands. This has been made precise recently by S. Csörgő, Horváth and Mason (1986). To formulate their result, for any $F \in D(\alpha)$ with $0 < \alpha < 2$ fix corresponding sequences $X_i$, $i \geq 1$, and $\mu_n, \sigma_n$, $n \geq 1$, such that (1.1) holds. For each $n \geq 1$ introduce the order statistics $X_{1,n} \leq \cdots \leq X_{n,n}$ of $X_1, \dots, X_n$ so that $S_n = \sum_{i=1}^n X_{i,n}$. Moreover, for a sequence $k_n$, $n \geq 1$, of nonnegative integers with $k_n + 1 \leq n - k_n$ for all $n \geq 1$, consider the sums

$$T_n(k_n) = \sum_{i=1}^{k_n} X_{i,n} + \sum_{i=n+1-k_n}^{n} X_{i,n}$$

of the $k_n$ largest and the $k_n$ smallest summands of $S_n$ and the sums

$$S_n(k_n) = S_n - T_n(k_n) = \sum_{i=k_n+1}^{n-k_n} X_{i,n}$$

of the remaining "middle portion" of the sample $\{X_1, \dots, X_n\} = \{X_{1,n}, \dots, X_{n,n}\}$. S. Csörgő, Horváth and Mason (1986) have shown that if $k_n \to \infty$ and $k_n/n \to 0$ as $n \to \infty$, then there exists a sequence $\nu_n(k_n)$, $n \geq 1$, of real centering constants such that

$$\sigma_n^{-1}(T_n(k_n) - \nu_n(k_n)) \to_{\mathscr{D}} G \quad \text{as } n \to \infty, \tag{1.2}$$

with $\sigma_n$ and $G$ from (1.1), and that there also exist two sequences $\mu_n(k_n), \sigma_n(k_n)$, $n \geq 1$, of real centering and norming constants with $\sigma_n(k_n) > 0$ for all $n \geq 1$ such that

$$\sigma_n(k_n)^{-1}(S_n(k_n) - \mu_n(k_n)) \to_{\mathscr{D}} N(0,1) \quad \text{as } n \to \infty, \tag{1.3}$$

where $N(0,1)$ denotes the standard normal distribution. Clearly, (1.2) says that the whole limit distribution $G$ of $S_n$ in (1.1) is produced entirely by the $2k_n$ extreme summands $X_{1,n}, \dots, X_{k_n,n}$ and $X_{n+1-k_n,n}, \dots, X_{n,n}$. For this, $k_n \to \infty$ is crucial; the corresponding statement for fixed $k$ is not true. If these summands appearing in $T_n(k_n)$ are discarded from $S_n$, then surprisingly one obtains a standard normal limit for the remaining trimmed sums $S_n(k_n)$, after appropriate centering and normalization. When looking at the whole sums, this central limit theorem (1.3) is hidden under (1.1) because $\sigma_n(k_n)/\sigma_n \to 0$ as $n \to \infty$. It should be stressed here that $S_n(k_n)$ in this central limit theorem is obtained by discarding an *asymptotically vanishing* proportion of the whole sample $X_1, \dots, X_n$ and *not* a *fixed* proportion, as is usually done when trimmed means are considered in statistics.

The asymptotic almost sure behavior of the trimmed sums $S_n(k_n)$ has been investigated in Haeusler and Mason (1987). It turns out that the rate at which $k_n$ converges to $\infty$ becomes crucial. The natural law of the iterated logarithm

(LIL) corresponding to the central limit theorem (1.3), namely

$$\limsup_{n\to\infty} \pm \frac{S_n(k_n) - \mu_n(k_n)}{(2\log_2 n)^{1/2}\sigma_n(k_n)} = 1 \quad \text{a.s.,} \tag{1.4}$$

where $\log_2 n = \log\log n$ for $n \geq 3$, holds for sequences $k_n$ with $k_n/n \to 0$ and $k_n/\log_2 n \to \infty$ as $n \to \infty$ (if also some technical monotonicity conditions are satisfied). If $k_n/\log_2 n \to 0$ as $n \to \infty$, then the centering and norming constants in (1.4) are not of the proper order of magnitude for a description of the asymptotic almost sure fluctuations of $S_n$ since for nonnegative summands $X_i$ we have

$$\limsup_{n\to\infty} \frac{S_n(k_n) - \mu_n(k_n)}{(2\log_2 n)^{1/2}\sigma_n(k_n)} = \infty \quad \text{a.s.,} \tag{1.5}$$

and

$$\liminf_{n\to\infty} \frac{S_n(k_n) - \mu_n(k_n)}{(2\log_2 n)^{1/2}\sigma_n(k_n)} = 0 \quad \text{a.s.} \tag{1.6}$$

Thus in this case the asymptotic almost sure behavior of $S_n(k_n)$ is radically different from the classical LIL behavior in (1.4). For a different type of trimming the problem was investigated by Griffin (1988b), who studied the more general case of distributions $F$ for which the full sums $S_n$ are stochastically compact in the sense of Feller (1967) and $F$ is not in the domain of partial attraction of the normal distribution. For nonnegative summands his trimming and the trimming considered here coincide, and proper centering and norming constants for $S_n(k_n)$ can be obtained from his results. Therefore the asymptotic almost sure behavior of $S_n(k_n)$ for nonnegative summands is completely known (and completely different) for trimming levels $k_n$ with $k_n/\log_2 n \to \infty$ or $\to 0$ as $n \to \infty$. The remaining sequences $k_n$ with $k_n \sim c\log_2 n$ as $n \to \infty$ for some $0 < c < \infty$ constitute a borderline case. For these sequences both Griffin (1988a) and Haeusler and Mason (1987) contain LIL's of the form

$$\limsup_{n\to\infty} \pm \frac{S_n(k_n) - \mu_n(k_n)}{(2\log_2 n)^{1/2}\sigma_n(k_n)} = M^{\pm} \quad \text{a.s.,} \tag{1.7}$$

for nonnegative summands $X_i$, with unspecified finite constants $M^{\pm}$ (that the lim sup must indeed be a constant almost surely follows directly from the Hewitt–Savage zero–one law). These results reveal a nonstandard LIL behavior: The centering and norming constants are the natural ones suggested by the central limit theorem (1.3), but the asymptotic constants may be different from one. The present paper completes the theory by providing a method for an actual computation of these constants $M^{\pm}$. In Haeusler and Mason (1987) a quantile function representation of the $X_i$ in combination with results from the asymptotic theory of uniform empirical processes was used for the proof of (1.4). We will show here that this methodology is also a suitable starting point for an approach to the present problem, but that it has to be extended and refined considerably by tools from the theory of order statistics and probabilities of large deviations for empirical measures, and by somewhat involved analytic considerations. It will be seen that the main feature of this approach is that it is a *computational* one, in the sense that no prior guess about the analytic form of $M^{\pm}$ is required. This is a major advantage of the quantile function methodology, which makes it a superior tool in the analysis for iid random variables.

Before we can state our main results, we have to fix the notation and setting for the rest of the paper. Since we will restrict ourselves to nonnegative random variables, we will always consider an $F \in D(\alpha)$ for $0 < \alpha < 2$ with $F(0-) = \lim_{x \uparrow 0} F(x) = 0$. Then the classical characterization of $D(\alpha)$ by the stable convergence criterion says that this is equivalent to

$$(F_{\alpha, l}) \qquad 1 - F(x) = x^{-\alpha} l(x) \quad \text{for all } x > 0 \text{ and } F(0-) = 0,$$

where $l$ is slowly varying at $\infty$, that is, $l(\lambda x)/l(x) \to 1$ as $x \to \infty$ for all $0 < \lambda < \infty$; cf. Gnedenko and Kolmogorov (1954), Section 35, Theorem 2, or Bingham, Goldie and Teugels (1987), Theorem 8.3.1 (for convenience we never distinguish notationally between probability distributions on the real line and their (right-continuous) distribution functions). The left-continuous quantile function $Q$ of $F$ is defined by

$$Q(u) = \inf\{x\colon F(x) \geq u\}, \qquad 0 < u < 1,$$

with $Q(0) = Q(0+) = \lim_{u \downarrow 0} Q(u)$. By the theory of asymptotic inverses and conjugate slowly varying functions, cf. Bingham, Goldie and Teugels (1987), Section 1.5.7, $(F_{\alpha, l})$ is equivalent to

$$(Q_{\alpha, L}) \quad Q(1 - u) = u^{-1/\alpha} L(u) \quad \text{for all } 0 < u < 1 \text{ and } Q(0+) \geq 0,$$

where $L$ is now slowly varying at 0. As before, let $X_i$, $i \geq 1$, be a sequence of iid random variables with distribution function $F$, with $X_{1,n} \leq \cdots \leq X_{n,n}$ denoting the order statistics of $X_1, \ldots, X_n$. Since $X_i \geq 0$, discarding the lower order statistics $X_{1,n}, \ldots, X_{k_n, n}$ from the entire partial sums as in (1.3)–(1.7) has no effect. Therefore from now on we will exclusively consider trimmed sums of the form

$$S_n(k_n) = \sum_{i=1}^{n-k_n} X_{i,n}, \tag{1.8}$$

always for sequences $k_n$ of nonnegative integers such that $k_n \sim c \log_2 n$ as $n \to \infty$ for some $0 < c < \infty$. Using the corresponding truncated variance function

$$\sigma^2(x) = \int_0^{1-x} \int_0^{1-x} (\min(u, v) - uv)\, dQ(u)\, dQ(v), \qquad 0 < x < 1,$$

appropriate centering and norming constants for $S_n(k_n)$ are given by

$$(1.9) \qquad \mu_n(k_n) = n\int_0^{1-k_n/n} Q(u)\,du \quad \text{and} \quad \sigma_n(k_n) = n^{1/2}\sigma(k_n/n),$$

that is, (1.3)–(1.7) hold with this choice; cf. S. Csörgő, Horváth and Mason (1986) and Haeusler and Mason (1987).

It turns out that the constant $M^+$ in (1.7) is determined by the Kummer series

$$\Phi(2-\alpha, 3-\alpha; \vartheta) = \sum_{k=0}^{\infty} \frac{(2-\alpha)}{(2-\alpha+k)} \frac{\vartheta^k}{k!}, \qquad 0 < \vartheta < \infty,$$

in the notation of Erdélyi, Magnus, Oberhettinger and Tricomi (1953). For given $0 < \alpha < 2$ and $0 < c < \infty$, we introduce the equation

$$(\mathrm{E}^+) \qquad \frac{1}{c} = \frac{\vartheta^2 e^{-\vartheta}}{2-\alpha} \Phi(2-\alpha, 3-\alpha; \vartheta),$$

which always has at least one solution $0 < \vartheta < \infty$ (cf. Lemma 5 below). An easy computation verifies that $(\mathrm{E}^+)$ is tantamount to

$$(1.10) \qquad \frac{1}{c} = \vartheta^{\alpha} e^{-\vartheta} \int_0^{\vartheta} x^{-\alpha+1} e^x\,dx;$$

cf. also Chapter 6 in Erdélyi, Magnus, Oberhettinger and Tricomi (1953), in particular (6.5.1).

THEOREM 1. *Assume $F \in D(\alpha)$ for some $0 < \alpha < 2$ and $F(0-) = 0$. Let $k_n$, $n \geq 1$, be a sequence of nonnegative integers with $k_n \sim c \log_2 n$ as $n \to \infty$ for some $0 < c < \infty$. Then*

$$\limsup_{n\to\infty} \frac{S_n(k_n) - \mu_n(k_n)}{(2\log_2 n)^{1/2}\sigma_n(k_n)} = M^+(\alpha, c) \quad a.s.,$$

*where*

$$M^+(\alpha,c) = \begin{cases} \dfrac{(2-\alpha)^{1/2}\alpha}{\alpha-1}\dfrac{c^{1/2}}{2}\left(1 + \max\left\{\left(\dfrac{1}{c\vartheta} - 1\right)e^{\vartheta/\alpha} :\right.\right. \\ \qquad\qquad \left.\left.\vartheta > 0 \text{ solves } (\mathrm{E}^+)\right\}\right), & \text{if } \alpha \neq 1, \\ \dfrac{c^{1/2}}{2}\left(\displaystyle\int_0^{\vartheta}\left(\log\frac{\vartheta}{x}\right)e^x\,dx + \vartheta\right) \text{ with } \vartheta > 0 \text{ solving } (\mathrm{E}^+), & \\ & \text{if } \alpha = 1. \end{cases}$$

REMARKS. 1. There exists a unique $3/2 < \alpha^* < 2$ such that

$$1 = \frac{\alpha^*}{2-\alpha^*} e^{-2\alpha^*} \Phi(2-\alpha^*, 3-\alpha^*; 2\alpha^*);$$

numerical evaluation gives $\alpha^* = 1.888802\ldots$.

2. If $0 < \alpha \le \alpha^*$, then for each $0 < c < \infty$ equation ($E^+$) has a unique solution $\vartheta > 0$, and the description of $M^+(\alpha, c)$ simplifies accordingly. This applies in particular for $\alpha = 1$ as already utilized in the statement of the theorem; in this case ($E^+$) becomes $1/c = \vartheta(1 - e^{-\vartheta})$.

3. If $\alpha^* < \alpha < 2$, then

$$1 = \frac{\vartheta - \alpha}{2 - \alpha} e^{-\vartheta} \Phi(2 - \alpha, 3 - \alpha; \vartheta)$$

has two solutions $0 < \vartheta_1 < \vartheta_2 < \infty$, depending on $\alpha$. For $0 < c < (\vartheta_1 - \alpha)/\vartheta_1^2$ and $(\vartheta_2 - \alpha)/\vartheta_2^2 < c < \infty$, equation ($E^+$) has a unique solution $\vartheta > 0$, for $c = (\vartheta_i - \alpha)/\vartheta_i^2$, $i = 1, 2$, it has two distinct solutions in $(0, \infty)$, and for $(\vartheta_1 - \alpha)/\vartheta_1^2 < c < (\vartheta_2 - \alpha)/\vartheta_2^2$, it has three distinct solutions; for these $\alpha$ we always have $(\vartheta_1 - \alpha)/\vartheta_1^2 < (\vartheta_2 - \alpha)/\vartheta_2^2$.

The description of the constant $M^-$ in (1.7) for nonnegative summands is of a similar structure but less involved because the equation corresponding to ($E^+$) has a unique solution for each $0 < \alpha < 2$ and $0 < c < \infty$. It is given by

$$(\mathrm{E}^-) \qquad \frac{1}{c} = \vartheta^{\alpha} e^{\vartheta} \Gamma(2 - \alpha; \vartheta)$$

with the incomplete gamma function $\Gamma(2 - \alpha; \vartheta) = \int_0^{\vartheta} x^{1-\alpha} e^{-x}\, dx$, $0 < \vartheta < \infty$.

THEOREM 2. *Let the assumptions of Theorem* 1 *be satisfied. Then*

$$\liminf_{n \to \infty} \frac{S_n(k_n) - \mu_n(k_n)}{(2 \log_2 n)^{1/2} \sigma_n(k_n)} = M^-(\alpha, c) \quad a.s.,$$

*with*

$$M^-(\alpha, c) = \begin{cases} \dfrac{(2 - \alpha)^{1/2} \alpha}{\alpha - 1} \dfrac{c^{1/2}}{2} \left(1 - \left(\dfrac{1}{c\vartheta} + 1\right) e^{-\vartheta/\alpha}\right), & \text{if } \alpha \ne 1, \\ \dfrac{c^{1/2}}{2} \left( \displaystyle\int_0^{\vartheta} \left(\log \frac{x}{\vartheta}\right) e^{-x}\, dx - \vartheta \right), & \text{if } \alpha = 1, \end{cases}$$

*where $\vartheta > 0$ is the unique solution of the equation* ($E^-$).

By an application of L'Hospital's rule it can be seen that for fixed $0 < c < \infty$ the quantities $M^+(\alpha, c)$ and $M^-(\alpha, c)$ are continuous functions of $\alpha$. Moreover, for fixed $0 < \alpha < 2$ it is easy to verify

$$(1.11) \qquad \lim_{c \to \infty} M^+(\alpha, c) = 1 \quad \text{and} \quad \lim_{c \to \infty} M^-(\alpha, c) = -1,$$

whereas

$$(1.12) \qquad \lim_{c \downarrow 0} M^+(\alpha, c) = \infty \quad \text{and} \quad \lim_{c \downarrow 0} M^-(\alpha, c) = 0.$$

Clearly, (1.11) shows that there is a smooth transition from the classical LIL in (1.4) for the case $k_n/\log_2 n \to \infty$, corresponding to the limits 1 and $-1$

in (1.11), to the nonstandard LIL of Theorems 1 and 2 for $k_n \sim c \log_2 n$. The reason is that in both cases similar exponential bounds are available for the crucial tail probabilities. For (1.4) they come from Bernstein's and Kolmogorov's classical exponential inequalities as detailed in Haeusler and Mason (1987). For Theorems 1 and 2 a large deviation behavior is decisive. It gives bounds which are of the same quality as the classical ones so that $(2 \log_2 n)^{1/2} \sigma_n(k_n)$ are still the proper normalizers, but leads to the constants $M^{\pm}(\alpha, c) \neq 1$. However, it is somewhat hidden, because it does not apply to $S_n(k_n)$ itself, but, roughly speaking, only to the const $\cdot \log_2 n$ largest summands of $S_n(k_n)$ after a suitable transformation of this portion of $S_n(k_n)$ into a classically trimmed sum. The remaining portion of $S_n(k_n)$ does not contribute to $M^{\pm}(\alpha, c)$ at all, as shown below by Proposition 1 in Section 2. This section also contains a detailed description of all the other steps in our computation of $M^{+}(\alpha, c)$ from Theorem 1 on the technical level, with the proofs postponed until Section 3. The computation of $M^{-}(\alpha, c)$ from Theorem 2 follows the same pattern and therefore will not be given here.

Statements (1.12) reveal a smooth transition from the nonclassical LIL behavior of $S_n(k_n)$ in Theorems 1 and 2 for $k_n \sim c \log_2 n$ to the results (1.5) and (1.6) for the case $k_n / \log_2 n \to 0$. But, of course, as mentioned already, (1.5) and (1.6) not give a proper description of the asymptotic almost sure fluctuations of $S_n(k_n)$ in this case. To obtain such a description, one has to use normalizers different from $(2 \log_2 n)^{1/2} \sigma_n(k_n)$, as shown by Griffin (1988b). This means that there is a jump in the almost sure behavior of $S_n(k_n)$ when going from trimming levels $k_n \sim c \log_2 n$ to $k_n$'s with $k_n / \log_2 n \to 0$. The reason is that in the ranges $k_n / \log_2 n \to \infty$ and $k_n \sim c \log_2 n$ for $0 < c < \infty$ the behavior of $S_n(k_n)$ is the result of the cumulative effect of many summands, namely, as explained above, of at least const $\cdot \log_2 n$ summands, whereas in the range $k_n / \log_2 n \to 0$ a very small number of large summands controls the behavior of $S_n(k_n)$. The most extreme case is given by trimming levels with $k_n / (\log_2 n)^{1/2} \to 0$, because then $S_n(k_n)$ and its largest summand $X_{n-k_n, n}$ have exactly the same behavior so that $S_n(k_n)$ is controlled by a *single* summand; cf. (1.6), (1.8) and (1.9) in Griffin (1988b). In consequence of this fact, classical or large deviation exponential bounds no longer apply in the range $k_n / \log_2 n \to 0$.

**2. The approach to Theorem 1.** Our aim is to evaluate the constant $M^{+}(\alpha, l, \{k_n\})$ in

$$\limsup_{n \to \infty} \frac{S_n(k_n) - \mu_n(k_n)}{(2 \log_2 n)^{1/2} \sigma_n(k_n)} = M^{+}(\alpha, l, \{k_n\}) \quad \text{a.s.,} \tag{2.1}$$

with $S_n(k_n)$ from (1.8) and $\mu_n(k_n)$ and $\sigma_n(k_n)$ from (1.9), under the conditions of Theorem 1, that is, for distribution functions $F$ satisfying $(F_{\alpha, l})$. As indicated by the notation, we have to take into account a possible dependence of this constant from $\alpha$ and $l$ appearing in $(F_{\alpha, l})$ and the sequence $k_n \sim c \log_2 n$ of trimming levels. The following observation is crucial for our approach: Vaguely speaking, the constant $M^+(\alpha, l, \{k_n\})$ is determined entirely by the summands $X_{n-i,n}$ of $S_n(k_n)$ with $i$ being of the order $v \log_2 n$ with $c \le v < \infty$. Precisely, we have the following proposition.

PROPOSITION 1. *For each* $1 < q < \infty$ *let* $m_n(q)$, $n \ge 1$, *be a sequence of nonnegative integers with* $m_n(q) \sim cq \log_2 n$ *as* $n \to \infty$. *Then*

$$\lim_{q\to\infty} \limsup_{n\to\infty} \left| \frac{\sum_{i=1}^{n-m_n(q)} X_{i,n} - n\int_0^{1-m_n(q)/n} Q(u)\,du}{(2\log_2 n)^{1/2}\sigma_n(k_n)} \right| = 0 \quad a.s.$$

For each $1 < q < \infty$ we now fix a sequence $m_n(q)$, $n \ge 1$, of nonnegative integers with $m_n(q) \sim cq \log_2 n$ as $n \to \infty$ and consider the constant in

$$\limsup_{n\to\infty} \frac{\sum_{i=n+1-m_n(q)}^{n-k_n} X_{i,n} - n\int_{1-m_n(q)/n}^{1-k_n/n} Q(u)\,du}{(2\log_2 n)^{1/2}\sigma_n(k_n)} = M^+(\alpha, l, \{k_n\}, \{m_n(q)\}) \quad \text{a.s.} \tag{2.2}$$

Again the Hewitt–Savage zero–one law implies that $M^+(\alpha, l, \{k_n\}, \{m_n(q)\})$ is indeed a constant, which may depend on the indicated parameters. From Proposition 1 we infer

$$M^+(\alpha, l, \{k_n\}) = \lim_{q\to\infty} M^+(\alpha, l, \{k_n\}, \{m_n(q)\}). \tag{2.3}$$

Therefore we can determine $M^+(\alpha, l, \{k_n\}, \{m_n(q)\})$ first and then evaluate the limit in (2.3) to obtain $M^+(\alpha, l, \{k_n\})$. The first step is to note that the centering and norming constants in (2.2) are of the same order of magnitude.

PROPOSITION 2. *For each* $1 < q < \infty$,

$$\lim_{n\to\infty} \frac{n\int_{1-m_n(q)/n}^{1-k_n/n} Q(u)\,du}{(2\log_2 n)^{1/2}\sigma_n(k_n)} = M(\alpha, c, q)$$

*with*

$$M(\alpha, c, q) = \begin{cases} \dfrac{(2-\alpha)^{1/2}\alpha}{\alpha - 1} \dfrac{c^{1/2}}{2}\left(q^{1-1/\alpha} - 1\right), & \textit{if } \alpha \ne 1, \\ \dfrac{c^{1/2}}{2} \log q, & \textit{if } \alpha = 1. \end{cases}$$

Because of Proposition 2 we only have to study the asymptotic behavior of the sums

$$\sum_{i=n+1-m_n(q)}^{n-k_n} X_{i,n} = \sum_{i=k_n+1}^{m_n(q)} X_{n+1-i,n},$$

that is, to determine the constant in

$$(2.4)\qquad \limsup_{n\to\infty} \frac{\sum_{i=k_n+1}^{m_n(q)} X_{n+1-i,n}}{(2\log_2 n)^{1/2}\sigma_n(k_n)} = \overline{M}(\alpha, l, \{k_n\}, \{m_n(q)\}) \quad \text{a.s.}$$

It turns out that this constant does not depend on the slowly varying function $l$. To see this, consider the *Pareto distribution functions* $F_\alpha$ given by $F_\alpha(x) = 1 - x^{-\alpha}$ for $1 \le x < \infty$ which satisfy $(F_{\alpha,l})$ with $l(x) = l_0(x) = \min(1, x^\alpha)$. Clearly, the quantile function $Q_\alpha$ pertaining to $F_\alpha$ is given by $Q_\alpha(1-u) = u^{-1/\alpha}$, $0 < u < 1$, that is, the corresponding function $L$ in $(Q_{\alpha,L})$ is identically equal to 1. In the sequel let $Y_i$, $i \ge 1$, always denote a sequence of iid random variables with common distribution function $F_\alpha$, and for each $n \ge 1$ let $Y_{1,n} \le \cdots \le Y_{n,n}$ be the order statistics of $Y_1, \dots, Y_n$. Also evaluating $\sigma_n(k_n)$ in the norming constants, we obtain the following proposition.

PROPOSITION 3. *For each* $1 < q < \infty$,

$$(2.5)\qquad \begin{aligned}&\limsup_{n\to\infty} \frac{\sum_{i=k_n+1}^{m_n(q)} Y_{n+1-i,n}}{n^{1/\alpha}(\log_2 n)^{1-1/\alpha}}\\ &\quad = \frac{2c^{1/2-1/\alpha}}{(2-\alpha)^{1/2}}\overline{M}(\alpha, l_0, \{k_n\}, \{m_n(q)\}) \quad a.s.,\end{aligned}$$

*and*

$$(2.6)\qquad \overline{M}(\alpha, l, \{k_n\}, \{m_n(q)\}) = \overline{M}(\alpha, l_0, \{k_n\}, \{m_n(q)\}).$$

As a consequence of Proposition 3, from now on we can and will drop the slowly varying function from the notation for the constants in (2.1)–(2.5). The next step is to demonstrate that $\overline{M}(\alpha, \{k_n\}, \{m_n(q)\})$ from (2.5) and hence the constants in (2.2)–(2.4) depend on the sequences $k_n \sim c\log_2 n$ and $m_n(q) \sim cq\log_2 n$ only through the parameters $c$ and $q$ so that $M^+(\alpha, \{k_n\})$ from (2.1) depends on $k_n$ only through $c$.

PROPOSITION 4. *For* $1 < q < \infty$ *and all sequences* $k_n, k_n'$, $n \ge 1$, *and* $m_n, m_n'$, $n \ge 1$, *of nonnegative integers satisfying* $k_n' \sim k_n \sim c\log_2 n$ *and* $m_n' \sim m_n \sim cq\log_2 n$ *as* $n \to \infty$,

$$\limsup_{n\to\infty} \frac{\sum_{i=k_n+1}^{m_n} Y_{n+1-i,n}}{n^{1/\alpha}(\log_2 n)^{1-1/\alpha}} = \limsup_{n\to\infty} \frac{\sum_{i=k_n'+1}^{m_n'} Y_{n+1-i,n}}{n^{1/\alpha}(\log_2 n)^{1-1/\alpha}} \quad a.s.$$

As a consequence of Proposition 4, from now on we can and will always write $M^+(\alpha, c)$, $M^+(\alpha, c, q)$ and $\overline{M}(\alpha, c, q)$ for the constants appearing in (2.1), (2.2), (2.4) and (2.5).

Summarizing Propositions 1–4 and (2.3), we see that it is enough to determine the constant in

$$\limsup_{n\to\infty} \frac{\sum_{i=k_n+1}^{m_n(q)} Y_{n+1-i,n}}{n^{1/\alpha}(\log_2 n)^{1-1/\alpha}} = \tilde{M}(\alpha, c, q) \quad \text{a.s.} \tag{2.7}$$

for *appropriately chosen* sequences $k_n \sim c \log_2 n$ and $m_n(q) \sim cq \log_2 n$ for all $1 < q < \infty$ and then to compute

$$M^+(\alpha, c) = \lim_{q\to\infty} \left\{ \frac{(2-\alpha)^{1/2}}{2c^{1/2-1/\alpha}} \tilde{M}(\alpha, c, q) - M(\alpha, c, q) \right\}. \tag{2.8}$$

To simplify matters somewhat, we will not present here a complete analytic description of $\tilde{M}(\alpha, c, q)$, which is possible; cf. Haeusler (1988). Instead, we will derive upper and lower bounds for $\tilde{M}(\alpha, c, q)$ which are sharp enough to evaluate the limit in (2.8). For this an appropriate version of the Rényi representation of exponential order statistics for the order statistics from the Pareto distributions $F_\alpha$ is crucial. Let $W_i$, $i \geq 1$, be iid random variables with a common exponential distribution with mean $1/\alpha$. It is well known that for each $n \geq 1$, with $W_{1,n} \leq \cdots \leq W_{n,n}$ being the order statistics of $W_1, \dots, W_n$, the random variables $\{W_{n+1-i,n}: 1 \leq i \leq n\}$ and $\{\sum_{j=i}^n W_j/j: 1 \leq i \leq n\}$ have the same joint distributions; cf., for example, Feller (1966), Section 1.6. When combined with the fact that the random variables $\{Y_{i,n}: 1 \leq i \leq n\}$ and $\{\exp(W_{i,n}): 1 \leq i \leq n\}$ also have the same joint distributions, this yields for all integers $0 \leq k < m \leq n$,

$$Y_{n-m,n}^{-1} \sum_{i=k+1}^{m} Y_{n+1-i,n} =_{\mathscr{D}} \sum_{i=1}^{m-k} Y_{i,m}. \tag{2.9}$$

Applying (2.9) with $k = k_n \sim c \log_2 n$ and $m = m_n \sim cq \log_2 n$, we see that (2.9) permits a transformation of the sums $\sum_{i=k_n+1}^{m_n(q)} Y_{n+1-i,n}$ of intermediate order statistics from $F_\alpha$ with sample size $n$ into the sums $\sum_{i=1}^{m_n(q)-k_n} Y_{i,m_n(q)}$ with sample size $m_n(q)$, with the additional factor $Y_{n-m_n(q),n}^{-1}$ whose influence can be easily controlled. Because of $k_n \sim m_n(q)/q$, however, the latter sums are trimmed sums in the classical statistical sense: A *fixed* proportion of the largest summands is discarded from the entire partial sums. This enables us to employ known properties of classically trimmed sums in the computation of $\tilde{M}(\alpha, c, q)$. The crucial tools are results on probabilities of large deviations of these sums, which have been derived by Groeneboom, Oosterhoff and Ruymgaart (1979) as an application of their general theory of probabilities of large deviations for empirical measures and functionals thereof. This will be detailed in the next section, together with the proofs of Propositions 1–4.

**3. Proofs.** For the proof of Proposition 1, we borrow the formulas

$$\int_x^1 u^{-\beta} L(u)\, du \sim \frac{1}{\beta - 1} x^{-\beta+1} L(x) \quad \text{as } x \downarrow 0 \tag{3.1}$$

for $1 < \beta < \infty$ from Section 1.5.6 in Bingham, Goldie and Teugels (1987) and

$$\sigma^2(x) \sim \frac{2}{2-\alpha} x^{1-2/\alpha} L^2(x) \quad \text{as } x \downarrow 0 \tag{3.2}$$

from Lemma 1 in S. Csörgő, Horváth and Mason (1986). As announced in Section 1, our approach will be based on a quantile function representation of the $X_i$. For this, let $U_i$, $i \geq 1$, be a sequence of iid random variables which are uniformly distributed on $(0, 1)$. For $n \geq 1$ let $U_{1,n} \leq \cdots \leq U_{n,n}$ be the order statistics and

$$G_n(u) = \frac{1}{n} \sum_{i=1}^{n} 1_{[0,u]}(U_i), \qquad 0 \leq u \leq 1,$$

the uniform empirical distribution function based on $U_1, \dots, U_n$. We require two results about the asymptotic almost sure behavior of $G_n$ and its quantile function $G_n^{-1}$. In both of them the function

$$h(x) = x \log x - x + 1, \qquad 0 < x < \infty, \tag{3.3}$$

plays a vital role. According to Theorem 3.2 of Csáki (1977), cf. also (3.18), (3.19) and (3.53) in his paper, for any $0 < c < \infty$,

$$\begin{aligned} \limsup_{n\to\infty} \sup_{0 \leq u \leq 1 - c(\log_2 n)/n} \left( \frac{n}{\log_2 n} \right)^{1/2} \frac{|G_n(u) - u|}{(1-u)^{1/2}} \\ = \max\bigl(2, c^{1/2}(\beta_c^+ - 1)\bigr) \quad \text{a.s.,} \end{aligned} \tag{3.4}$$

where $\beta_c^+$ is the unique solution in $(1, \infty)$ of the equation $h(\beta) = 1/c$. According to Theorem 5 of Wellner (1978), for any $0 < c < \infty$,

$$\limsup_{n\to\infty} \sup_{0 \leq u \leq 1 - c(\log_2 n)/n} \frac{1-u}{1 - G_n^{-1}(u)} = \gamma_c' \quad \text{a.s.,} \tag{3.5}$$

where $\gamma_c'$ is the unique solution in $(1, \infty)$ of the equation $h(\gamma) = \gamma/c$, and

$$\limsup_{n\to\infty} \sup_{0 \leq u \leq 1 - c(\log_2 n)/n} \frac{1 - G_n^{-1}(u)}{1-u} = \frac{1}{\gamma_c''} \quad \text{a.s.,} \tag{3.6}$$

where $\gamma_c''$ is the unique solution in $(0, 1)$ of the equation $h(\gamma) = \gamma/c$.

Now we are prepared to prove Proposition 1.

Proof of Proposition 1. To simplify the notation, we will write $m_n$ instead of $m_n(q)$ throughout the proof. It is well known that the two families $\{X_i: i \geq 1\}$ and $\{Q(U_i): i \geq 1\}$ of random variables have the same joint distributions, and, consequently, the two families consisting of the order statistics $\{X_{i,n}: 1 \leq i \leq n, n \geq 1\}$ and the transformed uniform order statistics $\{Q(U_{i,n}): 1 \leq i \leq n, n \geq 1\}$ also have the same joint distributions. Therefore w.l.o.g. we can and will assume that $X_{i,n} = Q(U_{i,n})$ holds for all $1 \leq i \leq n$ and $n \geq 1$.

Then, also employing the definition of $G_n$, we obtain the representation

$$\begin{aligned}\sum_{i=1}^{n-m_n} X_{i,n} - n\int_0^{1-m_n/n} Q(u)\,du &= n\int_0^{U_{n-m_n,n}} Q(u)\,dG_n(u)\\ &\quad - n\int_0^{1-m_n/n} Q(u)\,du\\ &= n\int_0^{1-m_n/n}(u - G_n(u))\,dQ(u)\\ &\quad + n\int_{U_{n-m_n,n}}^{1-m_n/n}\left(G_n(u) - 1 + \frac{m_n}{n}\right)dQ(u)\end{aligned}$$

by an integration by parts. Hence

$$\left|\frac{\sum_{i=1}^{n-m_n} X_{i,n} - n\int_0^{1-m_n/n} Q(u)\,du}{(2\log_2 n)^{1/2}\sigma_n(k_n)}\right| \le \frac{\sigma_n(m_n)}{\sigma_n(k_n)}(T_n + \Delta_n), \tag{3.7}$$

where

$$T_n = \frac{n}{(2\log_2 n)^{1/2}\sigma_n(m_n)}\int_0^{1-m_n/n}|u - G_n(u)|\,dQ(u)$$

and

$$\begin{aligned}\Delta_n &= \frac{n}{(2\log_2 n)^{1/2}\sigma_n(m_n)}\left|\int_{U_{n-m_n,n}}^{1-m_n/n}\left(G_n(u) - 1 + \frac{m_n}{n}\right)dQ(u)\right|\\ &\le \frac{n|G_n(1-m_n/n) - 1 + m_n/n|}{(2m_n\log_2 n)^{1/2}}\,\frac{Q(1-m_n/n) + Q(U_{n-m_n,n})}{m_n^{-1/2}\sigma_n(m_n)}\\ &\equiv \Delta_{1,n}\cdot\Delta_{2,n}.\end{aligned}$$

Recall that $m_n \sim cq\log_2 n$ with $q > 1$ so that $0 \le 1 - m_n/n \le 1 - c(\log_2 n)/n$ for all large $n$. Therefore (3.4) immediately implies

$$\limsup_{n\to\infty}\Delta_{1,n} \le \max\left(2^{1/2}, \left(\frac{c}{2}\right)^{1/2}(\beta_c^+ - 1)\right) \quad \text{a.s.} \tag{3.8}$$

To derive a bound for $\Delta_{2,n}$, apply $(Q_{\alpha,L})$ and (3.2) to obtain

$$\lim_{n\to\infty}\frac{Q(1-m_n/n)}{m_n^{-1/2}\sigma_n(m_n)} = \left(\frac{2-\alpha}{2}\right)^{1/2}. \tag{3.9}$$

From (3.5) and $0 \le 1 - m_n/n \le c(\log_2 n)/n$ for all large $n$, we conclude that $U_{n-m_n,n} \le 1 - \gamma m_n/n$ for each $0 < \gamma < 1/\gamma_c'$ with probability 1 for all large

$n$. By monotonicity of $Q$ and (3.2), we get with probability 1 for all large $n$,

$$\frac{Q(U_{n-m_n,n})}{m_n^{-1/2}\sigma_n(m_n)} \le \frac{Q(1-\gamma m_n/n)}{m_n^{-1/2}\sigma_n(m_n)}$$
$$\sim \left(\frac{2-\alpha}{\alpha}\right)^{1/2}\gamma^{-1/\alpha}\frac{L(\gamma m_n/n)}{L(m_n/n)} \to \left(\frac{2-\alpha}{\alpha}\right)^{1/2}\gamma^{-1/\alpha}$$

as $n \to \infty$, since $L$ is slowly varying at 0. Thus

$$\limsup_{n\to\infty}\frac{Q(U_{n-m_n,n})}{m_n^{-1/2}\sigma_n(m_n)} \le \gamma_c'^{1/\alpha}\left(\frac{2-\alpha}{\alpha}\right)^{1/2} \quad \text{a.s.},$$

and combining this with (3.8) and (3.9) we arrive at

$$\begin{aligned}\limsup_{n\to\infty}\Delta_n \le \left(\frac{2-\alpha}{\alpha}\right)^{1/2}&\left(1+\gamma_c'^{1/\alpha}\right)\\ &\times\max\left(2^{1/2},\left(\frac{c}{2}\right)^{1/2}(\beta_c^+-1)\right) \quad \text{a.s.}\end{aligned} \tag{3.10}$$

To obtain a bound for $T_n$, we write for all large $n$,

$$\begin{aligned}T_n \le \frac{1}{\sigma(m_n/n)}&\int_0^{1-m_n/n}(1-u)^{1/2}\,dQ(u)\\ &\times \sup_{0\le u\le 1-c(\log_2 n)/n}\left(\frac{n}{2\log_2 n}\right)^{1/2}\frac{|G_n(u)-u|}{(1-u)^{1/2}}.\end{aligned} \tag{3.11}$$

By an integration by parts and an application of (3.1), it is straightforward to determine the asymptotic behavior of the integral in (3.11) as $n \to \infty$, which when combined with (3.2) shows that the first two factors on the right-hand side of (3.11) converge to $(2/(2-\alpha))^{1/2}$. In view of (3.4) we get

$$\limsup_{n\to\infty} T_n \le \left(\frac{2}{2-\alpha}\right)^{1/2}\max\left(2^{1/2},\left(\frac{c}{2}\right)^{1/2}(\beta_c^+-1)\right) \quad \text{a.s.}$$

Combining this result with (3.10), we have shown that

$$\limsup_{n\to\infty}(T_n+\Delta_n) \le K(\alpha,c) < \infty, \tag{3.12}$$

where the constant $K(\alpha, c)$ depends on $\alpha$ and $c$, but not on $q$. From (3.2) we immediately obtain that $\sigma_n(m_n)/\sigma_n(k_n) \to q^{1/2-1/\alpha}$ as $n \to \infty$. But $1/2 - 1/\alpha < 0$ so that $q^{1/2-1/\alpha} \to 0$ as $q \to \infty$. Combining this with (3.7) and (3.12) concludes the proof. □

PROOF OF PROPOSITION 2. This result follows easily from the uniform convergence theorem for slowly varying functions; cf., for example, Bingham, Goldie and Teugels (1987), Theorem 1.2.1. □

REMARK. The proof of Proposition 1 also applies for $q = 1$ and then shows that the constants $M^{\pm}$ in (1.7) are finite. Together with Proposition 2 this entails that all $M$ constants introduced in Section 2 are indeed finite.

PROOF OF PROPOSITION 3. From (3.2) and slow variation of $L$ we obtain

$$\overline{M}(\alpha, l, \{k_n\}, \{m_n(q)\}) \tag{3.13}$$
$$= \frac{(2-\alpha)^{1/2}}{2c^{1/2-1/\alpha}} \limsup_{n\to\infty} \frac{\sum_{i=k_n+1}^{m_n(q)} X_{n+1-i,n}}{n^{1/\alpha}(\log_2 n)^{1-1/\alpha} L((\log_2 n)/n)} \quad \text{a.s.},$$

which in the special case $F = F_\alpha$ where $L \equiv 1$ reduces to (2.5). To verify (2.6), we have to show that the two lim sups in (2.5) and (3.13) are equal. For this, we compare the quantile function representations of the two sums in (2.5) and (3.13), that is, in addition to $X_{n+1-i,n} = Q(U_{n+1-i,n})$ we assume w.l.o.g. $Y_{n+1-i,n} = Q_\alpha(U_{n+1-i,n})$. Then

$$\begin{aligned} X_{n+1-i,n} &= (1 - U_{n+1-i,n})^{-1/\alpha} L(1 - U_{n+1-i,n}) \\ &= Y_{n+1-i,n} L(1 - U_{n+1-i,n}), \end{aligned}$$

and the desired result follows from the uniform convergence theorem for $L$ upon noting that, since $k_n \sim c \log_2 n$ and $m_n \sim cq \log_2 n$ as $n \to \infty$, by (3.5) and (3.6) with probability 1 for all large $n$,

$$\frac{c}{2\gamma'_c} \frac{\log_2 n}{n} \le 1 - U_{n-k_n,n} \le 1 - U_{n+1-m_n,n} \le \frac{2cq}{\gamma''_{cq}} \frac{\log_2 n}{n}. \tag{3.14}$$
□

PROOF OF PROPOSITION 4. For all integers $n \ge 1$ set $\underline{k}_n = \min(k_n, k'_n)$, $\overline{k}_n = \max(k_n, k'_n)$, $\underline{m}_n = \min(m_n, m'_n)$ and $\overline{m}_n = \max(m_n, m'_n)$. Then $\underline{k}_n \sim \overline{k}_n \sim c \log_2 n$ and $\underline{m}_n \sim \overline{m}_n \sim cq \log_2 n$ as $n \to \infty$. Using again the quantile function representation $Y_{n+1-i,n} = Q_\alpha(U_{n+1-i,n}) = (1 - U_{n+1-i,n})^{-1/\alpha}$ and the monotonicity of $Q_\alpha$, we obtain

$$\begin{aligned} &\left| \frac{\sum_{i=k_n+1}^{m_n} Y_{n+1-i,n}}{n^{1/\alpha}(\log_2 n)^{1-1/\alpha}} - \frac{\sum_{i=k'_n+1}^{m'_n} Y_{n+1-i,n}}{n^{1/\alpha}(\log_2 n)^{1-1/\alpha}} \right| \\ &\quad \le \frac{\sum_{i=\underline{k}_n+1}^{\overline{k}_n} Y_{n+1-i,n} + \sum_{i=\underline{m}_n+1}^{\overline{m}_n} Y_{n+1-i,n}}{n^{1/\alpha}(\log_2 n)^{1-1/\alpha}} \\ &\quad \le \frac{(\overline{k}_n - \underline{k}_n) Q_\alpha(U_{n-\underline{k}_n,n}) + (\overline{m}_n - \underline{m}_n) Q_\alpha(U_{n-\underline{m}_n,n})}{n^{1/\alpha}(\log_2 n)^{1-1/\alpha}} \\ &\quad = \frac{\overline{k}_n}{\log_2 n}\left(1 - \frac{\underline{k}_n}{\overline{k}_n}\right)\left(\frac{n(1 - U_{n-\underline{k}_n,n})}{\log_2 n}\right)^{-1/\alpha} \\ &\qquad + \frac{\overline{m}_n}{\log_2 n}\left(1 - \frac{\underline{m}_n}{\overline{m}_n}\right)\left(\frac{n(1 - U_{n-\underline{m}_n,n})}{\log_2 n}\right)^{-1/\alpha}, \end{aligned}$$

where the right-hand side converges to 0 with probability 1 as $n \to \infty$ on account of $\underline{k}_n/\overline{k}_n \to 1$ and $\underline{m}_n/\overline{m}_n \to 1$ in conjunction with $\overline{k}_n \sim c \log_2 n$, $\overline{m}_n \sim cq \log_2 n$ and (3.14) for $\underline{k}_n$ and $\underline{m}_n$. □

According to the program set up in Section 2, we will next consider $\tilde{M}(\alpha, c, q)$ from (2.7) for all $1 < q < \infty$ and begin with the derivation of an upper bound for $\tilde{M}(\alpha, c, q)$.

As explained in Section 2 a result on probabilities of large deviations for classically trimmed sums will be crucial. To state this result, we need some notation and terminology. Let $D$ denote the set of all probability measures on the real line, equipped with the topology $\mathscr{T}$ of setwise convergence. For $G, H \in D$ the Kullback–Leibler information number $K(H, G) \geq 0$ is given by

$$K(H,G) = \begin{cases} \int_{-\infty}^{\infty} h \log h \, dG = \int_{-\infty}^{\infty} \log h \, dH, & \text{if } H \ll G, \\ \infty, & \text{otherwise}, \end{cases}$$

where $h = dH/dG$, for which we may assume $0 \leq h < \infty$. Moreover, here and in the sequel the usual conventions $\log 0 = -\infty$, $0 \cdot (\pm\infty) = 0$ and $\log(a/0) = \infty$ for $a > 0$ apply. For $\Omega \subset D$ and $G \in D$ we write $K(\Omega, G) = \inf\{K(H, G): H \in \Omega\}$ for the distance between $\Omega$ and $G$, with $K(\varnothing, G) = \infty$. Let $D_1 \subset D$ denote the subset of all probability measures with support in $[1, \infty)$, and for $0 < \tau < 1$ let the functional $T_\tau: D_1 \to [1, \infty)$ be defined by

$$T_\tau(G) = \frac{1}{1-\tau} \int_0^{1-\tau} G^{-1}(u) \, du \quad \text{for all } G \in D_1.$$

LEMMA 1. *For $0 < \tau < 1$ and $-\infty < r < \infty$,*

$$\lim_{n \to \infty} \frac{1}{n} \log P\left( \frac{1}{n - [\tau n]} \sum_{i=1}^{n-[\tau n]} Y_{i,n} \geq r \right) = -K(\Omega_{\tau,r}, F_\alpha) \equiv -K_{\alpha,\tau}(r),$$

*where $\Omega_{\tau,r} = \{G \in D_1: T_\tau(G) \geq r\}$.*

Lemma 1 follows from a version of Theorem 6.3 in Groeneboom, Oosterhoff and Ruymgaart (1979), stated for a continuous distribution function with support in $[1, \infty)$ and classically trimmed sums with the trimming performed only in the upper tail, corresponding to the fact that the distribution is one-sided. The proof consists in obvious modifications of the proof given by Groeneboom, Oosterhoff and Ruymgaart (1979) for two-sided trimming.

The analytic properties of the rate function $K_{\alpha,\tau}$ in Lemma 1 will be important in the sequel. For $0 < \alpha < 2$ and $0 < \tau < 1$ we set

$$\rho_{\alpha,\tau} = \begin{cases} \dfrac{\alpha}{\alpha - 1} \dfrac{1 - \tau^{1-1/\alpha}}{1 - \tau}, & \text{if } \alpha \neq 1, \\ \dfrac{-\log \tau}{1 - \tau}, & \text{if } \alpha = 1. \end{cases}$$

By inspection, $1 < \rho_{\alpha,\tau} < \infty$ for all $\alpha$ and $\tau$.

LEMMA 2. *Let* $0 < \tau < 1$. *Then*

(3.15) $$K_{\alpha,\tau}(r) = 0 \quad \text{for } -\infty < r \leq \rho_{\alpha,\tau};$$

(3.16) $$0 < K_{\alpha,\tau}(r) < \infty \quad \text{for } \rho_{\alpha,\tau} < r < \infty;$$

(3.17) $K_{\alpha,\tau}$ *is continuous and strictly increasing on* $[\rho_{\alpha,\tau}, \infty)$.

PROOF. By elementary computation, $T_\tau(F_\alpha) = \rho_{\alpha,\tau}$; hence $F_\alpha \in \Omega_{\tau,r}$ for $-\infty < r \leq \rho_{\alpha,\tau}$, and (3.15) follows. For the proof of (3.16) notice that $T_\tau(F_\beta) = \rho_{\beta,\tau} \sim \beta\tau^{1-1/\beta}/(1-\tau) \to \infty$ as $\beta \downarrow 0$ so that for each $r$ we have $F_\beta \in \Omega_{\tau,r}$ for all sufficiently small $\beta > 0$. By computation, $K(F_\beta, F_\alpha) = \log(\beta/\alpha) + \alpha/\beta - 1 < \infty$, which proves $K_{\alpha,\tau}(r) < \infty$. For the proof of $K_{\alpha,\tau}(r) > 0$ observe that $T_\tau(F_\alpha) = \rho_{\alpha,\tau}$ implies $F_\alpha \notin \Omega_{\tau,r}$ for $\rho_{\alpha,\tau} < r < \infty$. The functional $T_\tau$ is continuous w.r.t. $\mathscr{T}$; cf. the beginning of the proof of Theorem 6.1 in Groeneboom, Oosterhoff and Ruymgaart (1979). Therefore, by Lemma 3.2 of that paper, there exists a $G \in \Omega_{\tau,r}$ such that $K_{\alpha,\tau}(r) = K(G, F_\alpha)$. Since $K(G, F_\alpha) = 0$ entails $G = F_\alpha$, we must have $K_{\alpha,\tau}(r) > 0$, which completes the proof of (3.16).

Left continuity of $K_{\alpha,\tau}$ follows from the continuity of $T_\tau$ and Lemma 3.3 in Groeneboom, Oosterhoff and Ruymgaart (1979), whereas right continuity is a consequence of $F_\alpha$ being continuous; cf. the proof of the implication "(a) and (b) $\Rightarrow$ (i)" in Theorem 6.1 of Groeneboom, Oosterhoff and Ruymgaart (1979). The "displacement of mass" argument used there can be easily adjusted to the one-sided situation considered here by minor changes in notation.

It remains to prove that $K_{\alpha,\tau}$ is strictly increasing on $(\rho_{\alpha,\tau}, \infty)$. For this, let $\rho_{\alpha,\tau} < r_1 < r_2 < \infty$ be fixed. By Lemma 3.2 in Groeneboom, Oosterhoff and Ruymgaart (1979), there exists a $G \in \Omega_{\tau,r_2}$ with $0 < K(G, F_\alpha) = K_{\alpha,\tau}(r_2) < \infty$, for which we must have $G \ll F_\alpha$. Fix $g = dG/dF_\alpha$, and for each $0 \leq t \leq 1$ let the probability distribution $G_t$ be defined by the density $g_t = 1 - t + tg$ w.r.t. $F_\alpha$. Notice that $0 < K(G, F_\alpha)$ implies $G \neq F_\alpha$; hence $F_\alpha(g_{t_1} \neq g_{t_2}) = F_\alpha(g \neq 1) > 0$ for $t_1 \neq t_2$, and this fact together with $x \log x$ being strictly convex on $[0, \infty)$ entails that

$$K(G_t, F_\alpha) = \int_1^\infty g_t(x) \log g_t(x)\, dF_\alpha(x), \qquad 0 \leq t \leq 1,$$

is a strictly convex function. Since $K(G_0, F_\alpha) = K(F_\alpha, F_\alpha) = 0$, this function is strictly increasing; hence for $0 \leq t \leq 1$,

(3.18) $$K(G_t, F_\alpha) < K(G_1, F_\alpha) = K(G, F_\alpha) = K_{\alpha,\tau}(r_2).$$

By construction we have $\sup_{-\infty<x<\infty}|G(x) - G_t(x)| \leq 2(1-t)$ for all $0 \leq t \leq 1$; hence $G_t^{-1}(u) \to G^{-1}(u)$ as $t \uparrow 1$ for all $0 < u < 1$ except perhaps for a countable number of discontinuity points of $G^{-1}$. Consequently, by dominated convergence and $G \in \Omega_{\tau,r_2}$, we obtain $\lim_{t\uparrow 1} T_\tau(G_t) = T_\tau(G) \geq r_2 > r_1$. Thus $G_t \in \Omega_{\tau,r_1}$ for all $t < 1$ sufficiently close to 1. In view of (3.18) for these $t$ we obtain that $K_{\alpha,\tau}(r_1) \leq K(G_t, F_\alpha) < K_{\alpha,\tau}(r_2)$. This completes the proof of (3.17) and the lemma. □

For $\rho_{\alpha,\tau} < r < \infty$ we require an explicit analytic description of $K_{\alpha,\tau}(r)$ which parallels the considerations in Groeneboom, Oosterhoff and Ruymgaart (1979) leading to formula (6.13) on the bottom of page 584. First, for $0 < \alpha < 2$ and $1 < b < \infty$ we get

$$f_\alpha(b) = \frac{\int_1^b x\,dF_\alpha(x)}{F_\alpha(b)} = \begin{cases} \dfrac{\alpha}{\alpha - 1}\dfrac{1 - b^{1-\alpha}}{1 - b^{-\alpha}}, & \text{if } \alpha \neq 1, \\ \dfrac{\log b}{1 - b^{-1}}, & \text{if } \alpha = 1. \end{cases}$$

By elementary calculus it can be seen that for $0 < \alpha \leq 1$ the function $f_\alpha$ is continuous, strictly increasing and one-to-one from $(1, \infty)$ onto itself, whereas for $1 < \alpha < 2$ it is also continuous and strictly increasing, but one-to-one from $(1, \infty)$ onto $(1, \alpha/(\alpha - 1))$. Notice that the generalized inverse $f_\alpha^{-1}(b) = \inf\{x\colon f_\alpha(x) \geq b\}$ with $\inf \varnothing = \infty$ equals the classical inverse for all $1 < b < \infty$ if $0 < \alpha \leq 1$, and for all $1 < b < \alpha/(\alpha - 1)$ if $1 < \alpha < 2$, whereas then $f_\alpha^{-1}(b) = \infty$ for all $\alpha/(\alpha - 1) \leq b < \infty$. Moreover, $b < f_\alpha^{-1}(b)$ for all $0 < \alpha < 2$ and $1 < b < \infty$.

For $0 < \alpha < 2$, $1 < b < \infty$ and $0 < s < \infty$, we set

$$\psi_{\alpha,b}(s) = \int_1^b x e^{sx}\,dF_\alpha(x) \Big/ \int_1^b e^{sx}\,dF_\alpha(x).$$

The function $\psi_{\alpha,b}$ is differentiable on $(0, \infty)$ with derivative

$$\psi'_{\alpha,b}(s) = \int_1^b x^2 \frac{e^{sx}\,dF_\alpha(x)}{\int_1^b e^{sx}\,dF_\alpha(x)} - \left(\int_1^b x \frac{e^{sx}\,dF_\alpha(x)}{\int_1^b e^{sx}\,dF_\alpha(x)}\right)^2 > 0,$$

because it is the variance of a nondegenerate probability distribution. If $1 < r < b < f_\alpha^{-1}(r)$, then $\lim_{s \downarrow 0} \psi_{\alpha,b}(s) = f_\alpha(b) < r$ and $\lim_{s \to \infty} \psi_{\alpha,b}(s) = b > r$ so that there exists a unique $0 < s_\alpha(r, b) < \infty$ with

$$\psi_{\alpha,b}(s_\alpha(r, b)) = r. \tag{3.19}$$

Notice that by the implicit function theorem for fixed $r$, the function $s_\alpha(r, b)$ is differentiable w.r.t. $b \in (r, f_\alpha^{-1}(r))$. For $1 < r < b < \infty$ with $f_\alpha^{-1}(r) \leq b$, we set $s_\alpha(r, b) = 0$.

Now we are prepared to formulate and prove the following lemma.

LEMMA 3. *Let $0 < \tau < 1$ and $\rho_{\alpha,\tau} < r < \infty$. Then*

$$K_{\alpha,\tau}(r) = \tau \log \tau + (1 - \tau)\log(1 - \tau) + \inf_{r < b < \infty} h_{\alpha,\tau,r}(b),$$

*where*

$$h_{\alpha,\tau,r}(b) = (1 - \tau)s_\alpha(r, b)r - (1 - \tau)\log\left(\int_1^b \exp(s_\alpha(r, b)x)\,dF_\alpha(x)\right) + \alpha\tau \log b$$

*defines a continuous function on* $(r, \infty)$ *with*

$$\inf_{r<b<\infty} h_{\alpha,\tau,r}(b) = \min_{b\in I} h_{\alpha,\tau,r}(b) \tag{3.20}$$

*for some compact interval* $I \subset (r, f_\alpha^{-1}(r))$.

PROOF. From continuity of $F_\alpha$, $F_\alpha(1) = 0$ and the definition of the Kullback–Leibler information number, it follows that any distribution function $G \in D$ with $K(G, F_\alpha) < \infty$ is also continuous with $G(1) = 0$. Therefore we can apply the change-of-variables formula $\int_a^b G^{-1}(u)\,du = \int_{G^{-1}(a)}^{G^{-1}(b)} x\,dG(x)$ for all $0 < a < b < 1$ from which it is easy to deduce that

$$K_{\alpha,\tau}(r) = \inf_{r<b<\infty} K(\Omega_b, F_\alpha),$$

where $\Omega_b = \{G \in D\colon \int_1^b x\,dG(x) \geq (1-\tau)r,\ G(b) = 1-\tau\}$. For $r < b < \infty$ we set

$$g_b(x) = \begin{cases} 0, & \text{if } x < 1, \\ (1-\tau)\exp(s_\alpha(r,b)x) \Big/ \int_1^b \exp(s_\alpha(r,b)x)\,dF_\alpha(x), & \text{if } 1 \leq x \leq b, \\ \tau/(1-F_\alpha(b)), & \text{if } b < x < \infty. \end{cases}$$

Then $g_b$ is a probability density w.r.t. $F_\alpha$ defining a $G_b \in D$ for which it can be shown by an application of the arguments on page 584 of Groeneboom, Oosterhoff and Ruymgaart (1979) that

$$K(\Omega_b, F_\alpha) = K(G_b, F_\alpha) = \tau \log \tau + (1-\tau)\log(1-\tau) + h_{\alpha,\tau,r}(b).$$

Consequently, we have

$$K_{\alpha,\tau}(r) = \tau \log \tau + (1-\tau)\log(1-\tau) + \inf_{r<b<\infty} h_{\alpha,\tau,r}(b),$$

and it remains to verify continuity of $h_{\alpha,\tau,r}$ and (3.20). The function $s_\alpha(r, b)$, $r < b < \infty$, is obviously continuous at all points $b \neq f_\alpha^{-1}(r)$. To prove continuity in $f_\alpha^{-1}(r)$, provided $f_\alpha^{-1}(r) < \infty$, it is enough to show that $s_\alpha(r, b) \to 0$ as $b \uparrow f_\alpha^{-1}(r)$. Notice that $f_\alpha(b) \to f_\alpha(f_\alpha^{-1}(r)) = r$ as $b \uparrow f_\alpha^{-1}(r)$; hence

$$\frac{\int_1^b x^2\,dF_\alpha(x)}{F_\alpha(b)} - rf_\alpha(b) \to \int_1^{f_\alpha^{-1}(r)} x^2 \frac{dF_\alpha(x)}{F_\alpha(f_\alpha^{-1}(r))} - \left(\int_1^{f_\alpha^{-1}(r)} x \frac{dF_\alpha(x)}{F_\alpha(f_\alpha^{-1}(r))}\right)^2$$
$$\equiv c_r > 0,$$

because $c_r$ is the variance of a nondegenerate probability distribution. Thus, with $c_r' = c_r/(2r)$ for all $b < f_\alpha^{-1}(r)$ sufficiently close to $f_\alpha^{-1}(r)$, we have

$$\tilde{s}(b) \equiv \frac{r - f_\alpha(b)}{\left(\int_1^b x^2\,dF_\alpha(x)/F_\alpha(b)\right) - (r + c_r')f_\alpha(b)} > 0,$$

and $\tilde{s}(b) \to 0$ as $b \uparrow f_\alpha^{-1}(r)$. Observe that $e^x \geq 1 + x$ for all real $x$ and $e^x \leq 1 + (1 + c_r'/r)x$ for all $0 \leq x \leq x_r$ and an appropriate $x_r > 0$. For $b < f_\alpha^{-1}(r)$ sufficiently close to $f_\alpha^{-1}(r)$, we have $0 < \tilde{s}(b) \leq x_r/f_\alpha^{-1}(r)$ so that for $1 \leq x \leq b$

we get $0 < \tilde{s}(b)x \le x_r$; hence

$$\psi_{\alpha,b}(\tilde{s}(b)) \ge \frac{\int_1^b x(1+\tilde{s}(b)x)\,dF_\alpha(x)}{\int_1^b (1+(1+c_r'/r)\tilde{s}(b)x)\,dF_\alpha(x)} = r$$

by definition of $\tilde{s}(b)$. Since $\psi_{\alpha,b}$ is strictly increasing, this implies $0 < s_\alpha(r,b) \le \tilde{s}(b)$ for all $b < f_\alpha^{-1}(r)$ sufficiently close to $f_\alpha^{-1}(r)$, and so $s_\alpha(r,b) \to 0$ as $b \uparrow f_\alpha^{-1}(r)$. Consequently, $s_\alpha(r,b)$ is continuous in $b \in (r,\infty)$, and therefore $h_{\alpha,\tau,r}$ is continuous, too.

To verify (3.20), we will first show that $h_{\alpha,\tau,r}(b) \to \infty$ as $b \downarrow r$. We have

$$\psi_{\alpha,b}(s) = \int_s^{sb} x^{-\alpha}e^x\,dx \Big/ s\int_s^{sb} x^{-\alpha-1}e^x\,dx \tag{3.21}$$

and for $0 < \nu < \infty$ by an integration by parts

$$\int_s^{sb} x^{-\nu}e^x\,dx = (sb)^{-\nu}e^{sb} - s^{-\nu}e^s + \nu\int_s^{sb} x^{-\nu-1}e^x\,dx. \tag{3.22}$$

Take $\nu = \alpha + 1$ in (3.22), followed by an application of (3.22) with $\nu = \alpha + 2$ to the right-hand side of the resulting equality, to obtain the expansion

$$\int_s^{sb} x^{-\alpha-1}e^x\,dx = (sb)^{-\alpha-1}e^{sb}(1 + O(s^{-1})) \quad \text{as } s \to \infty, \tag{3.23}$$

uniformly in $r < b < r + 1$. Take now $\nu = \alpha$ in (3.22) and substitute (3.23) on the right-hand side to arrive at

$$\int_s^{sb} x^{-\alpha}e^x\,dx = (sb)^{-\alpha}e^{sb}(1 + \alpha(sb)^{-1} + O(s^{-2})) \quad \text{as } s \to \infty, \tag{3.24}$$

uniformly in $r < b < r + 1$. Substitute now (3.24) for the numerator in (3.21) and with $-\alpha - 1$ instead of $-\alpha$ also for the denominator to obtain

$$\psi_{\alpha,b}(s) = b\frac{1 + \alpha(sb)^{-1} + O(s^{-2})}{1 + (\alpha+1)(sb)^{-1} + O(s^{-2})} = b - s^{-1} + O(s^{-2}) \quad \text{as } s \to \infty,$$

uniformly in $r < b < r + 1$. Hence for $0 < \varepsilon < 1$ there exists an $s_0 < \infty$ such that

$$b - \frac{1+\varepsilon}{s} \le \psi_{\alpha,b}(s) \le b - \frac{1-\varepsilon}{s} \tag{3.25}$$

for all $s \ge s_0$ and $r < b < r + 1$. For $b > r$ sufficiently close to $r$, we have

$$\bar{s}_\varepsilon(b) \equiv \frac{1+\varepsilon}{b-r} > \frac{1-\varepsilon}{b-r} \equiv \underline{s}_\varepsilon(b) \ge s_0,$$

so that $\psi_{\alpha,b}(\underline{s}_\varepsilon(b)) \le r \le \psi_{\alpha,b}(\bar{s}_\varepsilon(b))$ by (3.25). Thus, since $\psi_{\alpha,b}$ is strictly increasing and continuous, by definition of $s_\alpha(r,b)$ we must have $\underline{s}_\varepsilon(b) \le s_\alpha(r,b) \le \bar{s}_\varepsilon(b)$, that is, $1 - \varepsilon \le (b-r)s_\alpha(r,b) \le 1 + \varepsilon$. This proves $(b-r)s_\alpha(r,b) \to 1$ as $b \downarrow r$ and in particular $s_\alpha(r,b) \to \infty$. Consequently, taking (3.23) into account,

$$
\begin{aligned}
s_\alpha(r,b)r - \log\left(\int_1^b \exp(s_\alpha(r,b)x)\,dF_\alpha(x)\right)
&= s_\alpha(r,b)r - \alpha\log s_\alpha(r,b) - \log\left(\int_{s_\alpha(r,b)}^{s_\alpha(r,b)b} x^{-\alpha-1}e^x\,dx\right) - \log\alpha\\
&= s_\alpha(r,b)(r-b) + \log s_\alpha(r,b) + (\alpha+1)\log b\\
&\quad - \log\left(1 + O\left(s_\alpha(r,b)^{-1}\right)\right) - \log\alpha\\
&\to \infty \quad \text{as } b \downarrow r.
\end{aligned}
$$

This proves $h_{\alpha,\tau,r}(b) \to \infty$ as $b \downarrow r$.

To complete the proof of (3.20), we need some information about the behavior of $h_{\alpha,\tau,r}$ near $f_\alpha^{-1}(r)$. Clearly,

$$\varphi_{\alpha,r,b}(s) = sr - \log\left(\int_1^b e^{sx}\,dF_\alpha(x)\right), \qquad 0 \le s < \infty,$$

defines a continuous function with derivative $r - \psi_{\alpha,b}$ on $(0,\infty)$. If $r < b < f_\alpha^{-1}(r)(\le \infty)$, then we have $r - \psi_{\alpha,b}(s_\alpha(r,b)) = 0$, and taking $\psi'_{\alpha,b} > 0$ into account, we see that $\varphi_{\alpha,r,b}$ achieves its maximum on $[0,\infty)$ at $s_\alpha(r,b)$. If $f_\alpha^{-1}(r) \le b < \infty$, then $s_\alpha(r,b) = 0$ by definition. Consequently, for all $r < b < \infty$,

$$
\begin{aligned}
h_{\alpha,\tau,r}(b) &= (1-\tau)\varphi_{\alpha,r,b}(s_\alpha(r,b)) + \alpha\tau\log b\\
&\ge -(1-\tau)\log(1-b^{-\alpha}) + \alpha\tau\log b \equiv \underline{h}_{\alpha,\tau}(b).
\end{aligned}
\tag{3.26}
$$

Now, if $r$ is such that $f_\alpha^{-1}(r) = \infty$ holds, that is, $1 < \alpha < 2$ and $\alpha/(\alpha-1) \le r < \infty$, then (3.26) and $\underline{h}_{\alpha,\tau}(b) \to \infty$ as $b \to \infty$ together imply $h_{\alpha,\tau,r}(b) \to \infty$ as $b \uparrow f_\alpha^{-1}(r) = \infty$ which in combination with $h_{\alpha,\tau,r}(b) \to \infty$ as $b \downarrow r$ yields (3.20). It remains to deal with those $r$ for which $f_\alpha^{-1}(r) < \infty$ holds so that we have $s_\alpha(r,b) = 0$ for $f_\alpha^{-1}(r) \le b < \infty$ and hence equality in (3.26). By elementary calculus it can be seen that $\underline{h}_{\alpha,\tau}$ achieves its minimum on $(1,\infty)$ at $\tau^{-1/\alpha}$, is strictly increasing on $(\tau^{-1/\alpha},\infty)$ and strictly decreasing on $(1,\tau^{-1/\alpha})$. But $f_\alpha^{-1}(r) \le b$ is equivalent to $f_\alpha(b) \ge r > \rho_{\alpha,\tau} = f_\alpha(\tau^{-1/\alpha})$ which in turn is equivalent to $b > \tau^{-1/\alpha}$ so that we see that $h_{\alpha,\tau,r}$ equals $\underline{h}_{\alpha,\tau}$ on $[f_\alpha^{-1}(r),\infty)$ and hence it is strictly increasing on this interval. Taking also $h_{\alpha,\tau,r}(b) \to \infty$ as $b \downarrow r$ into account, we conclude

$$\inf_{r<b<\infty} h_{\alpha,\tau,r}(b) = \min_{r+\varepsilon \le b \le f_\alpha^{-1}(r)} h_{\alpha,\tau,r}(b)$$

for all sufficiently small $\varepsilon > 0$. Since $h_{\alpha,\tau,r}$ is continuous, it is enough to show that $h_{\alpha,\tau,r}$ is strictly increasing on an interval $(f_\alpha^{-1}(r) - \varepsilon, f_\alpha^{-1}(r))$ for some $\varepsilon > 0$. Recall that $s_\alpha(r,b)$ is differentiable in $b \in (r, f_\alpha^{-1}(r))$, whence the same is true for $h_{\alpha,\tau,r}$ with derivative given by

$$h'_{\alpha,\tau,r}(b) = (1-\tau)s'_\alpha(r,b)r + \alpha\tau/b$$

$$-(1-\tau)\frac{\int_1^b x\exp(s_\alpha(r,b)x)\,dF_\alpha(x)s'_\alpha(r,b) + \alpha b^{-\alpha-1}\exp(s_\alpha(r,b)b)}{\int_1^b \exp(s_\alpha(r,b)x)\,dF_\alpha(x)} \tag{3.27}$$

$$= -(1-\tau)\frac{\alpha b^{-\alpha-1}\exp(s_\alpha(r,b)b)}{\int_1^b \exp(s_\alpha(r,b)x)\,dF_\alpha(x)} + \frac{\alpha\tau}{b},$$

where the last equality is a consequence of (3.19). Thus we get as $b \uparrow f_\alpha^{-1}(r)$, observing that $s_\alpha(r,b) \to 0$,

$$h'_{\alpha,\tau,r}(b) \to -(1-\tau)\frac{\alpha\left(f_\alpha^{-1}(r)\right)^{-\alpha-1}}{F_\alpha\left(f_\alpha^{-1}(r)\right)} + \frac{\alpha\tau}{f_\alpha^{-1}(r)} = \underline{h}'_{\alpha,\tau}\left(f_\alpha^{-1}(r)\right) > 0,$$

because of $f_\alpha^{-1}(r) > \tau^{-1/\alpha}$. Therefore we have $h'_{\alpha,\tau,r}(b) > 0$ for all $b < f_\alpha^{-1}(r)$ sufficiently close to $f_\alpha^{-1}(r)$, which completes the proof. □

As a consequence of Lemma 3 and (3.26) and since $\underline{h}_{\alpha,\tau}$ is strictly increasing on $(\tau^{-1/\alpha},\infty)$, we have for all large $r$ that

$$K_{\alpha,\tau}(r) \ge \tau\log\tau + (1-\tau)\log(1-\tau) + \underline{h}_{\alpha,\tau}(r) \to \infty \quad \text{as } r\to\infty. \tag{3.28}$$

We are now prepared to derive the upper bound for $\tilde{M}(\alpha,c,q)$.

Proposition 5. *For $0<\alpha<2$, $0<c<\infty$ and $1<q<\infty$ let $r_{\alpha,c,q}$ be defined by*

$$K_{\alpha,1/q}(r_{\alpha,c,q}) = \frac{1}{cq}, \tag{3.29}$$

*where existence and uniqueness of $r_{\alpha,c,q}$ are guaranteed by Lemma 2 and (3.28). Then $\tilde{M}(\alpha,c,q) \le (cq)^{1-1/\alpha}\gamma_{cq}'^{1/\alpha}r_{\alpha,c,q}$.*

Proof. We will write $r = r_{\alpha,c,q}$ throughout the proof. Set $m_n = [cq\log_2 n]$ and $k_n = [m_n/q]$ for $n \ge 3$, fix $\varepsilon > 0$ arbitrarily and consider the eventually increasing sequence of integers $n_j = [\exp(j/\log j)]$, $j \ge 2$. For simplicity set $k'_j = k_{n_j}$ and $m'_j = m_{n_j}$. According to Proposition 4 and the definition of $\tilde{M}(\alpha,c,q)$ in (2.7), it is enough to show that

$$P\left(\max_{n_j\le n<n_{j+1}} \frac{\sum_{i=k_n+1}^{m_n} Y_{n+1-i,n}}{n^{1/\alpha}(\log_2 n)^{1-1/\alpha}} > (1+\varepsilon)(cq)^{1-1/\alpha}\gamma_{cq}'^{1/\alpha} r \text{ i.o.}\right) = 0. \tag{3.30}$$

From (3.5) we obtain by a quantile function argument that

$$\limsup_{j\to\infty}\left(\frac{\log_2 n_j}{n_j}\right)^{1/\alpha} Y_{n_{j+1}-m'_j,\,n_{j+1}} \le \left(\frac{\gamma'_{cq}}{cq}\right)^{1/\alpha} \quad \text{a.s.;} \tag{3.31}$$

hence (3.30) is a consequence of

$$P\left(\max_{n_j\le n<n_{j+1}}\frac{\sum_{i=k_n+1}^{m_n}Y_{n+1-i,n}}{(\log_2 n_j)Y_{n_{j+1}-m'_j,n_{j+1}}}>\left(1+\frac{\varepsilon}{2}\right)cqr\text{ i.o.}\right)=0.$$

This is implied by the Borel–Cantelli lemma provided that $\sum_{j=2}^{\infty}P_j<\infty$ for

$$P_j=P\left(\max_{n_j\le n<n_{j+1}}\frac{\sum_{i=k_n+1}^{m_n}Y_{n+1-i,n}}{Y_{n_{j+1}-m'_j,n_{j+1}}}>\left(1+\frac{\varepsilon}{2}\right)cqr\log_2 n_j\right).$$

Observe that $n_{j+1}/n_j\to 1$ as $j\to\infty$ so that

$$m'_{j+1}-m'_j\le cq\log_2 n_{j+1}-cq\log_2 n_j+1=1+cq\log\left(\frac{\log n_{j+1}}{\log n_j}\right)\to 1.$$

Hence $m'_{j+1}\le m'_j+1$ for all large $j$. We consider two cases.

*Case 1:* $m_{n_{j+1}-1}=m'_j$. Then, since the sequence $m_n$ is nondecreasing in $n$, we have $m_n=m'_j$ and $k_n=k'_j$ for all $n_j\le n<n_{j+1}$. Notice that $Y_{n+1-i,n}$ is nondecreasing in $n$ for fixed $i$ and that $n^{1/\alpha}(\log_2 n)^{1-1/\alpha}$ is also eventually nondecreasing in $n$. Therefore we have for all large $j$ that

$$P_j\le P\left(\frac{\sum_{i=k'_j+1}^{m'_j}Y_{n_{j+1}+1-i,n_{j+1}}}{Y_{n_{j+1}-m'_j,n_{j+1}}}>\left(1+\frac{\varepsilon}{2}\right)cqr\log_2 n_j\right).$$

*Case 2:* $m_{n_{j+1}-1}=m'_j+1$. Then $m'_{j+1}=m'_j+1$ and, since the sequence $m_n$ is nondecreasing in $n$, there exists a unique integer $n'$ with $n_j\le n'<n_{j+1}-1$ such that $m_n=m'_j$ and $k_n=k'_j$ for all $n_j\le n\le n'$ and $m_n=m'_{j+1}$ and $k_n=k'_{j+1}$ for all $n'<n<n_{j+1}$. Consequently, we obtain for all large $j$ that

$$\begin{aligned}P_j&\le P\left(\max_{n_j\le n\le n'}\frac{\sum_{i=k_n+1}^{m_n}Y_{n+1-i,n}}{Y_{n_{j+1}-m'_j,n_{j+1}}}>\left(1+\frac{\varepsilon}{2}\right)cqr\log_2 n_j\right)\\&\quad+P\left(\max_{n'<n<n_{j+1}}\frac{\sum_{i=k_n+1}^{m_n}Y_{n+1-i,n}}{Y_{n_{j+1}-m'_j,n_{j+1}}}>\left(1+\frac{\varepsilon}{2}\right)cqr\log_2 n_j\right)\\&\le P\left(\frac{\sum_{i=k'_j+1}^{m'_j}Y_{n_{j+1}+1-i,n_{j+1}}}{Y_{n_{j+1}-m'_j,n_{j+1}}}>\left(1+\frac{\varepsilon}{2}\right)cqr\log_2 n_j\right)\\&\quad+P\left(\frac{\sum_{i=k'_{j+1}}^{m'_{j+1}}Y_{n_{j+1}+1-i,n_{j+1}}}{Y_{n_{j+1}-m'_{j+1},n_{j+1}}}>\left(1+\frac{\varepsilon}{2}\right)cqr\log_2 n_j\right),\end{aligned}$$

where we employed the same monotonicity arguments as used in Case 1, and $Y_{n_{j+1}-m'_{j+1}, n_{j+1}} \le Y_{n_{j+1}-m'_j, n_{j+1}}$. This bound is greater than or equal to the bound obtained in Case 1 so that it constitutes a bound for $P_j$ in both cases. Applying now (2.9) to the right-hand side, we obtain for all large $j$ that

$$\begin{aligned} P_j &\le P\left(\sum_{i=1}^{m'_j - k'_j} Y_{i, m'_j} > \left(1 + \frac{\varepsilon}{2}\right) cqr \log_2 n_j\right) \\ &\quad + P\left(\sum_{i=1}^{m'_{j+1} - k'_{j+1}} Y_{i, m'_{j+1}} > \left(1 + \frac{\varepsilon}{2}\right) cqr \log_2 n_j\right) \\ &\le P\left(\frac{1}{m'_j - k'_j} \sum_{i=1}^{m'_j - k'_j} Y_{i, m'_j} > \left(1 + \frac{\varepsilon}{4}\right) r\right) \\ &\quad + P\left(\frac{1}{m'_{j+1} - k'_{j+1}} \sum_{i=1}^{m'_{j+1} - k'_{j+1}} Y_{i, m'_{j+1}} > \left(1 + \frac{\varepsilon}{4}\right) r\right), \end{aligned}$$

where in the last step we used

$$\frac{\log_2 n_j}{m'_j - k'_j} \to \frac{1}{c(q-1)} \quad \text{and} \quad \frac{\log_2 n_j}{m'_{j+1} - k'_{j+1}} \to \frac{1}{c(q-1)} \quad \text{as } j \to \infty.$$

From Lemma 1 we get for all large $j$ that

$$\begin{aligned} P_j &\le \exp\left(-m'_j K_{\alpha, 1/q}\left(\left(1 + \frac{\varepsilon}{4}\right) r\right)(1 + o(1))\right) \\ &\quad + \exp\left(-m'_{j+1} K_{\alpha, 1/q}\left(\left(1 + \frac{\varepsilon}{4}\right) r\right)(1 + o(1))\right). \end{aligned}$$

By construction $m'_j \sim m'_{j+1} \sim cq \log j$ as $j \to \infty$, and the definition of $r$ and the properties of $K_{\alpha, 1/q}$ established in Lemma 2 entail

$$cqK_{\alpha, 1/q}((1 + \varepsilon/4) r) > 1.$$

This clearly implies $\sum_{j=2}^{\infty} P_j < \infty$ and completes the proof of the proposition. □

In the derivation of *a lower bound for* $\tilde{M}(\alpha, c, q)$, the factor $Y_{n-m,n}^{-1}$ from (2.9) cannot be controlled by an argument like the one involving (3.31). Instead, we will use the following result on probabilities of large deviations. It is a variant of the theorem in Plachky and Steinebach (1975) and can be verified by the same proof.

LEMMA 4. *Let $Z_j$, $j \ge 1$, be nonnegative random variables (not necessarily independent) and $c_j$, $j \ge 1$, positive real constants with $c_j \to \infty$ as $j \to \infty$.*

*Assume that*:

(3.32) *there exists a* $0 < T \le \infty$ *such that* $M_j(t) \equiv E(\exp(tZ_j)) < \infty$ *for all* $0 \le t < T$ *and all large* $j$ *(possibly depending on* $t$*)*;

(3.33) *there exists a real-valued function* $c_0$ *on* $(0, T)$ *with continuous, strictly increasing derivative* $c_0'$ *such that for all* $0 < t < T$,

$$\lim_{j\to\infty} \frac{1}{c_j} \log M_j(t) = c_0(t).$$

*Then for each* $h \in (0, T)$ *and each sequence* $a_j$, $j \ge 1$, *of real constants with* $a_j \to c_0'(h)$ *as* $j \to \infty$,

$$\lim_{j\to\infty} \frac{1}{c_j} \log P\left(\frac{1}{c_j} Z_j > a_j\right) = c_0(h) - hc_0'(h).$$

PROPOSITION 6. *For* $0 < \alpha < 2$, $0 < c < \infty$ *and* $1 < q < \infty$, *let* $r_{\alpha,c,q}$ *be defined by* (3.29). *Then* $\tilde{M}(\alpha, c, q) \ge (cq)^{1-1/\alpha}(1 - 1/q)\gamma_{cq}''^{1/\alpha} r_{\alpha,c,q}$.

PROOF. Write $r = r_{\alpha,c,q}$ and set $m_n = [cq \log_2 n]$ and $k_n = [m_n/q]$ for $n \ge 3$. Fix $0 < \varepsilon < 1$ arbitrarily and consider the increasing sequence of integers $n_j = j^j$, $j \ge 1$. Also set $k_j' = k_{n_j}$ and $m_j' = m_{n_j}$. According to Proposition 4 and the definition of $\tilde{M}(\alpha, c, q)$ in (2.7), it is enough to show that

$$P^* \equiv P\left(\frac{\sum_{i=k_{j+1}'+1}^{m_{j+1}'} Y_{n_{j+1}+1-i,\, n_{j+1}}}{n_{j+1}^{1/\alpha}(\log_2 n_{j+1})^{1-1/\alpha}} \ge (1-\varepsilon)^2 (cq)^{1-1/\alpha}\left(1 - \frac{1}{q}\right)\gamma_{cq}''^{1/\alpha} r \text{ i.o.}\right) = 1.$$

For the proof set $\underline{n}_j = n_{j+1} - n_j$ and let $Y_{1,\underline{n}_j}^{(j)} \le \cdots \le Y_{\underline{n}_j,\underline{n}_j}^{(j)}$ denote the order statistics of $Y_{n_j+1}, \dots, Y_{n_{j+1}}$. Notice that $Y_{\underline{n}_j+1-i,\underline{n}_j}^{(j)} \le Y_{n_{j+1}+1-i,\, n_{j+1}}$ for all $1 \le i \le \underline{n}_j$ so that

$$P^* \ge P\left(\frac{\sum_{i=k_{j+1}'+1}^{m_{j+1}'} Y_{\underline{n}_j+1-i,\underline{n}_j}^{(j)}}{n_{j+1}^{1/\alpha}(\log_2 n_{j+1})^{1-1/\alpha}} \ge (1-\varepsilon)^2 (cq)^{1-1/\alpha}\left(1 - \frac{1}{q}\right)\gamma_{cq}''^{1/\alpha} r \text{ i.o.}\right). \tag{3.34}$$

Since the random variables $\sum_{i=k_{j+1}'+1}^{m_{j+1}'} Y_{\underline{n}_j+1-i,\underline{n}_j}^{(j)}$, $j \ge 1$, are independent by construction and the joint distributions of $\{Y_{i,\underline{n}_j}^{(j)}: 1 \le i \le \underline{n}_j\}$ and $\{Y_{i,\underline{n}_j}: 1 \le i \le \underline{n}_j\}$ are the same for each $j$, the probability on the right-hand side of (3.34) equals 1 by the Borel–Cantelli lemma provided that $\sum_{j=1}^{\infty} P_j = \infty$ holds with

$$P_j = P\left(\frac{\sum_{i=k_{j+1}'+1}^{m_{j+1}'} Y_{\underline{n}_j+1-i,\underline{n}_j}}{n_{j+1}^{1/\alpha}(\log_2 n_{j+1})^{1-1/\alpha}} \ge (1-\varepsilon)^2 (cq)^{1-1/\alpha}\left(1 - \frac{1}{q}\right)\gamma_{cq}''^{1/\alpha} r\right).$$

To verify this, we will derive a suitable lower bound for $P_j$. We have

$$P_j \geq P\left(\left\{\frac{\sum_{i=k'_{j+1}+1}^{m'_{j+1}} Y_{\underline{n}_j+1-i,\underline{n}_j}}{(\log_2 n_{j+1}) Y_{\underline{n}_j-m'_{j+1},\underline{n}_j}} > (1-\varepsilon)cq\left(1-\frac{1}{q}\right)r\right\}\right.$$

$$\left.\cap\left\{\frac{Y_{\underline{n}_j-m'_{j+1},\underline{n}_j}}{n_{j+1}^{1/\alpha}(\log_2 n_{j+1})^{-1/\alpha}} \geq (1-\varepsilon)(cq)^{-1/\alpha}\gamma_{cq}''^{1/\alpha}\right\}\right)$$

$$\geq P\left(\frac{\sum_{i=k'_{j+1}+1}^{m'_{j+1}} Y_{\underline{n}_j+1-i,\underline{n}_j}}{(\log_2 n_{j+1}) Y_{\underline{n}_j-m'_{j+1},\underline{n}_j}} \geq (1-\varepsilon)c(q-1)r\right)$$

$$-P\left(\frac{Y_{\underline{n}_j-m'_{j+1},\underline{n}_j}}{n_{j+1}^{1/\alpha}(\log_2 n_{j+1})^{-1/\alpha}} < (1-\varepsilon)(cq)^{-1/\alpha}\gamma_{cq}''^{1/\alpha}\right) \equiv P_{1,j} - P_{2,j}$$

and we will show that $P_1^* = \sum_{j=1}^{\infty} P_{1,j} = \infty$ and $P_2^* \equiv \sum_{j=1}^{\infty} P_{2,j} < \infty$, which implies the desired result. We consider $P_2^*$ first. Let $W_j$, $j \geq 1$, be iid random variables with common exponential distribution with mean 1, and for $n \geq 1$ let $W_{1,n} \leq \cdots \leq W_{n,n}$ be the order statistics of $W_1, \dots, W_n$. Then for each $n \geq 1$ the joint distributions of $\{Y_{n+1-i,n}: 1 \leq i \leq n\}$ and $\{(1-\exp(-W_{i,n}))^{-\alpha}: 1 \leq i \leq n\}$ are equal; hence by a straightforward calculation,

$$P_{2,j} = P\left(W_{m'_{j+1}+1,\underline{n}_j} > -\log\left(-\frac{cq}{(1-\varepsilon)^{\alpha}\gamma_{cq}''}\frac{\log_2 n_{j+1}}{n_{j+1}}\right)\right).$$

We will evaluate $P_{2,j}$ by an application of Lemma 4 to $Z_j = \underline{n}_j W_{m'_{j+1}+1,\underline{n}_j}$ and $c_j = m'_{j+1} + 1$. Utilizing the Rényi representation of exponential order statistics and $E(\exp(xW_1)) = 1/(1-x)$ for $0 < x < 1$, we obtain

$$M_j(t) = E\left(\exp\left(t\underline{n}_j W_{m'_{j+1}+1,\underline{n}_j}\right)\right) = E\left(\exp\left(t\underline{n}_j \sum_{i=1}^{m'_{j+1}+1} \frac{W_i}{\underline{n}_j + 1 - i}\right)\right) < \infty$$

for $0 < t < 1 - m'_{j+1}/\underline{n}_j \to 1$ as $j \to \infty$ so that (3.32) holds with $T = 1$. Moreover, for $0 < t < 1$ and all large $j$,

$$M_j(t) = \prod_{i=1}^{m'_{j+1}+1}\left(1 - \frac{t\underline{n}_j}{\underline{n}_j + 1 - i}\right)^{-1},$$

from which by an elementary analysis,

$$\lim_{j\to\infty} \frac{1}{m'_{j+1}+1} \log M_j(t) = \log\left(\frac{1}{1-t}\right) \equiv c_0(t).$$

The function $c_0$ is one-to-one from $(0,1)$ to $(1,\infty)$ with derivative $c_0'(t) = 1/(1-t)$ so that (3.33) is satisfied. Notice that $\underline{n}_j/(m'_{j+1}+1) \sim n_{j+1}/(cq\log_2 n_{j+1})$; hence

$$a_j \equiv -\frac{\underline{n}_j}{m'_{j+1}+1}\log\left(1-\frac{cq}{(1-\varepsilon)^{\alpha}\gamma''_{cq}}\frac{\log_2 n_{j+1}}{n_{j+1}}\right) = \frac{1}{(1-\varepsilon)^{\alpha}\gamma''_{cq}} + o(1)$$

as $j\to\infty$.

Let $0<h<1$ be defined by $((1-\varepsilon)^{\alpha}\gamma''_{cq})^{-1} = c'(h)$, that is, $h = 1-(1-\varepsilon)^{\alpha}\gamma''_{cq}$. Then from Lemma 4 we get

$$\lim_{j\to\infty}\frac{1}{m'_{j+1}+1}\log P_{2,j} = c_0(h) - hc_0'(h)$$

$$= -\log\left((1-\varepsilon)^{\alpha}\gamma''_{cq}\right) - \left((1-\varepsilon)^{\alpha}\gamma''_{cq}\right)^{-1} + 1.$$

Taking $m'_{j+1}+1 \sim cq\log_2 n_{j+1} \sim cq\log j$ into account, we obtain

$$P_{2,j} = \exp\left\{-(\log j)cq\left(\left((1-\varepsilon)^{\alpha}\gamma''_{cq}\right)^{-1} + \log\left((1-\varepsilon)^{\alpha}\gamma''_{cq}\right) - 1\right)(1+o(1))\right\}.$$

By definition of $\gamma''_{cq}$ in terms of the function $h$ from (3.3) and the fact that $x^{-1}+\log x - 1$ is strictly decreasing for $1<x<\infty$, we have

$$1 = cq\left(\gamma''^{-1}_{cq} + \log\gamma''_{cq} - 1\right) < cq\left(\left((1-\varepsilon)^{\alpha}\gamma''_{cq}\right)^{-1} + \log\left((1-\varepsilon)^{\alpha}\gamma''_{cq}\right) - 1\right).$$

This proves $P_2^* < \infty$. To verify $P_1^* = \infty$, we first employ (2.9) to obtain

$$P_{1,j} = P\left(\sum_{i=1}^{m'_{j+1}-k'_{j+1}} Y_{i,m'_{j+1}} > (1-\varepsilon)c(q-1)r\log_2 n_{j+1}\right)$$

and then $(\log_2 n_{j+1})/(m'_{j+1}-k'_{j+1}) \to 1/c(q-1)$ which for all large $j$ yields

$$P_{1,j} \ge P\left(\frac{1}{m'_{j+1}-k'_{j+1}}\sum_{i=1}^{m'_{j+1}-k'_{j+1}} Y_{i,m'_{j+1}} > \left(1-\frac{\varepsilon}{2}\right)r\right)$$

$$= \exp\left(-m'_{j+1}K_{\alpha,1/q}\left(\left(1-\frac{\varepsilon}{2}\right)r\right)(1+o(1))\right)$$

by Lemma 1. By construction $m'_{j+1} \sim cq\log j$ as $j\to\infty$, and the definition of $r$ and the properties of $K_{\alpha,1/q}$ established in Lemma 2 entail $cqK_{\alpha,1/q}((1-\varepsilon/2)r) < 1$. This clearly implies $P_1^* = \infty$ and completes the proof. □

From Propositions 3, 5 and 6 we now obtain the inequality

$$M_l^+(\alpha,c,q) \le \frac{(2-\alpha)^{1/2}}{2c^{1/2-1/\alpha}}\tilde{M}(\alpha,c,q) - M(\alpha,c,q) \le M_u^+(\alpha,c,q) \tag{3.35}$$

for all $1 < q < \infty$, where

$$M_l^+(\alpha, c, q) = (2-\alpha)^{1/2}\frac{c^{1/2}}{2} \times \left\{ q^{1-1/\alpha}\left( \left(1-\frac{1}{q}\right)\gamma_{cq}^{\prime\prime 1/\alpha} r_{\alpha,c,q} - \frac{\alpha}{\alpha-1}\right) + \frac{\alpha}{\alpha-1}\right\}$$

and

$$M_u^+(\alpha, c, q) = (2-\alpha)^{1/2}\frac{c^{1/2}}{2}\left\{ q^{1-1/\alpha}\left(\gamma_{cq}^{\prime 1/\alpha} r_{\alpha,c,q} - \frac{\alpha}{\alpha-1}\right) + \frac{\alpha}{\alpha-1}\right\}.$$

These bounds are crucial for *the evaluation of the limit in* (2.8) which completes the proof of Theorem 1. Since $\alpha$ and $c$ are fixed, from now on we will drop them from our notation. With this convention in mind, recall first that the function $r(q) = r_{\alpha,c,q}$, $1 < q < \infty$, is defined through (3.29). According to Lemma 3, for each $1 < q < \infty$ there exists at least one $b(q)$ with $r(q) < b(q) < f_\alpha^{-1}(r(q))$ and

$$\inf_{r(q) < b < \infty} h_{\alpha, 1/q, r(q)}(b) = h_{\alpha, 1/q, r(q)}(b(q)). \tag{3.36}$$

Choosing a particular such $b(q)$ for each $1 < q < \infty$, we can define a real-valued function $b(q)$, $1 < q < \infty$. Moreover, we define a third real-valued function $s(q)$, $1 < q < \infty$, by setting $s(q) = s_\alpha(r(q), b(q))$. Then by (3.29), Lemma 3 and (3.36) for each $1 < q < \infty$, we have

$$\begin{aligned} 1 = cq\Bigg\{ &\frac{1}{q}\log\frac{1}{q} + \left(1-\frac{1}{q}\right)\log\left(1-\frac{1}{q}\right) + \left(1-\frac{1}{q}\right)s(q)r(q) \\ &- \left(1-\frac{1}{q}\right)\log\left(\int_1^{b(q)} e^{s(q)x}\,dF_\alpha(x)\right) + \frac{\alpha}{q}\log b(q)\Bigg\}. \end{aligned} \tag{3.37}$$

Notice that $s(q) = s_\alpha(r(q), b(q))$ is defined through (3.19) because of $r(q) < b(q) < f_\alpha^{-1}(r(q))$. Consequently, by (3.19) for each $1 < q < \infty$,

$$r(q) = \int_1^{b(q)} x e^{s(q)x}\,dF_\alpha(x) \Big/ \int_1^{b(q)} e^{s(q)x}\,dF_\alpha(x). \tag{3.38}$$

To derive a third relation between $r(q)$, $b(q)$ and $s(q)$, recall from the proof of Lemma 3 that $h_{\alpha,1/q,r(q)}$ is differentiable on $(r(q), f_\alpha^{-1}(r(q)))$ with derivative

$$h'_{\alpha,1/q,r(q)}(b) = -\left(1-\frac{1}{q}\right)\frac{\alpha b^{-\alpha-1}\exp(s_\alpha(r(q),b)b)}{\int_1^b \exp(s_\alpha(r(q),b)x)\,dF_\alpha(x)} + \frac{\alpha}{qb};$$

cf. (3.27). Taking (3.36) into account, we see that for each $1 < q < \infty$ we must have $h'_{\alpha,1/q,r(q)}(b(q)) = 0$, that is,

$$\frac{1/q}{1-1/q} = \frac{b(q)^{-\alpha}e^{s(q)b(q)}}{\int_1^{b(q)} e^{s(q)x}\,dF_\alpha(x)}. \tag{3.39}$$

Taking logarithms and rearranging terms, we obtain

$$\log\left(\int_1^{b(q)} e^{s(q)x}\, dF_\alpha(x)\right) - \log\left(1 - \frac{1}{q}\right) = -\log\frac{1}{q} - \alpha\log b(q) + s(q)b(q),$$

which when utilized in (3.37) for each $1 < q < \infty$ gives

$$\begin{aligned}1 = c\Bigg\{ & q\log\left(1 - \frac{1}{q}\right) + q\left(s(q)r(q) - \log\left(\int_1^{b(q)} e^{s(q)x}\, dF_\alpha(x)\right)\right) \\ & \qquad -s(q)r(q) + s(q)b(q)\Bigg\} \\ \equiv {} & cR(q).\end{aligned} \tag{3.40}$$

Equations (3.38)–(3.40) are sufficient to determine the asymptotic behavior of $r(q)$, $b(q)$ and $s(q)$ as $q \to \infty$ and then to derive the form of $M^+(\alpha, c)$. These purely analytic considerations are somewhat tedious and will not be given here in full detail. In a first step one shows

$$0 < \liminf_{q\to\infty} s(q)b(q) \le \limsup_{q\to\infty} s(q)b(q) < \infty, \tag{3.41}$$

and if $q_k$, $k \ge 1$, is any sequence of constants with $q_k \to \infty$ and $s(q_k)b(q_k) \to \vartheta$ as $k \to \infty$ for some $0 < \vartheta < \infty$, then $\vartheta$ necessarily satisfies (1.10). Moreover, if $\alpha \neq 1$, then

$$\begin{aligned}\lim_{k\to\infty} M_l^+(\alpha, c, q_k) &= \lim_{k\to\infty} M_u^+(\alpha, c, q_k) \\ &= \frac{(2-\alpha)^{1/2}\alpha}{\alpha - 1}\frac{c^{1/2}}{2}\left(1 + \left(\frac{1}{c\vartheta} - 1\right)e^{\vartheta/\alpha}\right),\end{aligned} \tag{3.42}$$

whereas

$$\begin{aligned}\lim_{k\to\infty} M_l^+(1, c, q_k) &= \lim_{k\to\infty} M_u^+(1, c, q_k) \\ &= \frac{c^{1/2}}{2}\left(\int_0^{\vartheta}\left(\log\frac{\vartheta}{x}\right)e^x\, dx + \vartheta\right).\end{aligned} \tag{3.43}$$

The constant $M^+(\alpha, c)$ is now most easily determined in the case $\alpha = 1$. Then (1.10) is tantamount to $1/c = \vartheta(1 - e^{-\vartheta})$, the right-hand side being continuous and strictly increasing from 0 to $\infty$ for $0 \le \vartheta < \infty$. Hence, for each $0 < c < \infty$, (1.10) has a unique solution $\vartheta_c$ in $(0, \infty)$, and by a simple compactness argument based on (3.41) and (3.43),

$$\lim_{q\to\infty} M_l^+(1, c, q) = \lim_{q\to\infty} M_u^+(1, c, q) = \frac{c^{1/2}}{2}\left(\int_0^{\vartheta_c}\left(\log\frac{\vartheta_c}{x}\right)e^x\, dx + \vartheta_c\right),$$

which completes the proof of Theorem 1 for $\alpha = 1$ because of (2.8) and (3.35). This reasoning obviously applies for all $0 < \alpha < 2$ for which the function

$$v_\alpha(\vartheta) \equiv \vartheta^\alpha e^{-\vartheta}\int_0^{\vartheta} x^{-\alpha+1} e^x\, dx = \vartheta^2 e^{-\vartheta}\int_0^1 x^{-\alpha+1}e^{\vartheta x}\, dx, \qquad 0 \le \vartheta < \infty,$$

appearing in (1.10) is strictly increasing, because $v_\alpha$ is clearly continuous with $v_\alpha(0) = 0$ and $v_\alpha(\vartheta) \to \infty$ as $\vartheta \to \infty$. However, $v_\alpha$ is not strictly increasing for all $0 < \alpha < 2$. Its actual behavior is described in the following lemma.

LEMMA 5. *There exists a unique* $3/2 < \alpha^* < 2$ *such that* $4\alpha^* = v_{\alpha^*}(2\alpha^*)$. *If* $0 < \alpha \le \alpha^*$, *then* $v_\alpha$ *is strictly increasing from* $0$ *to* $\infty$, *and if* $\alpha^* < \alpha < 2$, *then the derivative of* $v_\alpha$ *has exactly two distinct zeros* $0 < \vartheta_1 < \vartheta_2 < \infty$ *(depending on* $\alpha$*), and* $v_\alpha$ *is strictly increasing on* $(0, \vartheta_1]$ *and* $[\vartheta_2, \infty)$ *and strictly decreasing on* $[\vartheta_1, \vartheta_2]$.

Lemma 5 implies that for $0 < \alpha \le \alpha^*$, (1.10) has a unique solution for each $0 < c < \infty$ so that we can complete the proof of Theorem 1 by the same reasoning as applied in the case $\alpha = 1$, using (3.42) instead of (3.43). The remaining case $\alpha^* < \alpha < 2$ is more complicated, because due to the fact that $v_\alpha$ is no longer strictly increasing there exist parameter values $c$ for which (1.10) has more than one solution $\vartheta$. However, the proper $\vartheta$ in the description of $M^+(\alpha, c)$ can be identified by using the fact that $M^+(\alpha, c)$ is a continuous function of $0 < c < \infty$, which completes the proof of Theorem 1; cf. Haeusler (1988) for the details of this analysis.

**4. Concluding remarks.** It is worthwhile to note that the statements of Theorem 1 and 2 are of a different nature for $0 < \alpha < 1$ and $1 \le \alpha < 2$. If $0 < \alpha < 1$, then it follows from (3.1) and (3.2) that the centering and norming constants in these theorems are of the same order of magnitude, more precisely, that

$$\frac{\mu_n(k_n)}{(2\log_2 n)^{1/2}\sigma_n(k_n)} \to \frac{(2-\alpha)^{1/2}\alpha}{1-\alpha}\,\frac{c^{1/2}}{2} \quad \text{as } n \to \infty.$$

Therefore, for $0 < \alpha < 1$, Theorems 1 and 2 are in fact statements about the lim sup and lim inf of $S_n(k_n)/(2\log_2 n)^{1/2}\sigma_n(k_n)$ without any centering. If $1 \le \alpha < 2$, however, then

$$\frac{\mu_n(k_n)}{(2\log_2 n)^{1/2}\sigma_n(k_n)} \to \infty \quad \text{as } n \to \infty,$$

so that the centering constants are of a larger order of magnitude than the norming constants, as it is the case in the classical Hartman–Wintner LIL and also in (1.4).

We considered only positive random variables, that is, one tail of a one-dimensional distribution. Our approach, however, is also suitable for the investigation of the two-sided trimmed sums $\sum_{i=k_n+1}^{n-k_n} X_{i,n}$ as considered in (1.4) for an underlying two-sided distribution in the domain of attraction of a stable law. In this case an analog of Proposition 1 shows that the constants corresponding to $M^{\pm}(\alpha, c)$ are again entirely determined by the extreme summands of $\sum_{i=k_n+1}^{n-k_n} X_{i,n}$. On the other hand, it is well known that the sample extremes from the two tails of a two-sided distribution are asymptotically independent. Utilizing this fact, it becomes possible to deal with the contributions to the constants coming from the two tails as if they were coming from two independent samples. Thus, in the proof of two-sided versions of Theorems 1 and 2, one has to use a result on probabilities of large deviations for an appropriate linear combination of $\sum_{i=1}^{n-[\tau n]} Y_{i,n}$ and $\sum_{i=1}^{n-[\tau n]} \tilde{Y}_{i,n}$, where $Y_i$, $i \geq 1$, and $\tilde{Y}_i$, $i \geq 1$, are now two independent sequences of iid standard Pareto random variables. This causes no problem in principle, but increases the amount of technicalities, which we will not detail.

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