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DYNAMICS OF THE MCKEAN–VLASOV EQUATION

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This note studies the deterministic flow of measures which is the limiting case as $n \to \infty$ of Dyson's model of the motion of the eigenvalues of random symmetric $n \times n$ matrices. Though this flow is nonlinear, highly singular and apparently of Wiener–Hopf type, it may be solved explicitly without recourse to Wiener–Hopf theory. The solution greatly clarifies the role of the famous Wigner semicircle law.

1. Some background to the problem. This note concerns a Wiener–Hopf type equation which arises from the study of the limiting McKean–Vlasov dynamics of random eigenvalue motions. In the limit as the number of eigenvalues becomes large, the occupation density of the eigenvalues converges to the solution of a deterministic measure-valued equation known as the McKean–Vlasov equation [see Chan (1992)]. Although the "flow" of the limiting measure is specified via the McKean–Vlasov equation, the fact that it is a weak equation which has to be integrated against suitable test functions means that it is difficult to visualize the flow directly from this equation. The aim of the present work is to find a more readily accessible way of studying the dynamics of the McKean–Vlasov equation.

One of the earliest motivations for studying random eigenvalues was in physics, particularly the statistical theory of energy levels pioneered by (among others) E. P. Wigner. For example, there is the following result due to Wigner: let $S^{(n)}$ be an $n \times n$ symmetric matrix whose entries $S_{ij}^{(n)}$, $i \le j$, $i, j = 1, \dots, n$ are i.i.d. $N(0, \sigma^2 n^{-1})$. Then as $n \to \infty$, the eigenvalues of $S^{(n)}$ are distributed according to the semicircle law with density $(2\pi\sigma^2)^{-1}\sqrt{4\sigma^2 - y^2}$, $|y| \le 2\sigma$. The semicircle density will again feature prominently in the following sections. The book by Mehta (1967) gives an excellent account of the importance of this result in the statistical theory of energy levels. Roughly speaking, the symmetric matrix $S^{(n)}$ plays the role of a finite-dimensional approximation to a Hamiltonian (which is a self-adjoint operator on some Hilbert space), and the study of the Schrödinger equation for $S^{(n)}$ naturally leads to consideration of its eigenvalues. The randomness comes in here because rather than considering systems with a particular interaction mechanism, one considers a whole *ensemble* of possible interaction mechanisms, each having a certain probability; this is in direct analogy with ordinary statistical mechanics, where one

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considers a system which could be in any one of a whole ensemble of states. Dyson, in a series of some four or five papers [see, e.g., Dyson (1962)], has also made a detailed study of statistical mechanical models of so-called Coulomb gases which involve spectral analysis of Brownian motions of symmetric matrices, as well as other matrix ensembles, such as unitary ensembles.

In an attempt to explain the result of Wigner cited above and to extend some of Dyson's ideas, an earlier paper, Chan (1992), considers a diffusion process valued on symmetric matrices which has the Normal distribution as its invariant measure (i.e., a matrix-valued Ornstein–Uhlenbeck process) and then goes on to consider the associated eigenvalue diffusion in $\mathbb{R}^n$. It turns out that the eigenvalues behave like charged particles on the line, and the earlier work Chan (1992) is closely related to that of Dawson (1983) and Dawson and Gärtner (1987) on diffusion models of interacting particles. In Dawson (1983) and Dawson and Gärtner (1987) the asymptotic behavior of the particle system (as the number of particles become large) is studied via the large deviations from the limiting McKean–Vlasov equation, while Chan (1992) takes a different approach by considering the equilibrium points of the McKean–Vlasov equation itself.

2. A nonlinear Wiener–Hopf equation. The following problem is studied in detail in Chan (1992). Consider the following finite system of n interacting particles in $\mathbb{R}$:

$$(2.1)\quad d\lambda_i = \frac{1}{\sqrt{n}}\,d\beta_i - \frac{1}{2}\lambda_i\,dt + \frac{1}{2n}\sum_{k\neq i}\frac{1}{\lambda_i - \lambda_k}\,dt, \qquad i = 1,\dots,n.$$

(Here β_i are independent Brownian motions on $\mathbb{R}$.) The associated measure-valued empirical process $\mu_n(t) := n^{-1}\sum_{i=1}^{n}\delta_{\lambda_i(t)}$ is shown to converge weakly as $n \to \infty$ to a deterministic measure-valued function μ_t satisfying

$$\frac{d}{dt}\langle\mu(t),\phi\rangle = \frac{1}{4}\iint\frac{\phi'(x)-\phi'(y)}{x-y}\mu_t(dx)\mu_t(dy) - \frac{1}{2}\int x\phi'(x)\mu_t(dx)$$

for all test functions ϕ. The space of test functions here consists of all bounded continuous twice-differentiable functions with bounded continuous derivatives. However, it is shown in Chan (1992) that if μ_t has no atomic component, then $\mu(\cdot)$ can be equivalently characterized as the weak solution to the McKean–Vlasov equation

$$(2.2)\qquad \frac{d}{dt}\int\phi(\lambda)\mu_t(d\lambda) = \frac{1}{2}\int\left\{⨍\frac{\mu_t(dy)}{\lambda - y} - \lambda\right\}\phi'(\lambda)\mu_t(d\lambda)$$

for all suitable test functions ϕ. (Here $⨍$ denotes Cauchy principal value.) Finally, it is shown in Chan (1992) that if for all t, μ_t has a density function belonging to L^2 such that $\int x\mu_t(x)\,dx < \infty$, and if the above equation is solved for the family of test functions $\phi_\theta(\lambda) = e^{-i\theta\lambda}$, $\theta \in \mathbb{R}$, then the Fourier transform of μ satisfies the following strange-looking nonlinear Wiener–Hopf type equation:

$$\frac{\partial \nu}{\partial t} = \frac{\theta}{4}\left\{\int_{-\infty}^{0} \nu_t(u)\nu_t(\theta - u)\,du - \int_{0}^{\infty} \nu_t(u)\nu_t(\theta - u)\,du\right\} - \frac{\theta}{2}\frac{\partial \nu}{\partial \theta}, \tag{2.3}$$

where

$$\nu(t,\theta) = \int_{-\infty}^{\infty} e^{-i\theta\lambda}\mu_t(d\lambda).$$

We assume that μ_t has an L^2 density function because in many ways L^2 is the natural setting for the method used in the next section to solve (2.3), although the L^2 assumption can be relaxed. The Hilbert transform of a function f is defined to be

$$\mathbf{H}f(x) = \frac{1}{\pi}⨍\frac{f(y)}{x-y}\,dy.$$

The key to deriving (2.3) from (2.2) is the following identity: for $f \in L^2$,

$$i(\mathbf{H}f)\hat{}\,(\theta) = \operatorname{sgn}(\theta) f\hat{}\,(\theta). \tag{2.4}$$

(Here $f\hat{}$ is the fourier transform of f.) If (2.3) has a solution in L^2 for all t, then (2.2) has a solution which has an L^2 density for all t. We shall therefore seek a solution to (2.3) in L^2.

The identity (2.4) also holds for functions in L^p for $1 < p < 2$, as does the Fourier inversion formula (provided it is interpreted in an appropriate sense) and there is also a corresponding H^p theory, so in many cases our method can be generalized to density functions in L^p, $p \in (1,2]$. Although the identity (2.4) can also be generalized to measures if it is interpreted in a suitable weak sense, the important point here is that when μ_t has a suitable density function, using the test functions $\phi_\theta(\lambda) = e^{-i\theta\lambda}$, $\theta \in \mathbb{R}$ one obtains an equation for the Fourier transform $\nu(t,\theta)$ which holds in the *strong* sense. The purpose in considering (2.3) rather than (2.2) is that the former provides a concrete means of interpreting the rather abstract weak equation (2.2). For instance, (2.3) can be solved numerically, and the results of some numerical solutions to (2.3) are shown in Chan (1992).

The main object of the earlier work Chan (1992) is to show that the Wigner semicircle law

$$\mu_*(y) := \pi^{-1}\sqrt{2-y^2}, \qquad |y| \le \sqrt{2}$$

is the unique equilibrium point of (2.2) having a Hölder-continuous density belonging to L^2, which is also the unique equilibrium point possessing finite moments of all orders. Moreover, if initially μ_0 possesses finite moments of all orders, then we actually have $\mu_t \Rightarrow \mu_*$ as $t \to \infty$. The object of the present article is to show how the Wiener–Hopf type equation (2.3) can be solved exactly and to investigate the asymptotic behavior of the solution and the way it relates to the semicircle law. It is surprising that an explicit solution exists at all, as on first sight (2.3) is probably among the most difficult Wiener–Hopf type equations one is every likely to encounter. Yet, as we shall see shortly, despite its nonlinearity, there are certain special features of (2.3) which allow an explicit solution to be obtained by very simple means.

The essential idea behind (2.3) is that by making an appropriate choice of a (separating) family of test functions (or perhaps even one special test function), one can obtain an equation for a functional of μ which gives a more concrete way of thinking about the dynamical evolution of the McKean–Vlasov limit and which in some sense may be more amenable to analysis. The functions $\phi_\theta(\lambda) = e^{-i\theta\lambda}$, $\theta \in \mathbb{R}$ are one such family but this is not necessarily the only, or even the best, choice of test function. One alternative approach may be based on the idea of Boltzmann's so-called "H-theorem." Firstly, observe that we are not restricted to test functions of one variable: for example we could use test functions of two variables and consider functionals of the form

$$F(\mu) = \int_{\mathbb{R}^2} f(x,y)\mu(dx)\mu(dy)$$

for bounded twice-differentiable functions f with bounded partial derivatives. The same treatment as in Chan (1992) will yield the following equation for the McKean–Vlasov limit:

$$\begin{aligned} \frac{d}{dt}\int f(x,y)\mu_t(dx)\mu_t(dy) &= \frac{1}{4}\iiint \frac{f_x(x,y) - f_x(z,y)}{x - z}\mu_t(dx)\mu_t(dy)\mu_t(dz) \\ &+ \frac{1}{4}\iiint \frac{f_y(x,y) - f_y(x,z)}{y - z}\mu_t(dx)\mu_t(dy)\mu_t(dz) \\ &- \frac{1}{2}\iint x f_x(x,y)\mu_t(dx)\mu_t(dy) \\ &- \frac{1}{2}\iint y f_y(x,y)\mu_t(dx)\mu_t(dy), \end{aligned}$$

and if for all t, μ_t has no atomic component, then

$$\begin{aligned} \frac{d}{dt}\int f(x,y)\mu_t(dx)\mu_t(dy) &= \frac{1}{2}\iint f_x(x,y)\left(⨍ \frac{\mu_t(dz)}{x - z} - x\right)\mu_t(dx)\mu_t(dy) \\ &+ \frac{1}{2}\iint f_y(x,y)\left(⨍ \frac{\mu_t(dz)}{y - z} - y\right)\mu_t(dx)\mu_t(dy). \end{aligned} \tag{2.5}$$

Now define a functional H by

$$H(\mu) = \int x^2\mu(dx) - \iint \log|x - y|\mu(dx)\mu(dy).$$

Taking as our test function $f(x, y) = \log|x - y|$, a formal calculation using (2.5)—freely interchanging orders of integration—then gives

$$\frac{d}{dt}H(\mu_t) = -\int\left(x - ⨍\frac{\mu_t(dy)}{x - y}\right)^2 \mu_t(dx) \le 0 \tag{2.6}$$

with equality only when μ satisfies

$$⨍\frac{\mu(dy)}{x - y} = x \quad \forall x \in \operatorname{supp}(\mu),$$

which is the characterization of (nonatomic) fixed points of (2.2) given in Chan (1992). In this way one also obtains the convergence to a fixed point which corresponds to a minimum of H, since the functional H is bounded from below. However, in order to justify the use of $\log|x - y|$ as a test function and make the above formal calculation rigorous, one would need to impose regularity conditions on the density of μ_t (for all t) which may or may not be reduced to regularity conditions on the initial measure μ_0. It would seem that such regularity conditions have to be much stronger than those imposed on μ_0 in Chan (1992), which only involve the moments of μ_0. (In order to obtain convergence to the fixed point, it is not necessary to assume that μ_0 even possesses a density.) In any case, it is not at all clear that (2.6) gives a better idea of the dynamics of μ_t than (2.2); the whole point of (2.3) is that it is an autonomous equation for the Fourier transform ν. More worrying perhaps, is the fact that although H may have a unique minimum, $H(\mu_1) = H(\mu_2)$ does not imply that $\mu_1 = \mu_2$. However, the idea of an "H-theorem" does seem to feature in the large deviations of the stationary empirical measure μ_n associated with the system (2.1) solved with its invariant (Gibbs) density as the initial condition—see Dawson and Gärtner (1988).

3. The solution. We assume that the initial measure μ_0 for (2.2) has an L^2 density function such that $\int x\mu_0(x)\,dx < \infty$. As with the classical Wiener–Hopf integral equation, we solve the present problem by considering the Fourier transform of (2.3). [The fact that in (2.3) ν is already a Fourier transform of something else does not greatly concern us for the moment.] First, define the "positive" and "negative" restrictions of ν as follows:

$$\nu^+(t,\theta) = \begin{cases} \nu(t,\theta), & \theta \ge 0, \\ 0, & \theta < 0, \end{cases}$$

$$\nu^-(t,\theta) = \begin{cases} 0, & \theta \ge 0, \\ \nu(t,\theta), & \theta < 0. \end{cases}$$

The definition of $\nu^\pm$ at $\theta = 0$ is not of any great importance, so long as we have $\nu = \nu^+ + \nu^-$.

The key to the solution lies in the following miraculous cancellations which allow us to write down separate *autonomous* equations for $\nu^\pm$. If we expand $\nu = \nu^+ + \nu^-$ in (2.3), we see that the first integral does not involve $\nu_t^+(u)$ and the second integral does not involve $\nu_t^-(u)$ since these are zero in the range of integration, and we get the following "cross-terms" which appear in each of the convolutions:

$$\text{(3.1a)} \qquad \int_{-\infty}^{0} \nu_t^-(u)\nu_t^+(\theta - u)\, du,$$

$$\text{(3.1b)} \qquad \int_{0}^{\infty} \nu_t^+(u)\nu_t^-(\theta - u)\, du.$$

Now, the integrand in (3.1a) is nonzero only for $u \le \theta$, so it may be written as

$$\int_{-\infty}^{\theta} \nu_t^-(u)\nu_t^+(\theta - u)\, du = \int_{0}^{\infty} \nu_t^-(\theta - y)\nu_t^+(y)\, dy,$$

making a change of variables by putting $y = \theta - u$. Thus we see that the two integrals in (3.1) are identical, and because they appear with opposite signs in (2.3), the "cross-terms" all vanish. Gathering up the remaining terms in (2.3) involving only ν^+ and only ν^-, respectively, we arrive at the following pair of equations:

$$\text{(3.2a)} \qquad \frac{\partial \nu^+}{\partial t} = -\frac{\theta}{4}\int_{0}^{\infty} \nu_t^+(u)\nu_t^+(\theta - u)\, du - \frac{\theta}{2}\frac{\partial \nu^+}{\partial \theta}, \qquad \theta > 0,$$

$$\text{(3.2b)} \qquad \frac{\partial \nu^-}{\partial t} = \frac{\theta}{4}\int_{-\infty}^{0} \nu_t^-(u)\nu_t^-(\theta - u)\, du - \frac{\theta}{2}\frac{\partial \nu^-}{\partial \theta}, \qquad \theta < 0.$$

Anyone who has had any experience of Wiener–Hopf problems will appreciate that it is very rare for a Wiener–Hopf equation to decompose into its respective halves like this; indeed the essential point of the theory of Wiener–Hopf factorizations is that it is a way of dealing with equations whose "positive" and "negative" parts do *not* separate. Of course, it is now apparent that in the present case, the reason for this cancellation and separation into two halves is the presence in (2.3) of *both* the "positive" and "negative" halves of the convolution with *opposite signs*.

The two equations at (3.2) can be solved separately by taking Fourier transforms. Let

$$N^{\pm}(t, x) = \int e^{-ix\theta}\nu^{\pm}(t, \theta)\, d\theta.$$

Then, from (3.2) we obtain the following pair of quasilinear PDE's for $N^{\pm}$ (with Cauchy data):

$$\text{(3.3a)} \qquad \frac{\partial N^+}{\partial t} = -\frac{1}{2}\left[\frac{\partial N^+}{\partial x}(iN^+ - x) - N^+\right], \qquad N^+(0, x) = n^+(x),$$

$$\text{(3.3b)} \qquad \frac{\partial N^-}{\partial t} = \frac{1}{2}\left[\frac{\partial N^-}{\partial x}(iN^- + x) + N^-\right], \qquad N^-(0, x) = n^-(x).$$

Equations (3.3) can be solved by the so called method of characteristics. For example, writing (3.3a) as

$$\frac{\partial N^+}{\partial t} + \frac{1}{2}(iN^+ - x)\frac{\partial N^+}{\partial x} = \frac{1}{2}N^+,$$

we seek to integrate the exact differentials on both sides by solving the pair of ODE's:

$$\frac{dX^+}{dt} = \frac{1}{2}(iU^+ - X^+), \qquad X^+(0) = x_0^+ \tag{3.4a}$$

$$\frac{dU^+}{dt} = \frac{1}{2}U^+, \qquad U^+(0) = y_0^+. \tag{3.4b}$$

This has solution

$$X^+(t) = X^+(t, x_0^+, y_0^+) = (x_0^+ - iy_0^+/2)e^{-t/2} + iy_0^+ e^{t/2}/2,$$
$$U^+(t) = U^+(t, x_0^+, y_0^+) = y_0^+ e^{t/2}.$$

We now have an implicit formula for the solution:

$$N^+(t, X^+(t, x_0^+, n^+(x_0^+))) = U^+(t, x_0^+, n^+(x_0^+)) = n^+(x_0^+)e^{t/2}. \tag{3.5a}$$

Similarly, we get an implicit solution to (3.3b):

$$N^-(t, X^-(t, x_0^-, n^-(x_0^-))) = U^-(t, x_0^-, n^-(x_0^-)) = n^-(x_0^-)e^{t/2}, \tag{3.5b}$$

where

$$X^-(t, x_0^-, y_0^-) = (x_0^- + iy_0^-/2)e^{-t/2} - iy_0^- e^{t/2}/2,$$
$$U^-(t, x_0^-, y_0^-) \doteq y_0^- e^{t/2}.$$

Let

$$f^{\pm}(t, x) = X^{\pm}(t, x, n^{\pm}(x)) = e^{-t/2}x \pm i\sinh(t/2)n^{\pm}(x).$$

We conclude this section by verifying that the functions $f^{\pm}$ are invertible and that the solution given by (3.5a, b) does indeed belong to L^2. We first make some important observations about $n^{\pm}(z) = \int e^{-iz\theta}\nu_0^{\pm}(\theta)\,d\theta$. Our assumption that $\int x\mu_0(x)\,dx < \infty$ implies that ν_0 is continuously differentiable. An integration by parts then yields

$$-izn^+(z) = -1 - \int_0^\infty e^{-i\theta z}\nu_0'(\theta)\,d\theta$$

for any z with a nonpositive imaginary part. Similarly,

$$izn^-(z) = -1 + \int_{-\infty}^0 e^{-i\theta z}\nu_0'(\theta)\,d\theta$$

for any z with nonnegative imaginary part. Putting $z = 0$ in the above yields $\int \nu_0'(\theta)\,d\theta < \infty$ and hence by the Riemann–Lebesgue lemma we deduce the asymptotic result

$$n^{\pm}(z) \sim \mp \frac{i}{z} \quad \text{as } |z| \to \infty. \tag{3.6}$$

Next, observe that since n^+ is the Fourier transform of a function which vanishes on $\mathbb{R}^-$, it is a Hardy function of class H^{2+} [see Dym and McKean (1976)]. In particular, [because we adopt a different sign convention from that of Dym and McKean (1976) in our definition of the Fourier transform], n^+ is analytic on the *lower* half-plane $\mathbb{H}^- = \{z\colon \Im z < 0\}$. For $\mu_0 \in L^2$, the recipe

$$z \mapsto \frac{1}{i}\int_{\mathbb{R}} \frac{\mu_0(y)}{y+z}\,dy$$

defines an analytic function on $\mathbb{H}^-$. Moreover, the mapping

$$\mu_0 \mapsto \frac{1}{i}\int_{\mathbb{R}} \frac{\mu_0(y)}{y+\cdot}\,dy$$

is a projection of L^2 onto H^{2+} and we have for $z \in \mathbb{H}^-$

$$n^+(z) = \frac{1}{i}\int_{\mathbb{R}} \frac{\mu_0(y)}{y+z}\,dy. \tag{3.7a}$$

[See subsection 2.4 of Dym and McKean (1976)—but again bear in mind the different sign conventions in our definitions of the Fourier transform and its inverse.] Similarly, n^- is an H^{2-} Hardy function analytic in the upper half-plane $\mathbb{H}^+ = \{z\colon \Im z > 0\}$ and for $z \in \mathbb{H}^+$

$$n^-(z) = -\frac{1}{i}\int_{\mathbb{R}} \frac{\mu_0(y)}{y+z}\,dy. \tag{3.7b}$$

Either by letting the imaginary part of z tend to zero in (3.7), as in subsection 2.11 of Dym and McKean (1976), or directly from (2.4) and the Fourier inversion formula $f^{\wedge\wedge}(x) = 2\pi f(-x)$, we see that for $x \in \mathbb{R}$,

$$n^{\pm}(x) = \pi\big(\mu_0(-x) \pm i(\mathbf{H}\mu_0)(-x)\big). \tag{3.8}$$

In particular, (3.7) and (3.8) show that $\Re n^{\pm}(x) \geq 0$ for $x \in \mathbb{R}$ and $\Re n^{\pm}(z) > 0$ for $z \in \mathbb{H}^{\mp}$.

PROPOSITION 3.1. *For each $x \in \mathbb{R}$ and $t \geq 0$, there exist unique complex numbers $g^+(t,x)$ in the closed lower half-plane $\{z\colon \Im z \leq 0\}$ and $g^-(t,x)$ in the closed upper half-plane $\{z\colon \Im z \geq 0\}$ such that $f^{\pm}(t, g^{\pm}(t,x)) = x$.*

PROOF. We prove the "+" version of the statement of the proposition; the proof of the "−" version is similar. The uniqueness assertion follows form the uniqueness of the solutions to the PDE's (3.3).

To show that for fixed t and x, $f^+(t,z) = x$ has a solution in $\overline{\mathbb{H}^-}$, it is enough to show that the image of a simple closed curve in $\mathbb{H}^-$ under $z \mapsto f^+(t,z)$ either passes through or winds around x, by the argument principle. Fix an arbitrary R, $\varepsilon > 0$ and let γ be the lower semicircle of radius R centered at $-i\varepsilon$, consisting of the diameter $D = \{z\colon z = a - i\varepsilon,\ a \in \mathbb{R},\ |a| \leq R\}$ and the lower circumference $C = \{z : z + i\varepsilon = Re^{i\theta}, \theta \in [\pi, 2\pi]\}$. (We want to avoid the real line itself because n^+ may be badly behaved at certain points on $\mathbb{R}$ since we are not assuming any regularity properties about μ_0.) If there is a point $z \in \gamma$ such that $f^+(t,z) = x$ then we have nothing more to prove, so we assume that the image of γ under f^+ does not pass through x. Since $\Re n^+ > 0$ on γ, by choosing ε sufficiently small, we can ensure that $\Im f^+(t,z) \geq 0$ along the diameter D. Also, (3.6) implies that as $R \to \infty$, the values of $|n^+|$ on γ remains bounded and it is therefore clear that for any fixed t, x, by taking R sufficiently large and ε sufficiently small we can ensure that the image of γ under $z \mapsto f^+(t,z)$ winds around x at least once and by the argument principle, γ must enclose at least one root of the equation $f^+(t,z) - x = 0$. □

Finally, to show that $\nu_t \in L^2$ for every t, where ν is the solution to (2.3), we need to check that $N^{\pm}(t, \cdot\,) \in L^2$. Since $N^{\pm}(t, f^{\pm}(t,x)) = n^{\pm}(x)e^{t/2}$ and since the initial condition $n^{\pm} \in L^2$, we have $N^{\pm}(t, f^{\pm}(t, \cdot\,)) \in L^2$ for all t. But (3.6) shows that for large x, $f^{\pm}(t,x) \sim e^{-t/2}x$ and the fact that $N^{\pm}(t, \cdot\,) \in L^2$ therefore follows immediately.

If we knew the initial conditions $n^{\pm}$ explicitly and if we could invert the functions $f^{\pm}(t,x) = X^{\pm}(t, x, n^{\pm}(x))$, then we would obtain explicit formulae for the solution. However, this is usually not possible.

4. Some example calculations. Although in general it is not possible to invert the flow of the ODE's (3.4) explicitly, one can nevertheless hope to say something about the asymptotic behavior of the implicit solution (3.5). In particular, since (3.5) essentially gives the density function of the measure-valued solution to the McKean–Vlasov equation (2.2) $[N(t,x) = N^+(t,x) + N^-(t,x)$ is the Fourier transform of the solution to (2.3), which in turn is the Fourier transform of the solution to (2.2)], one should be able to see how the Wigner semicircle density can arise as the limit of the flow of (2.2).

Suppose now that we solve the McKean–Vlasov equation (2.2) with an initial density μ_0 which is in L^2 and has finite first moment. We need to solve the Cauchy problems (3.3) with initial conditions

$$n^+(x) = \int_0^\infty e^{-i\theta x}\nu_0(\theta)\,d\theta,$$

$$n^-(x) = \int_{-\infty}^0 e^{-i\theta x}\nu_0(\theta)\,d\theta.$$

The main result of this Section is the following:

PROPOSITION 4.1. *Let $\mu_0 \in L^2$ such that $\int x\mu_0(x)\,dx < \infty$. Suppose further that the initial data for* (3.3), *$n^{\pm}$, have no zeros on the real line. Let $N(t,x) = N^+(t,x) + N^-(t,x)$, where $N^{\pm}$ are the solutions to* (3.3). *Then as $t \to \infty$, $N(t,x) \to 2\sqrt{2 - x^2}$ or* 0, *according as $|x| \leq \sqrt{2}$ or $|x| > \sqrt{2}$.*

PROOF. Let $g^{\pm}$ denote the inverses of $f^{\pm}$ given by Proposition 3.1; thus $f^{\pm}(t, g^{\pm}(t,x)) = x$. From the solution (3.5) we have $N^{\pm}(t, f^{\pm}(t,x)) =$

$n^{\pm}(x)e^{t/2}$, so that $N^{\pm}(t,x) = n^{\pm}(g^{\pm}(t,x))e^{t/2}$ is the explicit solution to (3.3). We are interested in $N(t,x) = N^{+}(t,x) + N^{-}(t,x) = e^{t/2}(n^{+}(g^{+}) + n^{-}(g^{-}))$, and from the fact that

$$(4.1)\quad x = f^{\pm}(t, g^{\pm}(t,x)) = e^{-t/2}g^{\pm}(t,x) \pm i\sinh(t/2)n^{\pm}(g^{\pm}(t,x))$$

we see that

$$(4.2)\quad n^{+}(g^{+}(t,x)) + n^{-}(g^{-}(t,x)) = \frac{e^{-t/2}[g^{-}(t,x) - g^{+}(t,x)]}{i\sinh(t/2)}.$$

From (3.7), $\Re n^{\pm} > 0$ in the (open) half-planes $\mathbb{H}^{\mp}$, so $n^{\pm}$ cannot have any roots there; the assumption that $n^{\pm}$ have no roots on the real line therefore means that $n^{\pm}$ have no roots in the closed half-planes $\overline{\mathbb{H}^{\mp}}$, respectively.

Consider first the "+" version of (4.1). The key now is to find the asymptotic behaviour of $g^{+}(t,x)$ as $t \to \infty$. If we fix x and let $t \to \infty$ in (4.1), we see that we cannot have $g^{+}(t,x) \to h(x)$ as $t \to \infty$ for some function h, otherwise the right-hand side of (4.1) would tend to infinity because of our assumption that n^{+} has no zeros in $\overline{\mathbb{H}^{-}}$; indeed, this assumption together with the property (3.6) imply that, for fixed x, we must have $|g^{+}(t,x)| \to \infty$ as $t \to \infty$. For large $|x|$, we have already seen at (3.6) that $|n^{+}(x)|$ decays like $|n^{+}(x)| \sim 1/|x|$. So multiplying both sides of (4.1) by $e^{t/2}$ shows that for large t, g^{+} behaves asymptotically as $g^{+}(t,x) \sim e^{t/2}h^{+}(x)$; in this case, letting $t \to \infty$ on the right-hand sides of (4.1) and (4.2) does indeed give sensible limits. It remains now to find the function h^{+}.

Letting $t \to \infty$ in (4.1), with $g^{+}(t,x) \sim e^{t/2}h^{+}(x)$ and taking into account (3.6), gives

$$h^{+}(x) + \frac{1}{2h^{+}(x)} = x,$$

and writing $h^{+}(x) = u(x) + iv(x)$, we have

$$(4.3a)\qquad u + \frac{u}{2(u^2 + v^2)} = x,$$

$$(4.3b)\qquad v - \frac{v}{2(u^2 + v^2)} = 0.$$

Suppose first that $v \not\equiv 0$. We then have $u(x) = x/2$ and more importantly, for $|x| \le \sqrt{2}$, $v(x) = -\sqrt{2 - x^2}/2$. Repeating the above calculations for $g^{-}(t,x)$ shows that $g^{-}(t,x) \sim e^{t/2}h^{-}(x)$ as $t \to \infty$, where h^{-} satisfies exactly the same equations as (4.3) by virtue of (3.6), except that this time, $v(x) = +\sqrt{2 - x^2}/2$ because g^{-} takes values in the upper half-plane. If we now substitute $g^{\pm}(t,x) \sim e^{t/2}h^{\pm}(x)$ into the right-hand side of (4.2), we see that this gives $N(t,x) \to 2\sqrt{2 - x^2}$ as $t \to \infty$.

Finally, for $|x| > \sqrt{2}$, (4.3a) has no solution with $u = x/2$, so we must have $v = 0$ and $h^{\pm} = h(x)$ is real. Since $e^{-t/2}g^{\pm}$ then tends to the same real function, (4.2) shows that $N(t,x) \to 0$ as required. □

Since $f^{\wedge\wedge}(x) = 2\pi f(-x)$, the previous result gives exactly the expected answer. Of course, depending on the choice of n^{+}, we are not always so fortunate as to have $g^+(t,x) \sim e^{t/2}h(x)$. In particular, if n^+ does have zeros in $\overline{\mathbb{H}^-}$, guessing the asymptotic growth of g^+ would be more difficult.

The crucial assumption in Proposition 4.1 is that $n^\pm$ have no zeros on $\mathbb{R}$. The following is a set of easy sufficient conditions for this to be the case, which covers a wide class of initial densities μ_0.

PROPOSITION 4.2. *Suppose that the support of the initial density function μ_0 is a connected (possibly infinite) interval $[a,b]$, where $-\infty \le a < b \le +\infty$ and that $\mu_0(x) > 0$ for all $x \in (a,b)$. Then $n^\pm$ have no roots on $\mathbb{R}$.*

PROOF. It is clear from the definition of the Hilbert transform that for $x \le a$, $\mathbf{H}\mu_0(x) < 0$ while for $x \ge b$, $\mathbf{H}\mu_0(x) > 0$. Therefore, from (3.8) we see that $n^\pm$ have no roots outside of (a,b) while the assumption that $\mu_0(x) > 0$ for all $x \in (a,b)$ means that $n^\pm$ have no roots in (a,b) either, again by virtue of (3.8). [Although strictly speaking, (3.8) only holds almost everywhere on $\mathbb{R}$, the identities (3.7) hold at every point of the respective half-planes and so the values of $n^\pm$ on $\mathbb{R}$ can be defined as the limits of (3.7) as the imaginary part of z tends to zero. These limits exist almost everywhere on $\mathbb{R}$ and where they do exist, they are given by (3.8); where they do not exist, $n^\pm$ are either not defined or infinite—in any case $n^\pm$ certainly cannot be zero.] □

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