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EXPONENTIAL WAITING TIME FOR A BIG GAP IN A ONE-DIMENSIONAL ZERO-RANGE PROCESS[1]

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The first time that the N sites to the right of the origin become empty in a one-dimensional zero-range process is shown to converge exponentially fast, as $N \to \infty$, to the exponential distribution, when divided by its mean. The initial distribution of the process is assumed to be one of the extremal invariant measures ν_ρ, $\rho \in (0, 1)$, with density $\rho/(1-\rho)$. The proof is based on the classical Burke theorem.

1. Introduction. The zero-range process can be informally described as follows. Particles are distributed on the integers. Associated with each site is an exponential clock independent of the others. When it rings, a particle, chosen at random among the ones sitting in the site if the site is occupied jumps to the next site to the left.

We consider this process starting from an equilibrium measure with fixed density. Define T_N as the first time that the sites $\{1, \dots, N\}$ become empty. We prove here that T_N/ET_N converges in law to an exponential random time with mean 1, exponentially fast in N. Furthermore, we compute both the scaling factor ET_N as well as the rate of convergence.

The main tool is the classical Burke theorem. The process can be represented as an infinite system of queues, which enables us to give an explicit representation of T_N as a geometric sum of independent random times. Indeed, T_N has the same distribution as the first time that a customer in a series of queues finds a line with more than N customers.

This kind of result has not been obtained up to now for conservative systems like the zero-range process. A number of results in this direction have been obtained for dissipative systems. The main difference between these two classes of processes is that dissipative systems lose memory much faster than conservative ones. On the other hand, this kind of result has been obtained for Harris recurrent chains. But the zero-range process is not Harris recurrent even on the set of configurations with fixed asymptotic density. (We thank Tom Liggett for showing us this fact.) As a matter of fact, this is believed to be the typical situation in the infinitely many particle systems context.

The ergodic properties of the zero-range process have been studied by Andjel (1982). The hydrodynamical limit was obtained by Andjel and Kipnis

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(1984). The equivalence with a system of queues was studied by Wick (1985) and Kipnis (1986).

The convergence to exponential for occurrence times of rare events seems to have been studied first by Bellman and Harris (1951) and Harris (1953). In the context of applied probability, the question was studied by Aldous (1982, 1989) and Aldous and Brown (1991a, b) for a finite state chain and by Korolyuk and Sil'vestrov (1984) and Cogburn (1985) for Harris recurrent chains. In statistical physics the question was considered in the so-called "pathwise approach to metastability" and studied for interacting spin flip systems and dynamical systems with or without random noise. [See the review paper of Schonmann (1992).] In the intermittency context the question was studied by Collet, Galves and Schmitt (1992). Occurrence times for large density fluctuations for equilibrium dissipative systems have been studied by Lebowitz and Schonmann (1987) and Galves, Martinelli and Olivieri (1989).

2. Results. The zero-range process under study is a particle system ξ_t with state space $\mathbf{X} = \mathbb{N}^{\mathbb{Z}}$. The generator of the process is given by

$$Lf(\xi) = \sum_{x \in \mathbb{Z}} 1\{\xi(x) > 0\}\left[f(\xi^{x,x-1}) - f(\xi)\right],$$

where

$$\xi^{x,x-1}(y) = \begin{cases} \xi(y), & \text{if } y \neq x, x-1, \\ \xi(x) - 1, & \text{if } y = x, \\ \xi(x-1) + 1, & \text{if } y = x - 1. \end{cases}$$

Let P^{ξ} and E^{ξ} be the probability and expectation with respect to the process starting with the configuration ξ. Let ν be a probability measure on $\mathbf{X}$. Define $E^{\nu} f(\xi_t) = \int \nu(d\xi) E^{\xi} f(\xi_t)$. We consider the product measure ν_ρ on $\mathbf{X}$ with marginals

$$\nu_\rho(\xi\colon \xi(x) = k) = \rho^k (1 - \rho), \qquad k \geq 0.$$

These measures are extremal invariant for the process [Andjel (1982)]. From now on we fix $\rho > 0$ and write P and E instead of P^{ν_ρ} and E^{ν_ρ}, respectively. Let $T_N = \inf\{t\colon \xi_t(x) = 0,\ x = 1, \dots, N\}$.

THEOREM. *The following holds*:

$$\sup_{t \geq 0} \left| P\big(\rho(1-\rho)^N T_N > t\big) - e^{-t}\right| \leq (N+3)(1-\rho)^{N/2}.$$

PROOF. First define the exit times from the box $\{1, \dots, N\}$. Let $S_0 = 0$ and for $n \geq 1$,

$$S_n = \inf\{t > S_{n-1}\colon \xi_{t^-}(1) = \xi_t(1) + 1\}.$$

For $n \geq 0$, define $Y_n = \min\{x \geq 1\colon \xi_{S_n}(x) > 0\}$, the length of the gap to the right of the origin by time S_n. Calling

$$J(N) = \min\{n \geq 0 : Y_n \geq N + 1\},$$

we have

$$T_N = S_{J(N)}.$$

Interpreting each site as a cashier and each particle as a customer, we may represent the zero-range process as a system of queues in series. For a more detailed description see Kipnis (1986) and Wick (1985). Then Burke's theorem says that for $n \geq 0$, $\{S_n\}$ is a Poisson point process with rate ρ, that is, $\{S_{n+1} - S_n\}$ are independent exponential times with mean $1/\rho$. Note that $S_{n+1} - S_n = \sum_{i=1}^{Y_n} W_{n,i}$, where $W_{n,i}$ is the time it takes for the first particle to the right of the origin at time S_n to perform its ith jump toward the origin. Burke's theorem and the fact that $\{W_{n,i}\}$ are independent mean-one exponential random variables implies that $\{Y_n\}$ are independent geometric random variables with parameter ρ, that is $P(Y_n = k) = \rho(1-\rho)^{k-1}$, $k \geq 1$, and $EY_n = 1/\rho$. Hence, denoting $\beta_N = (1-\rho)^N$,

$$P(J(N) = k) = \beta_N(1 - \beta_N)^k, \qquad k \geq 0,$$

where $E(J(N)) = (1 - \beta_N)/\beta_N$.

With these ingredients a straightforward computation shows that the law of $\beta_N T_N$ converges to an exponential distribution. To obtain the rate of convergence stated in the theorem, we compare, through a coupling, T_N with an exponential random time. The comparison is based on the following remark. The random time T_N is a double sum of exponential random variables. The number of terms of the sums is roughly geometrically distributed. This would give us an exponential random time if these numbers were independent. But since this is not the case, we couple T_N with another random time $\overline{T}_N$ constructed in the same way, but considering independent random numbers of terms. We have that

$$T_N = \sum_{n=0}^{J(N)-1} \sum_{i=1}^{Y_n} W_{n,i}$$

with the usual convention that $T_N = 0$ if $J(N) = 0$. On the other hand, with the same convention, let

$$\overline{T}_N = \sum_{n=0}^{J(N)-1} \sum_{i=1}^{\overline{Y}_n} W_{n,i},$$

where $\overline{Y}_n$ are defined in the following way. First consider an independent copy Z_n of Y_n. Then for each index $n < J(N)$, define

$$\overline{Y}_n = \begin{cases} Y_n, & \text{if } Z_n \leq N, \\ Z_n, & \text{if } Z_n > N. \end{cases}$$

Now it is easy to see the following results.

1. Given that $J(N) = k$, $\overline{Y}_0, \dots, \overline{Y}_{k-1}$ have independent geometric distributions with parameter ρ; indeed

$$P\big[J(N) = k, \overline{Y}_i = \alpha_i, 0 \le i \le k-1\big] = P[J(N) = k] \prod_{0 \le i \le k-1} P[Y_i = \alpha_i].$$

Therefore, the random time $\overline{T}_N$ is a mixture of two random variables: the constant 0 with probability $\beta_N = (1-\rho)^N$ and a geometric sum of exponential random times—which is itself an exponential random time—with probability $1 - \beta_N$.

2. Since $EJ(N) = (1-\beta_N)/\beta_N$, result 1 implies that

$$E\overline{T}_N = \frac{1-\beta_N}{\rho\beta_N}.$$

3. Let $M = \#\{n < J(N): \overline{Y}_n \ne Y_n\}$. Then a direct computation shows that

$$P(M = k) = \gamma_N^k(1-\gamma_N), \qquad k \ge 0,$$

where $\gamma_N = (1-\beta_N)/(2-\beta_N)$. Therefore, $E(M) = 1 - \beta_N$.

4. By definition, $\overline{T}_N \ge T_N$ and $E(\overline{T}_N - T_N) = E(M)E(\overline{Y}_1 - Y_1 | \overline{Y}_1 \ne Y_1)$. Since by direct computation $E(\overline{Y}_n - Y_n | \overline{Y}_n \ne Y_n) \le N + (1/\rho)$, it follows from result 3 that

$$E\big(\overline{T}_N - T_N\big) \le N + (1/\rho).$$

Now we can conclude the proof. By construction we have

$$\begin{aligned} e^{-t}(1-\beta_N) &= P\big(\rho\beta_N\overline{T}_N > t\big) \\ &\ge P(\rho\beta_N T_N > t) \\ &\ge P\big(\rho\beta_N T_N > t, \overline{T}_N - T_N \le \lambda_N\big). \end{aligned}$$

This last probability is greater than

$$\begin{aligned} &P\big(\rho\beta_N\overline{T}_N > t + \rho\beta_N\lambda_N\big) - P\big(\overline{T}_N - T_N > \lambda_N\big) \\ &\quad \ge (1-\beta_N)\exp(-t - \rho\beta_N\lambda_N) - \frac{E\big(\overline{T}_N - T_N\big)}{\lambda_N}, \end{aligned}$$

where in the last line we used the definition of $\overline{T}_N$ and the Markov inequality. Now, applying result 4 and fixing $\lambda_N = 1/\rho\sqrt{\beta_N}$, we get the bound of the theorem. □

The zero-range process under study is isomorphic to the asymmetric simple exclusion process as seen from a tagged particle [see Kipnis (1985) and Ferrari (1986), for instance]. With this isomorphism our theorem implies that the first time that the tagged particle sees a block of N particles to its right approaches an exponential random variable, when normalized by its mean.

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