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MARKOV CHAINS INDEXED BY TREES

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We study a variant of branching Markov chains in which the branching is governed by a fixed deterministic tree T rather than a Galton–Watson process.

Sample path properties of these chains are determined by an interplay of the tree structure and the transition probabilities.

For instance, there exists an infinite path in T with a bounded trajectory iff the Hausdorff dimension of T is greater than $\log(1/\rho)$ where ρ is the spectral radius of the transition matrix.

1. Introduction. Given a state space G, transition probabilities between the states and an initial state, the distribution of the corresponding Markov chain is a measure on the set of maps from $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ to G.

Replacing $\mathbb{N}_0$ by an infinite rooted tree T, we obtain a *tree-indexed Markov chain*; if $\sigma \in T$ is mapped to $x \in G$, then the images of the sons of σ are chosen independently according to the transition probabilities from x. See Section 2 for a formal definition. If T is the family tree of a supercritical Galton–Watson branching process, a branching random walk on G is obtained; we hope to convince the reader that there are quite different trees of considerable interest.

Our main focus is on recurrence notions for tree-indexed Markov chains. Such a chain is called *recurrent* if with positive probability infinitely many vertices of the tree visit the same state; it is called *ray-recurrent* if these vertices may be found along a single *ray* (i.e., an infinite non-self-intersecting path) of the tree. These notions are precisely defined and illustrated by examples in Section 3. In Section 3 we also consider the Green function in the tree-indexed setting; finiteness of this function precludes recurrence, but *not* vice versa.

The purpose of Section 4 is to show how dimensional characteristics of a tree T are reflected in the asymptotic behavior of T-indexed Markov chains. For instance, denote by $\rho(p)$ the spectral radius of the transition matrix p (cf. Section 2). The probability that a T-indexed chain will have a ray trapped in some finite subset of the state space is positive iff the Hausdorff dimension of the boundary of T is greater than $\log[1/\rho(p)]$. Recurrence and ray recurrence

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of T-walks are more closely related to the exponential growth rate and the Tricot packing dimension, respectively, rather than to the Hausdorff dimension.

In Section 5, viewing a T-indexed random walk on a graph G as a random graph homeomorphism from T to G, we attempt to extend this map to a mapping from the boundary of T to the end boundary of G. The natural extension exists iff the T-indexed random walk is not ray-recurrent; it is continuous essentially iff the walk is nonrecurrent. A sufficient condition for Hölder continuity of the boundary mapping is also obtained.

Section 6 begins with a sufficient condition for a strong form of recurrence. The highlight of that section is a construction of "incomparable trees": two trees T^0, T^1 for which there exist state spaces G^0, G^1 such that for $i = 0, 1$, the T^i-walk on G^i is recurrent but the T^i-walk on G^{1-i} is not. This construction is used to exhibit a T-walk which has a guaranteed return to its initial state, yet with positive probability has only finitely many such returns. Finally, Section 7 contains a diagram relating the different recurrence notions and a list of questions.

As the subject of tree-indexed processes is rather young, perhaps we should explain our motivation for studying it. A classical theme in the theory of random walks and other stochastic processes is that the most interesting problems and results are expressed in terms of the almost sure behavior of sample paths, while analytical manipulation of probabilities is usually relegated to the proofs. Contemporaneously, a central aim of the study of random walks on graphs and groups is to see how geometric characteristics of the graph (or group) are reflected in the random walk. However, if the graph is very large, the ordinary random walk on it will be transient and a typical sample path will visit only a tiny part of the graph. As a consequence, results like Kesten's criterion for amenability or the probabilistic characterization of expanders involve probabilities of return and rates of mixing, rather than sample path behavior. Tree-indexed random walks resolve these conflicting aims by providing sample paths which are sufficiently rich to reflect the geometry of a "large" graph, while maintaining the appealing Markov property of ordinary random walks. See Sections 4 and 5 and [4] for examples of this.

2. Preliminaries.

Some tree notation. By a *tree* we mean an infinite, locally finite, connected graph with a distinguished vertex 0 called the *root* and without loops or cycles. We *only consider trees without leaves*. That is, the degree of each vertex (except 0) is required to be at least 2. Let σ, τ be vertices of a tree. Write $\tau \le \sigma$ if τ is on the unique path connecting 0 to σ, and $|\sigma|$ for the number of edges on this path. For any two vertices σ, τ, denote by $\sigma \wedge \tau$ the vertex farthest from 0 satisfying

$$\sigma \wedge \tau \le \sigma \quad \text{and} \quad \sigma \wedge \tau \le \tau.$$

If $\sigma \ne 0$, then we let $\tilde{\sigma}$ stand for the vertex satisfying $\tilde{\sigma} \le \sigma$ and $|\tilde{\sigma}| = |\sigma| - 1$.

(We refer to σ as a son of $\tilde{\sigma}$.) The *boundary* ∂T of a tree T is the set of *rays* in T, that is, the set of infinite non-self-intersecting paths emanating from 0. A generalization of this definition, the end boundary of a graph, is discussed in Section 3. For rays ξ, η in T, let $\xi \wedge \eta$ be the vertex farthest from 0 which is on both rays. The metric

$$d(\xi, \eta) = e^{-|\xi \wedge \eta|} \tag{2.1}$$

makes ∂T into a compact metric space.

DEFINITION (Tree-indexed Markov chains). Let G be a countable state space, equipped with transition probabilities $\{p(x,y)|x,y \in G\}$ satisfying $\Sigma_y p(x,y) = 1$, and let T be a tree. The induced *T-indexed Markov chain* is a collection $\{S_\sigma|\sigma \in T\}$ of G-valued random variables, with finite-dimensional distributions defined inductively from

$$\begin{aligned} &P\big[S_\sigma = y|S_{\tilde{\sigma}} = x \text{ and } S_\tau \text{ for } |\tau \wedge \sigma| \le \tilde{\sigma}\big] \\ &\quad = P\big[S_\sigma = y|S_\sigma = x\big] = p(x,y) \end{aligned} \tag{2.2}$$

and an initial state, $S_0 \equiv x_0 \in G$. Thus the tree structure makes this class of Markov random fields particularly easy to construct. The special case in which G is a group and the transition probabilities are G-invariant allows further tools to be used and is studied separately in [4]. The relationship between tree-indexed Markov chains and historical processes attached to superprocesses is explained in [1], Section 6, and the references therein.

Since we are interested in recurrence, we shall always assume the irreducibility of the original Markov chain (G, p); namely for each $x, y \in G$ we have $p^n(x,y) > 0$ for some n. The reader is urged to concentrate on the case in which G is a locally finite connected graph, and the transition probabilities from each state in G to its neighbors are equal. In this case we speak of a T-indexed simple random walk on the graph G or, in short, a *T-walk on G* with initial state x_0. In the general case we refer to the *(T, p)-walk on G* (with initial state x_0).

The spectral radius $\rho(p)$. For any finite subset F of G, denote by p_F the substochastic matrix $\{p(x,y)\colon x, y \in F\}$ and by $\rho(p_F)$ its spectral radius. Let

$$\rho(p) = \sup_F \rho(p_F), \tag{2.3}$$

where the supremum is over finite subsets of G. If p_F is an irreducible matrix, then the Perron–Frobenius theorem, as in [13], Chapter 1, guarantees that for any proper subset F' of F we have

$$\rho(p_{F'}) < \rho(p_F). \tag{2.4}$$

In particular, for any finite F the strict inequality

$$\rho(p_F) < \rho(p) \tag{2.5}$$

holds, since p itself is irreducible.

For our purposes, the most useful representation of $\rho(p)$ is

$$\rho(p) = \limsup_{n\to\infty} [p^n(x,y)]^{1/n}, \tag{2.6}$$

where $p^n(x,y)$ is the n-step transition probability between the states $x, y \in G$.

If the Markov chain is reversible with stationary measure π, then $\rho(p)$ is exactly the spectral radius of p as an operator on $L^2(G, \pi)$.

When the transition probabilities $\{p(x,y)\}$ correspond to a simple random walk on the graph G, we write $\rho(G)$ for $\rho(p)$ and refer to it as the spectral radius of the graph G.

3. Recurrence and the Green function. We now introduce the concepts which are central in our development.

DEFINITIONS (Recurrence and ray recurrence). Consider a (T,p)-walk, $\{S_\sigma | \sigma \in T\}$, on the state space G, with initial state x_0.

(i) The (T,p)-walk is *recurrent* if for some y in G,

$$P\left(\sum_{\sigma\in T} 1_{[S_\sigma = y]} = \infty\right) > 0. \tag{3.1}$$

(ii) The (T,p)-walk is *ray-recurrent* if for some y in G,

$$P\left(\exists\, \xi \in \partial T,\ \sum_{\sigma\in\xi} 1_{[S_\sigma = y]} = \infty\right) > 0. \tag{3.2}$$

In fact, recurrence implies that (3.1) holds for *all* $y \in G$, as the following lemma shows.

LEMMA 3.1. *Let $\{S_\sigma | \sigma \in T\}$ be a (T,p)-walk on G and let $y, z \in G$. Then*

$$P\left(\sum_{\sigma\in T} 1_{[S_\sigma = y]} = \infty \text{ and } \sum_{\tau\in T} 1_{[S_\tau = z]} < \infty\right) = 0. \tag{3.3}$$

PROOF. Since G is connected, $p^k(y,z) = \delta > 0$ for some $k \geq 1$. Let $m, N \geq 1$. The probability that there exist vertices $\sigma_1, \dots, \sigma_N$ of T with $S_{\sigma_i} = y$, $|\sigma_{i+1}| > |\sigma_i| + k$ and $|\sigma_1| > m$ but $S_\tau \neq z$ whenever $|\tau| > m$ is at most $(1-\delta)^N$. For this our standing assumption that T has no leaves is crucial. Letting N tend to ∞,

$$P\left(\sum_{\sigma\in T} 1_{[S_\sigma = y]} = \infty \text{ and } \sum_{|\tau|>m} 1_{[S_\tau = z]} = 0\right) = 0.$$

Finally, taking the union of the events in the last formula over all values of m gives (3.3). □

In Section 6 we shall see there is no 0–1 law for recurrence (or ray recurrence) of T-walks, that is, the probabilities of (3.1) and (3.2) may lie strictly between 0 and 1, even if T is a binary tree. At present we give a few examples illustrating the definitions above.

EXAMPLE 3.2 (The binary tree and Galton–Watson trees). If T is a binary tree and G is the d-dimensional lattice $\mathbb{Z}^d$ (with the usual graph structure), then the T-walk on G is ray-recurrent. This follows easily from branching process considerations (see also Corollary 4.6). Alternatively, it is a consequence of Proposition 6.1. The latter only gives recurrence; note, however, the following claim.

CLAIM. Let T be a binary tree. If the (T, p)-walk on an arbitrary state space G (starting from x_0) is recurrent, then it is ray-recurrent. The same holds if T is replaced by (almost all) family trees of a *nondegenerate* Galton–Watson branching process *without death*.

PROOF. Using the (T, p)-walk $\{S_\sigma\}$ on G (starting from $x_0 \in G$), define a random subset T^* of the vertex set of T by $T^* = \{\sigma \in T: S_\sigma = x_0\}$. This set is itself a tree when endowed with the partial order $\leq$ induced from T (except that a vertex may have infinitely many sons). Explicitly, we connect two vertices $\sigma_1, \sigma_2 \in T^*$ if $\sigma_1 \leq \sigma_2$ and for every vertex $\tau \in T$ such that $\sigma_1 \leq \tau \leq \sigma_2$, we have $\tau \notin T^*$.

The levels of T^* form a Galton–Watson process (possibly with an infinite number of offspring with positive probability). Observe that $0 \in T^*$, and denote by T_1^* the first level of T^*. The proof is completed by noting that the probability that T^* is infinite equals the probability that it contains an infinite line of descent. This is immediate if $P[T_1^* \text{ is infinite}] = 0$, but also holds if this probability is positive, since in this case the branching process is supercritical and if some vertex in T^* has an infinite number of offspring, survival is guaranteed. □

REMARK. Our assumptions on the original branching process generating T above are that the offspring distribution is concentrated on $\{1, 2, 3, \dots\}$ but *not* on $\{1\}$. We assume this whenever we mention Galton–Watson trees in the sequel. This is not a serious restriction, as trees arising from any supercritical Galton–Watson process, conditioned on nonextinction, may be "trimmed" to satisfy it, by discarding all vertices that have a finite number of descendents (see [2], Chapter 1). Returning to our choice of T as a *binary* tree, consider the T-walk on G when G is a k-tree (each state in G has precisely k sons). If $k \geq 15$, then the Green function condition, described in the next section, shows that the T-walk is nonrecurrent (see Proposition 4.3). If $k < 15$, then the T-walk on G is ray-recurrent. This follows from branching process considerations, analogous to those in [5], or from Proposition 4.5.

EXAMPLES 3.3 "Exploding" trees and the 3–1 tree—see Figures 1 and 2). An "exploding" tree is constructed from an increasing sequence of nonnegative integers $\{m_i\}_{i=1}^{\infty}$ as follows.

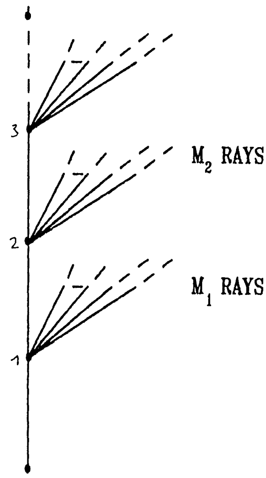

FIG. 1. *An exploding tree.*

Start with a single ray (a copy of $\mathbb{N}_0 = \{0, 1, 2, \dots\}$), which is called the *pivotal* ray. For each $i \geq 1$, add m_i disjoint rays emanating from the vertex at distance i from the root on the pivotal ray. If the state space G is a k-tree and T is the exploding tree constructed above with $m_i > k^i$, then Proposition 6.1 shows that the T-walk on G is recurrent.

However, the boundary ∂T is countable; therefore, since the ordinary random walk on G is nonrecurrent, the T-walk on G is *not* ray-recurrent.

A tree which behaves qualitatively like the exploding trees but has bounded degrees is the 3–1 tree, which appeared previously in [11] and [3]. The 3–1 tree

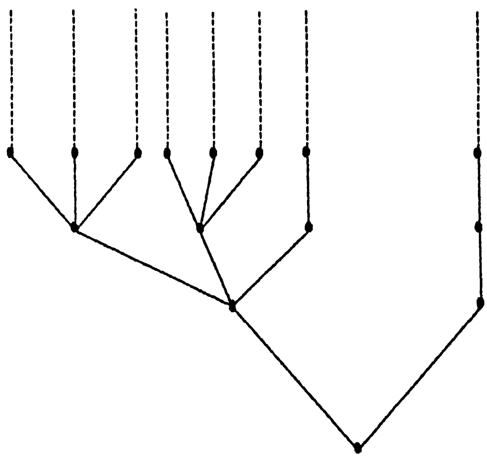

FIG. 2. *The* 3–1 *tree.*

has 2^n vertices at the nth level. The root has two sons in level 1. For each $n \geq 1$ assume the 2^n vertices at the nth level are ordered from left to right; then let the left half of them have three sons and the right half just one son. Order the resulting 2^{n+1} vertices from left to right, in a manner compatible with the order of their fathers, and continue inductively. The leftmost ray of the 3–1 tree is called its pivotal ray; each vertex off this ray has only finitely many rays containing it. The 3–1 tree has a countable boundary and therefore the random walk on $\mathbb{Z}^3$ indexed by it is not ray-recurrent; it is recurrent by Proposition 6.1.

Green function. Let T be a tree. The Green function $g_T(x, y)$ for the (T, p)-walk $\{S_\sigma | \sigma \in T\}$ on the state space G is defined by

$$g_T(x,y) = \sum_{\sigma \in T} P_x[S_\sigma = y],$$

where the subscript x is meant to indicate the initial state: $S_0 \equiv x$. Denoting by A_n the cardinality of the nth level $\{\sigma \colon |\sigma| = n\}$ of T, we may write

$$g_T(x,y) = \sum_{n=0}^{\infty} A_n p^n(x,y).$$

The Borel–Cantelli lemma implies that if $g_T(x, x) < \infty$, then the (T, p)-walk above is nonrecurrent. However, the converse is false.

EXAMPLE 3.4 (A nonrecurrent T-walk with infinite Green function). Let T be the exploding tree with $m_i = 2^i$ (Example 3.3).

Consider a (T, p)-walk on the integers with initial state $S_0 = 0$ and transition probabilities

$$p(n, n+1) = \alpha = 1 - p(n, n-1) \tag{3.4}$$

for all $n \in \mathbb{Z}$, where $\frac{1}{2} < \alpha < 1$. Denote by $\{\sigma_n\}_{n \geq 0}$ the pivotal ray of T, where $|\sigma_n| = n$. By the strong law of large numbers, almost surely

$$\lim_{n \to \infty} \frac{1}{n} S_{\sigma_n} = 2\alpha - 1. \tag{3.5}$$

Almost surely there is an N such that $S_{\sigma_n} > 0$ for $n \geq N$. Let $k \geq N$ and let ξ be a ray of T which intersects the pivotal ray precisely in $\{\sigma_1, \dots, \sigma_k\}$ (there are $m_k = 2^k$ such rays). By the standard formula for ordinary asymmetric random walks on the integers,

$$P\left[\exists\, \tau \in \xi \colon \tau \geq \sigma_k,\, S_\tau = 0 | \{S_{\sigma_n}\}_{n \geq 1}\right] = \left(\frac{1-\alpha}{\alpha}\right)^{S_{\sigma_k}}. \tag{3.6}$$

Now if α is chosen to satisfy

$$2\left(\frac{1-\alpha}{\alpha}\right)^{2\alpha - 1} < 1 \tag{3.7}$$

(i.e., $\alpha > 0.7772818947\dots$), then by (3.5), almost surely

$$\sum_k 2^k \left(\frac{1-\alpha}{\alpha} \right)^{S_{\sigma_k}} < \infty.$$

In conjunction with (3.6) and using the Borel–Cantelli lemma with conditioning on the values of $\{S_{\sigma_n}\}$, this guarantees the (T, p)-walk is nonrecurrent. However, the Green function for this (T, p)-walk is given by

$$g_T(0,0) = 1 + \sum_{n=1}^{\infty} (2^{2n} - 1) \binom{2n}{n} \alpha^n (1-\alpha)^n,$$

which is infinite iff

$$16\alpha(1-\alpha) \geq 1 \tag{3.8}$$

(i.e., $\alpha \leq \frac{1}{2} + \sqrt{3}/4 = 0.9330127019\dots$). Any α satisfying (3.7) and (3.8) provides an example of a nonrecurrent (T, p)-walk with an infinite Green function. □

REMARKS.

(i) Let T be a general exploding tree defined from some sequence $\{m_i\}_{i \geq 1}$. Consider the (T, p)-walk on $\mathbb{Z}$, with p given by (3.4). As we increase $\alpha > \frac{1}{2}$, the walk passes from recurrence to nonrecurrence; the critical α for recurrence satisfies

$$\left(\frac{\alpha}{1-\alpha} \right)^{2\alpha - 1} = \limsup_i m_i^{1/i}, \tag{3.9}$$

while the critical α for divergence of the sum defining the Green function is determined by

$$[4\alpha(1-\alpha)]^{-1/2} = \limsup_i m_i^{1/i}.$$

The second assertion is straightforward. The first is established as in the proof above, with one additional observation. If the left-hand side in (3.9) is smaller than the right-hand side, then the series $\sum_k m_k((1-\alpha)/\alpha)^{S_{\sigma_k}}$ almost surely diverges. The events in (3.6) for different rays ξ are independent given $\{S_{\sigma_n}\}_{n \geq 1}$, so the (T, p)-walk is recurrent in this case by the second Borel–Cantelli lemma.

(ii) Let T be the 3–1 tree. The (T, p)-walk on $\mathbb{Z}$ given by (3.4) with $\alpha = 11/12$, say, is nonrecurrent yet has an infinite Green function. The proof follows the same lines.

(iii) Let G be the Cayley graph of a free group on six generators, that is, G is a regular tree of degree 12. If T is the exploding tree used in Proposition 3.4 (or the 3–1 tree), then the T-walk $\{S_\sigma | \sigma \in T\}$ on G is nonrecurrent, yet has an infinite Green function. This follows from Proposition 3.4 since the distances $d(S_\sigma, x_0)$ from the initial state x_0 perform a (T, p)-walk on $\mathbb{N}_0 = \{0, 1, 2, \dots\}$

with $p(0,1) = 1$ and

$$p(n, n+1) = 11/12 = 1 - p(n, n-1) \quad \text{for } n \geq 1.$$

Formulas (3.7) and (3.8) are satisfied by $\alpha = 11/12$, and the change from $\mathbb{Z}$ to $\mathbb{N}_0$ does not affect the result (see [9], Section 4.5b, for the asymptotics of the n-step transition probabilities for a random walk on a free group).

4. Dimension notions, expanders and bounded trajectories.

DEFINITIONS (Dimension). Let T be a tree and ∂T its boundary.

(i) The *Hausdorff dimension* $\dim(\partial T)$ is defined as usual, using the metric on ∂T given in Section 2. Explicitly,

$$\dim(\partial T) = \sup\left\{\beta > 0 \,\middle|\, \inf_{\Pi} \sum_{\sigma \in \Pi} e^{-\beta|\sigma|} > 0\right\}, \tag{4.1}$$

where the inner infimum is over *cutsets* Π of T, that is, sets of vertices which intersect each ray of T.

(ii) The (upper) *Minkowski dimension* $\mathrm{Mdim}(\partial T)$ is simply the exponential growth rate of T:

$$\mathrm{Mdim}(\partial T) = \limsup_{n \to \infty} \frac{1}{n} \log A_n, \tag{4.2}$$

where A_n is the cardinality of the nth level of T.

(iii) The *packing dimension* $\mathrm{Pdim}(\partial T)$ may be defined by

$$\mathrm{Pdim}(\partial T) = \inf\left\{\sup_{k \geq 1} \mathrm{Mdim}(\partial T^{(k)})\right\}, \tag{4.3}$$

where the infimum is taken over all sequences $T^{(1)}, T^{(2)}, \dots$ of subtrees of T, which are all rooted at 0 and satisfy

$$\partial T = \bigcup_{k=1}^{\infty} \partial T^{(k)}.$$

REMARKS. The Hausdorff dimension of a tree was introduced in [7] and employed in [10] and [11] to study probabilistic properties of the tree. The original definition of packing dimension, due to Tricot, is more elaborate and involves packings of disjoint balls. Its equivalence to a formula of the type (4.3) was established by Taylor and Tricot [14]; we will only use (4.3). For any tree T, the inequalities

$$\dim(\partial T) \leq \mathrm{Pdim}(\partial T) \leq \mathrm{Mdim}(\partial T)$$

are easily verified. When T is a Galton–Watson tree, all of these dimensional quantities coincide and equal the logarithm of the expected offspring; in the general case they may differ. For instance, the 3–1 tree (Example 3.3) has $\mathrm{Pdim}(\partial T) = 0$ but $\mathrm{Mdim}(\partial T) = \log 2$. To exhibit another strict inequality, we need the following example.

EXAMPLE 4.1 (The iterated 3–1 tree). Let $\{n_j\}$ be an increasing sequence of positive integers. Construct a tree T_{iter} as follows.

The first n_1 levels of T_{iter} are as in the 3–1 tree. To each vertex σ at level n_1 of T_{iter} attach a copy of the first $n_2 - n_1$ levels of the 3–1 tree, with σ as its root. Continue by attaching a copy of the first $n_3 - n_2$ levels of the 3–1 tree to each vertex at level n_2, and so on. For any choice of $\{n_j\}$, the resulting tree T_{iter} has positive packing dimension. This may be seen directly, but also follows from the proof of Proposition 4.4. In fact, $\text{Pdim}(\partial T_{\text{iter}}) = \log 2$. However, if $\{n_j\}$ increase sufficiently rapidly, then $\dim(\partial T_{\text{iter}}) = 0$ (a sharper result is given in [3], Example 5.3). This construction was originally suggested by B. Weiss.

Before presenting some quantitative results, perhaps the qualitative statements contained in the next theorem will convey how the dimension notions just defined are reflected in the asymptotic behavior of tree-indexed random walks.

Let G be an infinite connected graph, with bounded degrees. Then G is called an *expander* if there exists $c > 0$ such that any finite subset of G of cardinality $N \geq 1$ has at least cN neighbors outside it. The greatest such c is called the *isoperimetric constant* of G. What is important for us is that a graph of bounded degree G is an expander iff $\rho(G) < 1$ (see [8]).

THEOREM 4.2. *Let T be a tree.*

(i) *There exists an expander G such that the T-walk on G is recurrent iff* $\text{Mdim}(\partial T) > 0$.

(ii) *There exists an expander G such that the T-walk on it is ray-recurrent iff* $\text{Pdim}(\partial T) > 0$.

(iii) *There exists an expander G for which the T-walk on G has (with positive probability) a ray with a bounded trajectory iff* $\dim(\partial T) > 0$.

The proof will be given in the three propositions below; each of them is a sharpened version of the corresponding part of Theorem 4.2. We note that despite the similar appearance of the three parts of this theorem, there is an important difference. Referring to part (iii), if $\dim(\partial T) > 0$, then for any G with spectral radius sufficiently close to 1, there exists a bounded ray (see Proposition 4.5), while the analogous statements for parts (i) and (ii) are false. For instance, if T is the 3–1 tree and G is a 100-tree with a ray added (emanating from the root of G), then $\rho(G) = 1$ but the T-walk on G is not recurrent.

It will be convenient to have some concrete expanders in mind.

Denote by G_r the tree in which vertices σ with $|\sigma| \equiv 0 \bmod r$ have two sons, and all other vertices have just one son. G_r is an expander; indeed it is easy to verify that the isoperimetric constant of G_r is $1/(2r - 1)$.

PROPOSITION 4.3.

(a) *Let p be a transition matrix on G. If* $\mathrm{Mdim}(\partial T) < \log[1/\rho(p)]$, *then the (T, p)-walk on G is nonrecurrent.*

(b) *If* $\mathrm{Mdim}(\partial T) > 0$, *then for large enough r, the T-walk on G_r is recurrent. In fact, this holds if r satisfies*

$$\frac{\log 2}{3r^2} < \mathrm{Mdim}(\partial T). \tag{4.4}$$

PROOF.

(a) This is an immediate consequence of the Green function condition (Proposition 3.4) and the expression (2.6) of $\rho(p)$ in terms of the probabilities of return:

$$g_T(x, x) = \sum_{n=0}^{\infty} A_n p^n(x, x) < \infty,$$

since

$$\limsup_n \left[A_n p^n(x, x)\right]^{1/n} < 1.$$

(b) We assume $S_0 \equiv 0$ and denote by $|S_\sigma|$ the distance from 0 to S_σ in G_r. From the properties of an asymmetric ordinary random walk on the integers, for each $\sigma \in T$ we have

$$P[\exists\, \tau: \sigma \le \tau,\ S_\tau = 0 | S_\sigma] \ge 2^{-1-|S_\sigma|/r}. \tag{4.5}$$

Therefore,

$$\begin{aligned} P\left[\bigcap_{|\tau| \ge n} \{S_\tau \ne 0\} | S_\sigma, |\sigma| = n\right] &\le \prod_{|\sigma| = n} \left(1 - 2^{-1-|S_\sigma|/r}\right) \\ &\le \exp\left(-\frac{1}{2} \sum_{|\sigma| = n} 2^{-|S_\sigma|/r}\right). \end{aligned} \tag{4.6}$$

Certainly $|S_\sigma| \le |\sigma|$, so (4.6) implies

$$P\left[\bigcap_{|\tau| \ge n} \{S_\tau \ne 0\}\right] \le \exp\left(-\frac{1}{2} A_n 2^{-n/r}\right).$$

The events on the left-hand side are increasing in n; applying the last formula with $n_1, n_2, \ldots$ satisfying

$$\frac{1}{n_i} \log A_{n_i} \to \mathrm{Mdim}(\partial T) > 0 \tag{4.7}$$

and sufficiently large r, we find that for any fixed n,

$$P\left[\bigcap_{|\tau| \ge n} \{S_\tau \ne 0\}\right] = 0. \tag{4.8}$$

This implies a strong form of recurrence, namely the set of vertices $\{\tau | S_\tau = 0\}$ is finite with probability 1. To see that recurrence holds whenever r satisfies (4.4), better estimates of $|S_\sigma|$ (for many σ) are needed. Let $\varepsilon > 0$ and denote $T_n = \{\sigma \in T\colon |\sigma| = n\}$. By the law of large numbers, with probability tending to 1, for most $\sigma \in T_n$ the number of ancestors $\theta \le \sigma$ for which

$$S_\theta \equiv 0 \mod r$$

is smaller than $(1+\varepsilon)n/r^2$. Utilizing the law of large numbers again, we find that when n is sufficiently large, with probability at least $1-\varepsilon$,

$$|S_\sigma| \le \frac{(1+\varepsilon)^2 n}{3r}$$

for most $\sigma \in T_n$. Apply (4.5) only to these σ and obtain

$$P\Bigg[\bigcap_{|\tau| \ge n} \{S_\tau \ne 0\}\Bigg] \le \varepsilon + \exp\Big[-\tfrac{1}{4} A_n 2^{-(1+\varepsilon)^2 n/3r^2}\Big].$$

Choosing ε small and a sequence $\{n_i\}$ satisfying (4.7), we see that (4.4) implies recurrence in the strong form (4.8). □

Remark. The condition (4.4) is optimal, that is, if T is an exploding tree with $\mathrm{Mdim}(\partial T) < \log 2/(3r^2)$, then the T-walk on G_r is not recurrent. To see this, observe that the ratio of the two sides in the inequality (4.5) is bounded by 4, and that as $|\sigma| \to \infty$ along the pivotal ray of T,

$$\frac{|S_\sigma|}{|\sigma|} \to \frac{1}{3r}$$

with probability 1.

Proposition 4.4.

(a) *Let p be a transition matrix on G. If* $\mathrm{Pdim}(\partial T) < \log[1/\rho(p)]$, *then the (T, p)-walk on G is not ray-recurrent.*

(b) *If* $\log 2/3r^2 < \mathrm{Pdim}(\partial T)$, *then the T-walk on G_r is ray-recurrent (G_r was defined just before Proposition* 4.3).

Proof.

(a) The hypothesis means that there is a countable collection $\{T^{(j)}\}$ of subtrees of T, such that $\mathrm{Mdim}(T^{(j)}) < \log[1/\rho(p)]$ for all j, and $\partial T = \bigcup_j T^{(j)}$. Let $y \in G$. By Proposition 4.3, with probability 1 for each j the set $\{\sigma \in T^{(j)}\colon S_\sigma = y\}$ is finite. Since every ray of T is a ray of some $T^{(j)}$, ray recurrence is impossible.

(b) For each vertex σ of T denote by $T(\sigma)$ the subtree $\{\tau \in T\colon \sigma \le \tau\}$ of T, rooted at σ. Consider the subtree $\tilde{T} = \{\sigma \in T\colon \mathrm{Pdim}[\partial T(\sigma)] > \log 2/3r^2\}$ of T, rooted at 0. Clearly $\mathrm{Pdim}(\partial\tilde{T}) > \log 2/3r^2$ and moreover, for each $\sigma \in \tilde{T}$,

$$\mathrm{Pdim}\left[\partial\tilde{T}(\sigma)\right] > \frac{\log 2}{3r^2}.$$

The $\tilde{T}$-walk on G_r satisfies (4.8), that is, with probability 1 some $0 < \sigma_1 \in \tilde{T}$ satisfies $S_{\sigma_1} = 0$. Conditioning on σ_1 and using $\mathrm{Mdim}[\partial\tilde{T}(\sigma_1)] > \log 2/3r^2$, with probability 1 there is $\sigma_2 \in \tilde{T}(\sigma_1)$ for which $\sigma_2 > \sigma_1$ and $S_{\sigma_2} = 0$. Continuing inductively, we find, with probability 1, vertices

$$\sigma_1 < \sigma_2 < \sigma_3 < \cdots$$

of $\tilde{T}$ satisfying $S_{\sigma_i} = 0$ for all $i \geq 1$, which is a strong form of ray recurrence. $\square$

PROPOSITION 4.5. *Consider a (T, p)-walk on a state space G, with initial state $x_0 \in G$.*

(a) *If* $\dim(\partial T) \leq \log[1/\rho(p)]$, *then with probability* 1, *each ray $\xi \in \partial T$ has an unbounded trajectory* $\{S_\sigma\colon \sigma \in \xi\}$.

(b) *If* $\dim(\partial T) > \log[1/\rho(p)]$, *then with positive probability there is a ray ξ of T, with a bounded trajectory visiting x_0 along an infinite arithmetic progression.*

PROOF.

(a) Since G has only countably many finite subsets, it suffices to prove that for each subset F with probability 1 the trajectory of every ray $\xi \in \partial T$ eventually exits F. Fix a finite subset F of G. Recall from Section 2 that the restriction p_F of p to F satisfies $\rho(p_F) < \rho(p)$. Choosing β such that

$$\dim(\partial T) < \beta < \log\frac{1}{\rho(p_F)}, \tag{4.9}$$

we may find a sequence $\{\Pi_n\}$ of cutsets of T such that

$$\sum_{\sigma \in \Pi_n} e^{-\beta|\sigma|} \to 0$$

as $n \to \infty$. By (4.9),

$$\rho(p_F) < e^{-\beta},$$

so the basic properties of the spectral radius imply that

$$\sum_{y \in F}\sum_{\sigma \in \Pi_n} p_F^{|\sigma|}(x_0, y) \overset{n\to\infty}{\to} 0.$$

For each n, the left-hand side here is an upper bound for

$$P[\exists\, \xi \in \partial T\colon \forall\, \sigma \in \xi,\, S_\sigma \in F],$$

which must vanish, as we claimed.

(b) The proof depends on the results of Lyons [11] concerning percolation on trees. By our hypothesis and (2.6), there exists $k \geq 2$ such that

$$p^k(x_0, x_0) > \exp[-k \dim(\partial T)]. \tag{4.10}$$

Consider a "squashed" version of T:

$$T^k = \{\sigma \in T: |\sigma| \equiv 0 \bmod k\},$$

with the partial order induced from T. It is easily verified that

$$\dim(\partial T^k) = k \dim(\partial T). \tag{4.11}$$

Considering the random subgraph of T^k

$$\{\sigma \in T^k: S_\sigma = x_0\}$$

defines a (quasi-Bernoulli) percolation process on T^k. Theorem 3.1 of [10], in conjunction with (4.10) and (4.11), implies that with positive probability the component of 0 in the random graph above is infinite, which means there exists a ray ξ of T such that

$$\sigma \in \xi, |\sigma| \equiv 0 \mod k \Rightarrow S_\sigma = x_0,$$

as asserted. If G, as a graph, is locally finite, then it follows that the trajectory of ξ is bounded. To obtain this in the general case, observe that in (4.10) the left-hand side, $p^k(x_0, x_0)$, is a sum over closed paths of length k starting at x_0. We can replace $p^k(x_0, x_0)$ by a finite sum, while still satisfying (4.10). In other words, there exists a finite subset F of G such that

$$p_F^k(x_0, x_0) > \exp[-k \dim(\partial T)].$$

Now one defines a new percolation process on the squashed tree T^k by considering the random subgraph $\{\sigma \in T^k\colon\ S_\sigma = x_0$ and $\tau \in T,\ \tau \leq \sigma \Rightarrow S_\tau \in F\}$. With positive probability the component of 0 in *this* subgraph is infinite, and we are done. □

COROLLARY 4.6. *Let T be a Galton–Watson tree with expected offspring $m > 1$. A (T, p)-walk is ray-recurrent if $m\rho(p) < 1$.*

PROOF. Since $\operatorname{Mdim}(\partial T) = \log m = \dim(\partial T)$ (see [11]), the assertion follow from Propositions 4.5 and 4.3. □

5. The boundary correspondence. In this section the graph structure of the state space G plays a prominent role, as we attempt to extend the random mapping from T to G given by a T-walk to a map of the boundaries.

DEFINITION (The end boundary [6]). Let G be a connected, locally finite, infinite graph with a distinguished vertex x_0. Denote by B_n the ball of radius n around x_0 in G (with the usual graph metric). For each n the subgraph $G \setminus B_n$ has finitely many connected components and the inclusion map takes them to components of $G \setminus B_{n-1}$. The inverse limit of these finite sets, equipped with the projective topology, is a compact space called *the end boundary of* G and denoted ∂G. Concretely, an element of ∂G (called an *end* of G) is a sequence $(C_1, C_2, C_3, \dots)$ where for each n, C_n is a connected component of $G \setminus B_{n-1}$ and $C_n \supset C_{n+1}$.

REMARKS.

(i) Replacing x_0 by another distinguished vertex yields an isomorphic end boundary.

(ii) When G is a *tree* we have already defined its boundary in Section 2 and it is isomorphic to the end boundary as defined here. Indeed mapping a ray $(x_0, x_1, x_2, \dots)$ of G to $(C_1, C_2, \dots)$, where C_n is the component of x_n in $G \setminus B_{n-1}$, yields the desired identification. We will continue to refer to the elements of ∂T, where T is the indexing tree, as rays.

(iii) It will be useful to consider the union $G \cup \partial G$ as a topological space, where the induced topology on G is discrete, and a basic neighborhood of an end $(C_1, C_2, \dots)$ consists of all ends with the same first k coordinates, together with $C_k \subset G$.

(iv) Let $\{p(x, y)\}$ be (reversible) transition probabilities on G, which determine a graph structure on G. Assume the corresponding (ordinary) Markov chain $\{Y_n\}_{n \geq 1}$ is transient. For each k, the chain eventually exits the ball B_k and stays in one component of the complement; thus $\{Y_n\}$ converges, with probability 1, to an end of G. The *exit measure* μ_x is defined on Borel subsets of ∂G by

$$\mu_x(A) = \Big[\lim_{n \to \infty} Y_n \in A\Big]. \tag{5.1}$$

The function $x \to \mu_x(A)$ is p-harmonic. The random function from $\mathbb{N} \cup \{\infty\}$ to $G \cup \partial G$, taking n to Y_n and ∞ to $\lim_n Y_n$, is a.s. defined and continuous.

Next, we extend these considerations to tree-indexed chains.

LEMMA 5.1. *Let* $\{S_\sigma : \sigma \in T\}$ *be a* (T, p)*-walk on the graph* G. *Then*

$$P\Big[\forall\, \xi \in \partial T,\ \lim_{\sigma \in \xi} S_\sigma \text{ exists}\Big] = 1$$

(*where all limits are in* ∂G) *iff the* (T, p)*-walk is not ray-recurrent.*

PROOF. If the walk is not ray-recurrent, then a.s. for each ray ξ of T and every state $y \in G$, the set $\{\sigma \in \xi : S_\sigma = y\}$ is finite, which implies the trajectory $\{S_\sigma : \sigma \in \xi\}$ converges to an end. The converse is immediate. □

DEFINITION (The boundary mapping). Let $\{S_\sigma\}$ be a (T, p)-walk on G which is not ray-recurrent. The (random) *boundary mapping*

$$S^* : \partial T \to \partial G$$

is defined a.s. by

$$S^*(\xi) = \lim_{\sigma \in \xi} S_\sigma. \tag{5.2}$$

THEOREM 5.2. *Let $\{S_\sigma\}$ be a (T, p)-walk on G which is not ray-recurrent, with boundary mapping S^*.*

(i) *If this (T, p)-walk is nonrecurrent, then the mapping*

$$T \cup \partial T \to G \cup \partial G,$$

taking $\sigma \in T$ to S_σ and $\xi \in \partial T$ to $S^*(\xi)$, is continuous a.s.

(ii) *Conversely, assume the (T, p)-walk on G is recurrent and for some $x \in G$ the (ordinary) exit measure μ_x, defined in (5.1), is not concentrated on one end of G. Then S^* is discontinuous with positive probability.*

PROOF.

(i) Since the (T, p)-walk is nonrecurrent, with probability 1 for each $n \geq 1$ there is an $m(n)$, such that

$$\sigma \in T, |\sigma| > m(n) \Rightarrow S_\sigma \notin B_n,$$

where B_n is the ball of radius n around x_0. Thus if ξ, η are rays with $|\xi \wedge \eta| > m(n)$, then the first n coordinates of the ends $S^*(\xi)$ and $S^*(\eta)$ agree; the case of $\xi \in \partial T$ and $\sigma \in T$ is similar, yielding a.s. continuity.

(ii) By assumption, there are two disjoint Borel sets $A_1, A_2 \subset \partial G$ with $\mu_x(A_i) > 0$ for $i = 1, 2$. Passing to subsets, we may assume A_1, A_2 are closed. Since the ordinary Markov chain with transition matrix p is transient, there is a positive probability that its trajectory, starting at x, will converge to an end in A_i without returning to x. It easily follows that for each $i \in \{1, 2\}$ and $\tau \in T$,

$$P\left[\left\{\sum_{\sigma \geq \tau} 1_{[S_\sigma = x]} = \infty\right\} \cap \{\forall\, \xi \in \partial T, \tau \in \xi \Rightarrow S^*(\xi) \notin A_i\}\right] = 0. \tag{5.3}$$

The union of the events in (5.3) over $i = 1, 2$ and $\tau \in T$ contains the event

$$\left\{\sum_{\sigma \in T} 1_{[S_\sigma = x]} = \infty\right\} \cap \{S^* \text{ is continuous}\},$$

the probability of which must vanish. The asserted discontinuity now follows from the recurrence assumption. □

Now we restrict attention to simple T-walks on a graph G. If $\operatorname{Mdim}(\partial T) < \log[1/\rho(G)]$, then Proposition 4.3 ensures the T-walk is nonrecurrent, so the boundary mapping S^* is almost surely continuous. When G has bounded degrees this statement may be sharpened. Equip ∂G with the metric

$$d(\{C_j\}, \{C_j'\}) = \exp[-\max\{n : C_n = C_n'\}]. \tag{5.4}$$

When G is a tree, this agrees with the metric introduced in Section 2.

PROPOSITION 5.3. *If* Mdim$(\partial T) < \log[1/\rho(G)]$ *and G has bounded degrees, then the boundary map $S^*\colon \partial T \to \partial G$ is a.s. Hölder continuous.*

PROOF. Let p be the transition matrix for the simple random walk on G, with stationary measure π assigning to each vertex its degree, and $\sup_x \pi(x) = D < \infty$. We shall verify the Hölder continuity for any positive exponent

$$\alpha < \left[\log \frac{1}{\rho(G)} - \mathrm{Mdim}(\partial T)\right] \Big/ \log D.$$

The n-step transition probabilities satisfy

$$p^n(x,y) \le \rho^n,$$

where $\rho = \rho(G)$. Therefore,

$$\sum_{\sigma \in T} P\big[S_\sigma \in B_{\alpha|\sigma|}\big] = \sum_{n=0}^{\infty} \sum_{y \in B_{\alpha n}} A_n p^n(x,y) \le 2 \sum_{n=0}^{\infty} A_n D^{\alpha n} \rho^n. \tag{5.5}$$

Since $\mathrm{Mdim}(\partial T) + \log(D^\alpha \rho) < 0$, the sum in (5.5) converges, so that a.s. $\{\sigma\colon S_\sigma \in B_{\alpha|\sigma|}\}$ is a finite set; this means that for some $N_0 \ge 1$, any two rays $\xi, \eta \in \partial T$ with

$$|\xi \wedge \eta| \ge N_0$$

satisfy

$$d(S^*(\xi), S^*(\eta)) \le \exp[-\alpha|\xi \wedge \eta|] = d(\xi,\eta)^\alpha.$$

This establishes Hölder continuity. □

6. Strong recurrence and incomparable trees.

DEFINITIONS.

(i) The (T,p)-walk $\{S_\sigma\}$ on G, starting at x_0, is *strongly recurrent* if for some $y \in G$,

$$P\left\{\sum_{\sigma \in T} 1_{[S_\sigma = y]} = \infty\right\} = 1. \tag{6.1}$$

(ii) The (T,p)-walk is called *strongly ray-recurrent* if for some $y \in G$,

$$P\left\{\exists\, \xi \in \partial T\colon \sum_{\sigma \in \xi} 1_{[S_\sigma = y]} = \infty\right\} = 1.$$

(iii) The (T,p)-walk has a *guaranteed return* if

$$P\big[\exists\, \sigma \in T\colon S_\sigma = x_0,\ |\sigma| > 0\big] = 1.$$

REMARK. By Lemma 3.1, if a (T,p)-walk is strongly recurrent, then (6.1) holds for *all* $y \in G$. We believe the corresponding statement for strong ray recurrence is also true, but cannot prove it.

Later in this section, we shall show that the five recurrence notions (recurrence, ray recurrence and the three just defined) are all different. We start, however, with a sufficient condition for strong recurrence.

DEFINITIONS. A graph G is called *spherically symmetric* around $x_0 \in G$ if for any two vertices $x, y \in G$ which are at the same distance from x_0, there is an automorphism of G which fixes x_0 and takes x to y. Note that a tree G is spherically symmetric iff any two vertices at the same distance from x_0 have identical degrees.

PROPOSITION 6.1. *Let T be a tree and let G be a spherically symmetric graph around $x_0 \in G$, of bounded degree. Denote by $G(x_0, n)$ the set of vertices of G which are at distance n from x_0. Also, denote $T_n = \{\sigma \in T: |\sigma| = n\}$ and $A_n = |T_n|$. If*

$$\limsup_{n\to\infty} \frac{A_n}{|G(x_0, n)|} > 0 \tag{6.2}$$

(where here the $|\cdot|$ measures cardinality), then the T-walk on G, started at x_0, is strongly recurrent.

To prove the proposition, we need an inequality pertaining to ordinary reversible Markov chains.

LEMMA 6.2. *Let $\{Y_n\}_{n=0}^{\infty}$ be a reversible Markov chain on the state space G, with transition matrix p, initial state x_0 and a stationary measure π satisfying*

$$\sup\left\{\frac{\pi(x)}{\pi(y)}: x, y \in G\right\} = \beta < \infty. \tag{6.3}$$

Then there is a positive α such that any finite subset F of G satisfying

$$P_{x_0}[\exists\ n: Y_n \in F] = 1$$

also satisfies

$$P_{x_0}[\exists\ m, n: m < n, Y_m \in F, Y_n = x_0] \ge \frac{\alpha}{|F|}.$$

PROOF. We may safely assume the chain is transient. Denote by $g(x, y)$ the expected number of visits to y given $Y_0 = x$. By reversibility

$$\pi(x)g(x, y) = \pi(y)g(y, x),$$

so (6.3) implies that for all x, y,

$$g(x, y) \le \beta g(y, x). \tag{6.4}$$

Next, denote

$$f(x, y) = P_x[\exists\ n \ge 0: Y_n = y].$$

The identity

$$g(x,y) = f(x,y)g(y,y),$$

in conjunction with (6.4), gives

$$f(x_0,y) \le \beta \frac{g(y,x_0)}{g(y,y)} = \beta \frac{f(y,x_0)g(x_0,x_0)}{g(y,y)}.$$

Since $g(y,y) \ge 1$ we can set

$$\alpha = [\beta g(x_0,x_0)]^{-1}$$

and obtain

$$\alpha f(x_0,y) \le f(y,x_0). \tag{6.5}$$

For each $y \in F$ we have $f(x_0,y) \ge \mu(x_0,y)$, where $\mu(x_0,\cdot\,)$ is the entrance measure to F for the Markov chain started at x_0. Therefore,

$$\begin{aligned} P_{x_0}&[\exists\ m,n\colon m<n,\ Y_m \in F,\ Y_n = x_0] \\ &= \sum_{y\in F} \mu(x_0,y)\,f(y,x_0) \ge \alpha \sum_{y\in B} \mu(x_0,y)\,f(x_0,y) \\ &\ge \alpha \sum_{y\in F} \mu(x_0,y)^2 \ge \frac{\alpha}{|F|}, \end{aligned}$$

as claimed. $\square$

Proof of Proposition 6.1. Choose an increasing sequence $\{m_j\}$ such that for some $c > 0$,

$$\forall j \ge 1, \qquad A_{m_j} \ge c\big|G(x_0,m_j)\big|. \tag{6.6}$$

If $\{Y_n\}$ denotes the ordinary random walk on G, then by Lemma 6.2 for some $\alpha > 0$ and all m,

$$P_{x_0}\big[\exists\ n_1,n_2\colon n_1<n_2,\ Y_{n_1} \in G(x_0,m),\ Y_{n_2} = x_0\big] \ge \frac{\alpha}{|G(x_0,m)|}.$$

Passing to a subsequence of the $\{m_j\}$ if necessary, we can ensure that for all j the entrance measure from $x \in G(x_0,m_j)$ to $\{x_0\} \cup G(x_0,m_{j+1})$ assigns at least the mass $\alpha/(2|G(x_0,m_j)|)$ to x_0 (here we are using spherical symmetry). Returning to the tree-indexed setting, for each j and $\sigma \in T_{m_j}$ pick a ray $\xi = \xi(\sigma)$ through σ and consider the event Ω_σ that the trajectory of $\xi(\sigma)$ hits $G(x_0,m_j)$ and then returns to x_0 before hitting $G(x_0,m_{j+1})$. The spherical symmetry of G implies that all the events Ω_σ for $\sigma \in T_{m_j}$, $j \ge 1$ are independent. By the previous discussion,

$$|\sigma| = m_j \Rightarrow P[\Omega_\sigma] \ge \frac{\alpha}{2|G(x_0,m_j)|}.$$

Now (6.6) ensures that

$$\sum_{j=1}^{\infty} \sum_{\sigma \in T_{m_j}} P[\Omega_\sigma] = \infty,$$

so that the Borel–Cantelli lemma gives strong recurrence. □

EXAMPLE 6.3 (Strong recurrence depends on the initial state). Let T be a tree such that every vertex $\sigma \in T$ with $|\sigma| = n$ has a_n sons, where $\{a_n\}_{n \geq 0}$ increases very rapidly. Take G (rooted at x_0) to be isomorphic to T. If the T-walk on G is started at x_0, then it is *strongly ray-recurrent*, as repeated application of Proposition 6.1 shows. Denote $A_n = \prod_{j=0}^{n} a_j$ and assume $\{a_n\}$ satisfies $\sum[(A_{n-1})/a_n] < \infty$. Then

$$\prod_n \left(\frac{a_n}{a_n + 1} \right)^{A_{n-1}} > 0$$

and consequently the T-walk on G with any initial vertex *other than* x_0, has *no guaranteed return*.

In the following examples, to avoid the pathology in Example 6.3, we restrict ourselves to state-space graphs of *bounded degree*.

EXAMPLE 6.4 (Ray recurrence without guaranteed return—see Figure 3). Let T be a binary tree and let G be a regular tree of degree 16 rooted at x_0, with two added vertices x_1, x_2 where x_1 is connected only to x_0, x_2 and x_2 is a leaf. We know that the T-walk on the regular tree of degree 16 is nonrecurrent (by the remark following Example 3.2) and since T is binary, has no guaranteed return. It follows that with positive probability that T-walk *on* G never

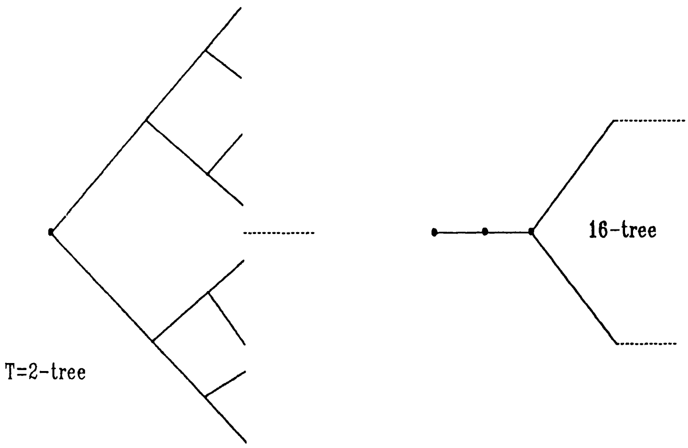

FIG. 3. *Ray recurrence without guaranteed return.*

returns to x_0. On the other hand, with positive probability T has a ray which visits x_2 every two steps (compare to a branching process).

Next, we describe the highlight of this section.

EXAMPLE 6.5 (Incomparable trees—see Figure 4). Let T^0 be a 3-tree (every vertex has three sons). For an increasing sequence $\{h_j\}$, to be specified later, let T^1 be an exploding tree (Example 3.3) in which 100^{h_j} rays are attached to the pivotal ray at height h_j.

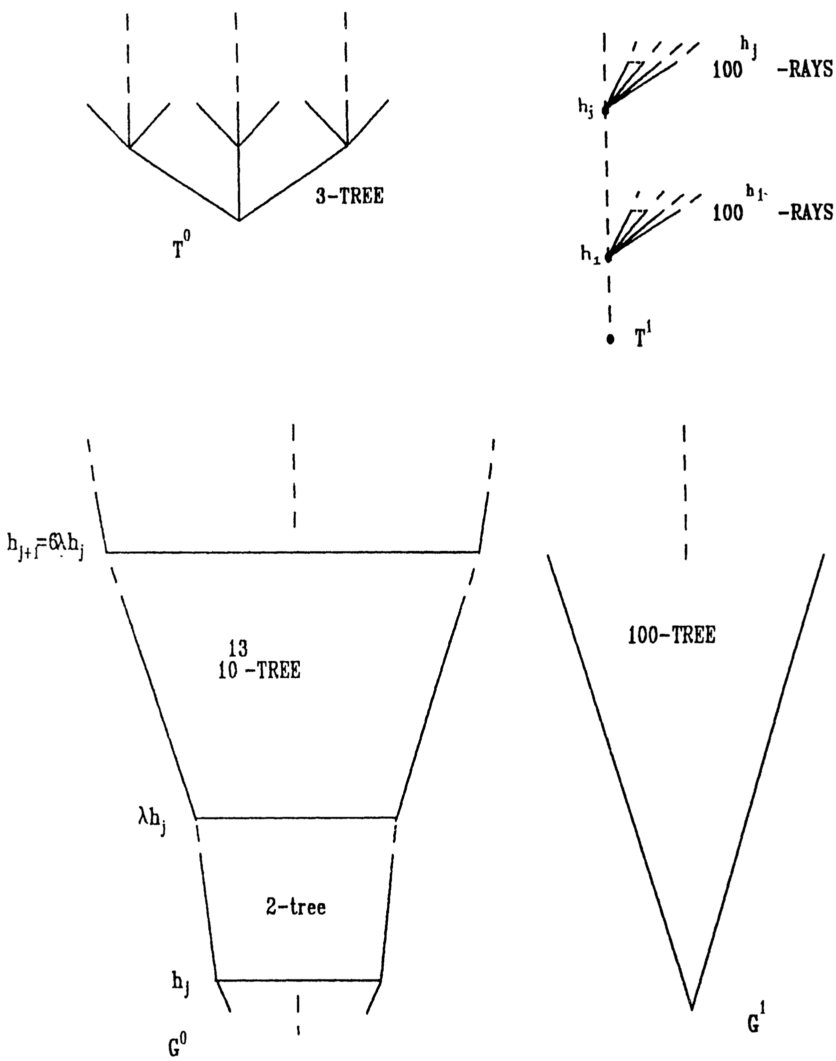

FIG. 4. *Incomparable trees.*

Next, we define state-space trees: G^0 is a tree in which every vertex between levels h_j and λh_j for $j \geq 1$ has two sons and all other vertices have $L = 10^{13}$ sons. Here $\lambda > 1$ is chosen to satisfy

$$(3/2)^{\lambda} > 3L^5/2^6 \tag{6.7}$$

and we take $h_j = (6\lambda)^j$.

Finally, take G^1 to be a 100-tree.

CLAIM. For $i = 0, 1$ the T^i-walk on G^i is strongly recurrent, while the T^i-walk on G^{1-i} is nonrecurrent and without guaranteed return.

PROOF. Note that G^0, G^1 are both spherically symmetric. For each $j \geq 1$, the space tree G^0 has less than $3^{\lambda h_j}$ vertices at level λh_j. This follows inductively from the inequality

$$3^{\lambda h_j} L^{h_{j+1} - \lambda h_j} 2^{\lambda h_{j+1} - h_{j+1}} < 3^{\lambda h_{j+1}},$$

which is a rewriting of (6.7). Thus, by Proposition 6.1, the T^0-walk on G^0 is strongly recurrent. Similarly, by comparing levels $h_j + 1$ of T^1 and G^1, we see that the T^1-walk on G^1 is strongly recurrent. Next, Proposition 4.3 implies that the T^0-walk on G^1 is not recurrent, because

$$\mathrm{Mdim}(T^0) = \log 3 < \log \frac{100}{2\sqrt{99}} = \log \frac{1}{\rho(G^1)}$$

(see [9], Section 4.5b). Since T^0 is a 3-tree, this walk has no guaranteed return.

The fact that the T^1-walk on G^0 is nonrecurrent is more recondite. Consider an ordinary random walk $\{Y_n\}_{n \geq 0}$, with $Y_0 \equiv 0$, on G^1. The law of large numbers implies that almost surely there exists (a random) N such that

$$n \geq N \Rightarrow |Y_n| \geq \frac{n}{3}. \tag{6.8}$$

We employ (6.8) for the trajectory of the pivotal ray ξ of T^1. A random walk starting at level $2\lambda h_j$ of G^1 has probability at most $L^{-\lambda h_j}$ of ever returning to 0. Thus, conditioning on the minimal N for which the trajectory of the pivotal ray ξ satisfies (6.8), we find that

$$\sum_{\{\eta : |\eta \wedge \xi| \geq N\}} P\{\exists\, \sigma \in \eta : S_\sigma = 0 | N\} \leq \sum_{h_{j+1} \geq N} 100^{h_{j+1}} L^{-\lambda h_j},$$

which is finite by our choice of $L = 10^{13}$. Now the Borel–Cantelli lemma implies nonrecurrence, since the finitely many rays η of T^1 satisfying $|\eta \wedge \xi| < N$ cannot change the type of the T^1-walk. □

EXAMPLE 6.6 (Guaranteed return without strong recurrence—see Figure 5). This is an application of the previous example. From the trees T^i, G^i used there, construct G by identifying the roots of G^0, G^1. Construct T by identifying the roots of T^0, T^1 and connecting this vertex to an additional

FIG. 5. *Guaranteed return without strong recurrence.*

vertex designated as a new root. By conditioning on the first step of the T-walk, we verify the guaranteed return. Since with positive probability the T^i-walk on G^{1-i} never returns to 0, it follows that the T-walk on G is *not* strongly recurrent. The same example also shows that the guaranteed return for a T-walk can depend on the initial state, even for state-space graphs of bounded degree.

7. A recurrence diagram and unresolved questions.

First, we recall the different recurrence notions for tree-indexed walks (formal definitions are given in Sections 3 and 6).

Recurrence. With positive probability, there is a state in G which is visited infinitely often by the tree-indexed walk.

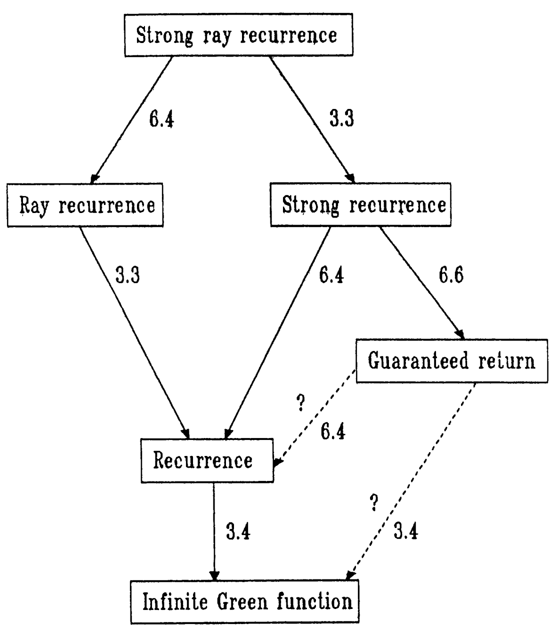

FIG. 6. *Note: Ray recurrence is incomparable to strong recurrence and to guaranteed return (cf. Examples* 3.3 *and* 6.4).

Strong recurrence. As above, but with probability 1.

Ray recurrence. With positive probability, there is a state in G which is visited infinitely often along a single ray of the tree.

Strong ray recurrence. Like ray recurrence, but with probability 1.

Guaranteed return. With probability 1, the tree-indexed walk returns to its initial state.

Infinite Green function. The expected number of visits of the tree-indexed walk to its initial state is infinite.

In Figure 6 the arrows indicate implications which *cannot* be reversed. The relevant examples are indicated next to the arrows (the implications themselves follow from the definitions). The broken lines indicate unresolved implications. We are indebted to the referee for suggesting the inclusion of this diagram.

Questions and remarks.

1. Does the guaranteed return of a T-walk on a graph G imply recurrence (or at least an infinite Green function)? This is most interesting when G is the Cayley graph of a finitely generated group. (In [4] it is shown that for a T-walk on a Cayley graph, recurrence implies strong recurrence.)
2. Can an example of "incomparable trees" exist if we restrict attention to Cayley graphs G? If not, we could define an interesting semiorder on the set of trees: T dominates T' if for any Cayley graph on which the T'-walk is recurrent, so is the T-walk. In view of Example 6.6, this question is closely related to the previous one.
3. Does recurrence of a tree-indexed chain dependent on the initial state x_0 in G? Similarly for ray recurrence. (It is elementary to check that the finiteness of the Green function does *not* depend on the initial state; Example 6.3 shows that strong ray recurrence, strong recurrence and guaranteed return depend on it.)
4. Does the event that y is visited infinitely often along some single ray of the tree and z is not have probability 0 for any two states y, z in G? (Compare with Lemma 3.1.)
5. Which graphs G have the property that for any tree T such that the T-walk on G has an infinite Green function, this T-walk is necessarily recurrent? In [4] it is shown that the lattices $\mathbb{Z}^d$ and more generally Cayley graphs of polynomial growth have this property, while Cayley graphs on which there exist nonconstant bounded harmonic functions do not.

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