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HITTING SPHERES FROM THE EXTERIOR

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By using an appropriate martingale, we compute the joint distribution of the hitting time and place of a sphere by d-dimensional Brownian motion, when the process starts outside the sphere.

Let (B_t) be a standard d-dimensional Brownian motion, $d \geq 2$, and

$$T = \inf\{t > 0: |B_t| = R\}$$

be the hitting time of a sphere of radius R centered at the origin, $S(0, R)$. A "Laplace–Gegenbauer" transform of the joint distribution of T and $B(T)$, the position at time T, was computed in [4] when the starting point of the process is either inside or outside the sphere. At the end of [4] it is described how the joint distribution may be obtained by using an exponential martingale when the process starts inside the sphere, and it is asserted that no similar approach is known for the exterior problem. In [1], by considering an appropriate martingale, we proved that such an approach is indeed possible in the three-dimensional case. We will extend the result to the d-dimensional case in this article.

We begin by introducing some notation and preliminary results. We let K_ν and I_ν will denote the Bessel functions of "purely imaginary argument" (see [3], pages 77 and 78).

The relationship between K_ν and the Laplace transform of the Brownian semigroup is well known (see [3], page 183); in fact, one has

$$\begin{aligned} u_\alpha(|x-y|) &= \int_0^\infty \frac{e^{-\alpha t} e^{-|x-y|/2t}}{(2\pi t)^{d/2}}\, dt \\ &= 2 \cdot (2\pi)^{-d/2} \left(\frac{\sqrt{2\alpha}}{|x-y|} \right)^h K_h\left(\sqrt{2\alpha}\,|x-y|\right), \end{aligned} \tag{1}$$

with $h = (d-2)/2$.

The following addition theorem for K_ν, due to MacDonald (see [3], page 365), will be basic to our proof: If $w = \sqrt{Z^2 + z^2 - 2Zz\cos\phi}$, then

$$\frac{K_\nu(w)}{(w)^\nu} = 2^\nu \Gamma(\nu) \sum_{m=0}^{\infty} (\nu + m) \frac{K_{\nu+m}(Z)}{Z^\nu} \frac{I_{\nu+m}(z)}{z^\nu} C_m^\nu(\cos\phi). \tag{2}$$

Here C_n^ν denotes the Gegenbauer polynomial of degree n and order $\nu > 0$ (see

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[3], page 50). Recall that $C_n^\nu(t)$ is the coefficient of α^n in the expansion of $(1 - 2\alpha t + \alpha^2)^{-\nu}$ in ascending powers of α. We let Θ_t denote the angle between B_0, 0 and B_t [for us B_0 will always be outside $S(0, R)$, so $B_0 \neq 0$]. By rotational symmetry, we need only give the joint distribution of T and Θ_T, which will be determined by

$$E^x\left[e^{-\beta T} \cos n\Theta_T\right].$$

Since $\cos n\theta$ can be written as a linear combination of the $C_k^h(\cos\theta)$ (see [2]), it suffices to compute

$$E^x\left[e^{-\beta T} C_m^h(\cos\Theta_T)\right].$$

Let x be a point outside the sphere $S(0, R)$. Then one has the following result (see [4] for a different proof).

THEOREM 1. *For $|x| > R$ we have*

$$E^x\left[e^{-(\lambda^2/2)T} C_m^h(\cos\Theta_T)\right] = \left(\frac{R}{|x|}\right)^h \frac{K_{h+m}(\lambda|x|)}{K_{n+m}(\lambda R)} C_m^h(1), \tag{3}$$

with $h = (d-2)/2$ and $\lambda > 0$.

PROOF. Let y be a point inside $S(0, R)$, that is, $|y| < R < |x|$. Consider the function

$$F(t, x) = e^{-\lambda^2 t/2} \frac{K_h(\lambda|x-y|)}{(\lambda|x-y|)^h},$$

which satisfies

$$\frac{\partial F}{\partial t} + \frac{1}{2}\Delta F = 0 \quad \text{at } (t, x) \in (0, \infty) \times \left(\mathbb{R}^d - \{y\}\right),$$

and consider the process

$$Z_t = F(t, B_t).$$

It follows from Itô's formula that $F(t, B_t)$ is a martingale on $[0, T]$ with respect to $P^x(|x| > R)$. Actually, since $\{y\}$ is polar for B_t, $F(t, B_t)$ is a martingale on $[0, \infty]$ with respect to P^x. From the martingale property for Z_t, we obtain

$$E^x\left[e^{-(\lambda^2/2)T} \frac{K_h(\lambda|B(T) - y|)}{(\lambda|B(T) - y|)^h}\right] = \frac{K_h(\lambda|x-y|)}{(\lambda|x-y|)^h}.$$

We now use MacDonald's addition formula (2) on both sides of (3) and equate coefficients of $I_{h+m}(\lambda|y|)$ to obtain

$$E^x\left[e^{-(\lambda^2/2)T} \frac{K_{h+m}(\lambda|B(T)|)}{(\lambda|B(T)|)^h} C_m^h(\cos\Theta_T)\right] = \frac{K_{h+m}(\lambda|x|)}{(\lambda|x|)}^h C_m^h(\cos\phi),$$

where $\phi = \angle y0x$.

Observing that $|B(T)| = R$ and choosing y so that $\phi = 0$, we get the result. □

COMMENT. The same martingale and basically the same argument allow us to reobtain Wendel's results for the case $|x| < R$.

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