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# A SOLUTION OF THE COMPUTER TOMOGRAPHY PARADOX AND ESTIMATING THE DISTANCES BETWEEN THE DENSITIES OF MEASURES WITH THE SAME MARGINALS [1]

BY L. A. KHALFIN[2] AND L. B. KLEBANOV

*Russian Academy of Sciences*

We give estimates of the distances between the densities of measures having the same finite number of the same marginals. These estimates give a solution of the computer tomography paradox of Gutman, Kemperman, Reeds and Shepp, and open the possibility for construction of a new method of inversion of the Radon transformation.

**1. Introduction.** The traditional methods of computer tomography (CT; see, for example, the excellent survey [4] and recent report [5]) are useful in many branches of science, medicine and technology. CT is based on the inversion of the Radon transformation, which allows one to reconstruct (in a unique way) the density of a measure. However, such unique reconstruction is possible only if one knows all (an infinite number) of the marginals. In practice one can have only a finite number of marginals; unique reconstruction of the density is impossible in this case.

Let $f(x)$ be a probability density function on the Euclidean plane $\mathbb{R}^2$ with a compact support $D \subset \mathbb{R}^2$. Let $\theta_1, \dots, \theta_N$ be $N$ directions ($N$ unit vectors) in the plane $\mathbb{R}^2$. It was proved in [3] that for any density $f$: $0 \le f(x) \le 1$, $x \in D$, and for any finite number of directions $\theta_1, \dots, \theta_N$ ($N < \infty$) there exists another density $f_0(x)$, $x \in D$, such that $f_0$ has the same marginals in the directions $\theta_1, \dots, \theta_N$ as $f$ and such that $f_0$ has only two values: 0 or 1. This result gives the following CT paradox (see [3]): It implies that for any human object and corresponding projection data there exist many different reconstructions, in particular, a reconstruction consisting only of bone and air (density 0 or 1), but still having the same projection data as the original object. Related nonuniqueness results are familiar [6–8] in tomography and are usually ignored because CT machines seem to produce useful images. It is likely that the explanation of this apparent paradox is that point reconstruction in tomography is impossible. CT machines produce useful images because all functions $0 \le f(x) \le 1$ with the same line integrals have (essentially) the same integrals over "nice" sets. In other words, it is likely that all functions $f(x)$ with $0 \le f(x) \le 1$ and with the same line integrals have nearly identical integrals over pixels that are not too small. However, we have neither a proof nor a precise statement of this heuristic idea.

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[2]Current address: Department of Physics, Clarkson University, Potsdam, New York 13676.

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In the present work we given a full quantitative and qualitative solution of this CT paradox. The solution is based on new estimates of the distances between corresponding functionals of the densities with a finite number ($N$) of the same marginals (Section 1). In Section 2 we give corresponding estimates of the distances between densities having $\varepsilon$-identical marginals in a finite number of directions. In Section 3 we sketch the proofs of the main theorems of this work. In Section 4 we discuss possible applications of derived mathematical results to CT problems.

**2. On estimation of a distance between the densities with a finite number of identical marginals.** Let $\mathbb{F}_D$ be the set of all probabilistic densities with compact support in the square

$$D = \{x = (x_1, x_2) \colon |x_1| \leq 1,\ |x_2| \leq 1\}.$$

Let $\varphi_\sigma$ be the Gaussian density

$$\varphi_\sigma(x) = \left(2\pi\sigma^2\right)^{-1} \exp\left(-\left(x_1^2 + x_2^2\right)/\left(2\sigma^2\right)\right), \qquad x \in \mathbb{R}^2,$$

and let $f * \varphi_\sigma(x)$ and $g * \varphi_\sigma(x)$ ($f, g \in \mathbb{F}_D$) be the convolutions of $\varphi_\sigma$ with $f, g \in \mathbb{F}_D$.

THEOREM 1. *Let $n \geq 2$ be a natural number. There exist $N = 2n$ directions $\theta_1, \ldots, \theta_N$ on the Euclidean $\mathbb{R}^2$ plane such that if the densities $f, g \in \mathbb{F}_D$ have the same marginals in the directions $\theta_1, \ldots, \theta_N$, then*

$$\sup_{x \in \mathbb{R}^2} |f * \varphi_\sigma(x) - g * \varphi_\sigma(x)| \leq 1 \Big/ \left(\sqrt{\pi}\sigma^{n+2} 2^{(3n-4)/2} \Gamma((n+1)/2)\right), \tag{1}$$

*where $\Gamma(z)$ is the gamma function. The vectors $\theta_1, \ldots, \theta_N$ can be chosen as follows:*

$$\theta_j = (v_j, -1)/\left(v_j^2 + 1\right)^{1/2}, \qquad j = 1, \ldots, n,$$
$$\theta_j = (1, v_{j-n})/\left(v_{j-n}^2 + 1\right)^{1/2}, \qquad j = n+1, \ldots, 2n,$$

*where*

$$v_k = \cos\left(\pi(2k-1)/(2n)\right), \qquad k = 1, \ldots, n.$$

Applying this theorem to the densities $f$ and $f_0$, mentioned in the Introduction (see Theorem 2 in [3]), we get full quantitative and qualitative solutions of the CT paradox. Although the sup norm distance between $f$ and $f_0$ is equal to 1, the distance between the smoothed densities $f * \varphi_\sigma$ and $f_0 * \varphi_\sigma$ is small for sufficiently large $N$.

Theorem 1 gives an estimate of the closeness of $g * \varphi_\sigma$ and $f * \varphi_\sigma$ in terms of homogeneous weak metrics. Without any a priori restrictions posed on the set of admissible densities, this estimate looks quite sufficient. For concrete a priori restrictions on the set of admissible densities $f$ (such restrictions would be defined by concrete problems for which CT will be used) some other metrics might be more useful. For such metrics, the estimate of the closeness between $f*\varphi_\sigma$ and $g*\varphi_\sigma$ can be obtained by the method used for the proof of the Theorem 1 (see Section 3). We will discuss the corresponding results in a separate work.

If in addition to the hypotheses of Theorem 1 we assume that $f(x)$ is differentiable, then it is possible to get an estimator for $f(x)$ from $g*\varphi_\sigma(x)$.

THEOREM 2. *Let $f \in \mathbb{F}_D$ be a differentiable density function. Under the hypotheses of Theorem 1 we have*

$$\sup_{x\in\mathbb{R}^2} |f(x) - g*\varphi_\sigma(x)| \le C_f\left(2/\pi\right)^{1/2}\sigma + 1\Big/\left(\left(2^{3n-4}\pi\right)^{1/2}\sigma^{n+2}\Gamma\left((n+1)/2\right)\right), \tag{2}$$

*where*

$$C_f = \sup_{x\in\mathbb{R}^2} \|\mathrm{grad}\, f(x)\|.$$

For $\sigma = \sigma(n) = (n)^{-1/2}$ we get from (2) the inequality

$$\sup_{x\in\mathbb{R}^2} |f(x) - g*\varphi_\sigma(x)| \le \widetilde{C}n^{-1/2}, \tag{2'}$$

where $\widetilde{C}$ depends only on $C_f$.

**3. An estimate of a distance between the densities with $\varepsilon$-identical marginals.** Let us suppose that $\theta_1, \ldots, \theta_N$ are the same directions as in Theorem 1.

THEOREM 3. *If the integrals of the densities $f$ and $g$ ($f, g \in \mathbb{F}_D$) over all straight lines with directions $\theta_j$ ($j = 1, \ldots, N$) $\varepsilon$-coincide, that is, for some fixed $\varepsilon \in (0,1)$,*

$$\left|\int_{-\infty}^{\infty} f(x+t\theta_j)\,dt - \int_{-\infty}^{\infty} g(x+t\theta_j)\,dt\right| \le \varepsilon, \qquad j = 1,\ldots,N,\ x\in\mathbb{R}^2,$$

*then*

$$\begin{aligned}&\sup_{x\in\mathbb{R}^2} |f*\varphi_\sigma(x) - g*\varphi_\sigma(x)|\\ &\quad\le \pi\sigma^{-(n+2)}\Big[\varepsilon\left(8 + (4/\pi)\log n\right)2^{(n+2)/2}\Gamma\left((n+2)/2\right)\\ &\qquad + 1\Big/\left(\pi 2^{(3n-4)/2}\Gamma\left((n+1)/2\right)\right)\Big].\end{aligned} \tag{3}$$

**4. Proofs of main theorems.** The proofs of all the preceding theorems are based on the following main lemma.

LEMMA 1. *Under the hypotheses of Theorem* 1 *we have, for every positive* $A$,

$$\sup_{\|\tau\|_\infty \le A} \left|\widehat{f}(\tau_1,\tau_2) - \widehat{g}(\tau_1,\tau_2)\right| \le A^n/\left(2^{n-2}n!\right), \tag{4}$$

*where* $\widehat{f}$ *and* $\widehat{g}$ *are the Fourier transforms of* $f$ *and* $g$, *respectively, and* $\|\tau\|_\infty = \max(|\tau_1|,|\tau_2|)$.

PROOF. Let us fix arbitrary $a \in [0,A]$ and consider the difference

$$h_a(\tau_1) = \widehat{f}(\tau_1,a) - \widehat{g}(\tau_1,a)$$

as a function of $\tau_1 \in (-a,a)$. Because the densities $f$ and $g$ have the same marginals in the directions $\theta_1,\dots,\theta_N$, we have $h_a(\tau_1) = 0$, $\tau_1 \in \{av_1,\dots,av_n\}$. Here

$$\begin{aligned} \theta_j &= (v_j,-1)/\left(v_j^2+1\right)^{1/2}, \qquad j = 1,\dots,n,\\ \theta_j &= (1,v_{j-n})/\left(v_{j-n}^2+1\right)^{1/2}, \qquad j = n+1,\dots,2n = N,\\ v_s &= \cos\left(\pi(2s-1)/(2n)\right), \qquad s = 1,\dots,n, \end{aligned}$$

It is easy to see that the points $av_1,\dots,av_n$ are the Tchebyshev points of interpolation for the interval $(-a,a)$. Because the densities $f$ and $g$ have compact support in the unit square $D$, then $\widehat{f}(\tau_1,\tau_2)$ and $\widehat{g}(\tau_1,\tau_2)$ are infinitely differentiable functions and

$$\begin{aligned} \left|(\partial^{s+t}/\partial\tau_1^s\,\partial\tau_2^t)\widehat{f}(\tau_1,\tau_2)\right| &\le 1,\\ \left|(\partial^{s+t}/\partial\tau_1^s\,\partial\tau_2^t)\widehat{g}(\tau_1,\tau_2)\right| &\le 1 \end{aligned} \tag{5}$$

for any natural numbers $s,t$.

By using (4) and (5) we get from an error estimate for Tchebyshev interpolation (see [9], pages 81 and 82) that

$$|h_a(\tau_1)| = |\widehat{f}(\tau_1,a) - \widehat{g}(\tau_1,a)| \le a^n/\left(2^{n-2}n!\right) \tag{6}$$

for $|\tau_1| \le a$. By the same arguments we derive

$$|\widehat{f}(\tau_1,-a) - \widehat{g}(\tau_1-a)| \le a^n/\left(2^{n-2}n!\right), \qquad |\tau_1| \le a, \tag{7}$$

$$|\widehat{f}(\pm a,\tau_2) - \widehat{g}(\pm a,\tau_2)| \le a^n/\left(2^{n-2}n!\right), \qquad |\tau_2| \le a. \tag{8}$$

From (6)–(8) it follows that

$$\sup_{\|\tau\|_\infty \le a} |\widehat{f}(\tau_1,\tau_2) - \widehat{g}(\tau_1,\tau_2)| \le a^n/\left(2^{n-2}n!\right), \qquad a \in (0,A]. \tag{9}$$

The inequality (4) in Lemma 1 is an immediate consequence of the inequality (9). □

PROOF OF THEOREM 1. Now, as a typical example, we give the main idea of the proof of Theorem 1 on the basis of Lemma 1. The other theorems in this paper have similar proofs with some additional arguments.

By using Lemma 1, we get

$$\begin{aligned}
&|f * \varphi_\sigma(x) - g * \varphi_\sigma(x)| \\
&= (2\pi)^{-2} \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\bigl(-i(t_1 x_1 + t_2 x_2)\bigr) \bigl(\widehat{f}(t_1, t_2) \right. \\
&\qquad \left. - \widehat{g}(t_1, t_2)\bigr) \exp\Bigl(-\sigma^2 \bigl(t_1^2 + t_2^2\bigr)/2\Bigr)\, dt_1\, dt_2 \right| \\
&\le (2\pi)^{-2} \int_0^{2\pi} d\varphi \int_0^{\infty} |\widehat{f}(r\cos\varphi, r\sin\varphi) - \widehat{g}(r\cos\varphi, r\sin\varphi)| \\
&\qquad \times \exp\bigl(-\sigma^2 r^2/2\bigr) r\, dr \\
&\le (2\pi)^{-1} \int_0^{\infty} \Bigl(r^n \exp\Bigl(-\sigma^2 r^2/2\Bigr) / \bigl(2^{n-2} n!\bigr)\Bigr) r\, dr \\
&= \Gamma\bigl((n+2)/2\bigr) / \bigl(2^{n-2/2} \pi \sigma^{n+2} n!\bigr).
\end{aligned} \tag{10}$$

However,

$$n! = \Gamma(n+1) = \Gamma\bigl(2(n+1)/2\bigr) = \bigl(2^{n-1}/\sqrt{\pi}\bigr)\Gamma\bigl((n+1)/2\bigr)\Gamma\bigl((n+2)/2\bigr) \tag{11}$$

and after putting expression (11) into (10), we get the main inequality (1) in Theorem 1. □

PROOF OF THEOREM 2. It is easy to see that

$$\begin{aligned}
\sup_{x \in \mathbb{R}^2} |f(x) - f * \varphi_\sigma(x)| &\le \int_{\mathbb{R}^2} |f(x) - f(x+u)| \varphi_\sigma(u)\, du \\
&\le C_f \int_{\mathbb{R}^2} \|u\| \varphi_\sigma(u)\, du = \sigma C_f \bigl(2/\pi\bigr)^{1/2}.
\end{aligned}$$

Inequality (2) follows now from the estimate in Theorem 1. □

PROOF OF THEOREM 3. We use here the same notation as in Lemma 1. Because the densities $f$ and $g$ have $\varepsilon$-identical marginals in the directions $\theta_1, \dots, \theta_N$,

$$|h_a(\tau_1)| \le \varepsilon, \qquad \tau_1 \in \{av_1, \dots, av_n\}. \tag{12}$$

Let $P_{\hat{f}}(\tau_1)$ and $P_{\hat{g}}(\tau_1)$ be Tchebyshev polynomials of interpolation of degree $n+1$, that is,

$$P_{\hat{f}}(\tau_1) = \sum_{k=1}^{n} l_k^{(n)}(\tau_1) \widehat{f}(av_k), \tag{13}$$

$$P_{\hat{g}}(\tau_1) = \sum_{k=1}^{n} l_k^{(n)}(\tau_1) \widehat{g}(av_k). \tag{14}$$

It follows from (14) that

(15) $$\sup_{\tau_1 \in [-a,a]} |P_{\hat{f}}(\tau_1) - \widehat{f}(\tau_1, a)| \leq a^n 2^{1-n}/n!,$$

(16) $$\sup_{\tau_1 \in [-a,a]} |P_{\hat{g}}(\tau_1) - \widehat{f}(\tau_1, a)| \leq a^n 2^{1-n}/n!.$$

Put

$$\lambda_n = \sup_{\tau_1} \sum_{k=1}^{n} \left| l_k^{(n)}(\tau_1) \right|.$$

It is known (see [9]) that

(17) $$\lambda_n \leq 8 + \left(4/\pi\right) \log n.$$

Therefore, we get from (12), (13), (14) and (17) that

(18) $$\sup_{|\tau_1| \leq a} \left|P_{\hat{f}}(\tau_1) - P_{\hat{g}}(\tau_1)\right| \leq \varepsilon\Big(8 + \left(4/\pi\right) \log n\Big).$$

From (15), (16) and (18) we obtain

$$\sup_{|\tau_1| \leq a} |\widehat{f}(\tau_1, a) - \widehat{g}(\tau_1, a)| \leq a^n 2^{2-n}/n! + \varepsilon\Big(8 + \left(4/\pi\right) \log n\Big). \qquad \square$$

**5. Discussion.** It is possible to use mathematical results that we derived, for CT problems. As was discussed in [1, 2] (and in more detail in recent reports [10–12]), on the basis of the main inequality (1) it is possible to give a new definition of the space resolution power and the sensitivity (the resolution power in density) of CT. Moreover, on the basis of these results [1, 2, 10–12] we proposed a new method of reconstruction in CT which unlike commonly used methods, yields no mathematical artifacts. On the basis of stabilization properties (see Section 2) it was proved [10–12] that this new method is well posed. Different concrete problems for which CT is used correspond to choosing different kernels of a convolution (not only the Gaussian and the Valle–Poissin [10–12]).

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## REFERENCES

[1] KHALFIN, L. A. and KLEBANOV, L. B. (1990). A solution of the computer tomography paradox and estimating the distances between the densities of measures with the same marginals. Preprint, Clarkson Univ., Potsdam, NY.

[2] KHALFIN, L. A. and KLEBANOV, L. B. (1991). On some problems of the computer tomography. Solvay Internationaux Instituts de Physique et de Chimie, Brussels.

[3] GUTMAN, S., KEMPERMAN, J. H. B., REEDS, J. A. and SHEPP, L. A. (1990). Existence of probability measures with given marginals. Preprint, AT & T Bell Laboratories.

[4] SHEPP, L. A. and KRUSKAL, J. B. (1978). *Amer. Math. Monthly* **85** 42.

[5] CORMACK, A. (1986). *Notices Amer. Math. Soc.* **33**(5) 716.

[6] LEAHY, J. V., SMITH, K. T. and SOLOMON, D. C. (1979). Uniqueness, non-uniqueness and inversion in the x-ray and Radon problems. In *Proceedings of the International Symposium on Ill-Posed Problems*. Newark.

[7] GRUNBAUM, F. A. (1981). Reconstruction with arbitrary directions: dimensions two and three. In *Mathematical Aspects of Computerized Tomography* (G. T. Herman and F. Nutterer, eds.). Springer, Berlin.

[8] LOUIS, A. K. (1981). *Math. Methods Appl. Sci.* **3** 1.

[9] GONCHAROV, V. L. (1933). *Theory of Interpolation and Extrapolation of Functions.* Nauka, Moscow (in Russian).

[10] KHALFIN, L. A. and KLEBANOV, L. B. (1993). On some problems of computerized tomography. Report on Kolmogorov Probability and Statistics Semester, Stochastic Analysis and Connected Questions, Euler International Mathematical Institute, St. Petersburg, Russia.

[11] KHALFIN, L. A. and KLEBANOV, L. B. (1993). On some stability problems in computer and diffractional tomography. Report on 1993 AMS–SIAM Summer Seminar in Applied Mathematics, The Mathematics of Tomography, Impedance Imaging, and Integral Geometry, June 1993, South Hadley.

[12] KHALFIN, L. A. (1993). On the sensitiveness and the resolution power of tomography. Report on 1993 AMS–SIAM Summer Seminar in Applied Mathematics, The Mathematics of Tomography, Impedance Imaging and Integral Geometry, June 1993, South Hadley.

STEKLOV MATHEMATICAL INSTITUTE
RUSSIAN ACADEMY OF SCIENCES
FONTANKA 27
ST. PETERSBURG 191011
RUSSIA

INSTITUTE OF MATHEMATICAL GEOLOGY
RUSSIAN ACADEMY OF SCIENCES
SHPALERNAYA 12
ST. PETERSBURG 191187
RUSSIA