The Annals of Probability
1994, Vol. 22, No. 4, 1707–1714

LOWER ESTIMATES OF THE CONVERGENCE RATE FOR U-STATISTICS[1]

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Recent results on the Berry–Esséen bound for U-statistics assumed the following conditions: Suppose a U-statistic (of degree 2) is nondegenerate. Then the rate of convergence in the CLT is of the order $O(n^{-1/2})$ provided that

$$\mathbb{E}\Big|\mathbb{E}\Big\{h(X_1,X_2)\mid X_1\Big\}\Big|^3<\infty,\quad \mathbb{E}|h(X_1,X_2)|^{5/3}<\infty,$$

where h is a symmetric kernel corresponding to the U-statistic. It follows from our results that these moment conditions are final. In particular, the last moment condition cannot be replaced by a moment of order $5/3-\varepsilon$ for any $\varepsilon>0$. Similar results hold for von Mises statistics.

U-Statistics. Let $X, X_1, X_2, \ldots \in \mathbb{X}$ be a sequence of independent identically distributed (briefly i.i.d.) random variables (r.v.'s) taking values in a measurable space $\mathbb{X}$. Let h be a symmetric function of two variables and such that $\mathbb{E}h(X, X_1) = 0$.

The U-statistic corresponding to h is defined [see Hoeffding (1948)] by

$$U_n = U_n(h) = \sum_{1\le i<j\le n} h(X_i, X_j).$$

Let the function $g\colon \mathbb{X} \to \mathbb{R}$ be given by $g(x) = \mathbb{E}\,h(x, X)$. Thoughout we shall assume that $0 < \mathbb{E}g^2(X) < \infty$. Therefore, without loss of generality we may assume that $\mathbb{E}g^2(X) = 1$.

If we define the symmetric function

$$\psi(x,y) = h(x,y) - g(x) - g(y),$$

then $\mathbb{E}\psi(x, X) = 0$ and

$$U_n = (n-1)\sum_{1\le i\le n} g(X_i) + \sum_{1\le i<j\le n} \psi(X_i, X_j), \tag{1}$$

which is the Hoeffding decomposition of U_n.

Let $\tau_n^2 = n(n-1)^2$ be the variance of the first term on the right-hand side in (1) and let Φ stand for the standard normal distribution function. The

Received November 1992.
[1]Research supported by the SFB 343 at Bielefeld.
AMS 1991 *subject classifications.* Primary 60F05; secondary 62E20.
Key words and phrases. U-statistics, Berry-Esséen bound, convergence rate, lower bound.

following Berry–Esséen result is well known (see discussion below for historical references).

THEOREM 1. *There exists a universal constant c such that*

$$\sup_{x\in\mathbb{R}} \left|P\{\tau_n^{-1}U_n < x\} - \Phi(x)\right| \le cn^{-1/2}\{\mathbb{E}|g(X)|^3 + \mathbb{E}|\psi(X,X_1)|^{5/3}\} \tag{2}$$

for $n \ge 2$.

REMARK. Obviously, the estimate (2) is equivalent (up to the value of the absolute constant c) to

$$\sup_{x\in\mathbb{R}} \left|P\{\tau_n^{-1}U_n < x\} - \Phi(x)\right| \le cn^{-1/2}\{\mathbb{E}|g(X)|^3 + \mathbb{E}|h(X,X_1)|^{5/3}\} \tag{3}$$

The Berry–Esséen type results

$$\sup_{x\in\mathbb{R}} \left|P\{\tau_n^{-1}U_n < x\} - \Phi(x)\right| = O(n^{-1/2})$$

in growing strength and generality were obtained by Grams and Serfling (1973), Bickel (1974), Chan and Wierman (1977) and Callaert and Janssen (1980). Helmers and van Zwet (1982) proved it under the condition: there exists $\varepsilon > 0$ such that $\mathbb{E}|\psi(X,X_1)|^{5/3+\varepsilon} < \infty$. The result without an ε in the moment condition can be found in Koroliuk and Borovskih (1989), Friedrich (1989), Bolthausen and Götze (1989) and Götze (1991). Our main result is that the moment condition $\mathbb{E}|\psi(X,X_1)|^{5/3} < \infty$ in Theorem 1 is final, that is, it cannot be replaced by $\mathbb{E}|\psi(X,X_1)|^{5/3-\varepsilon} < \infty$ for any $\varepsilon > 0$.

Throughout we denote by $\eta, \eta_1, \eta_2, \dots$ a sequence of i.i.d. standard normal r.v.'s, by $Z, Z_1, Z_2, \dots$ a sequence of i.i.d. real r.v.'s, and assume that all r.v.'s are independent.

A natural way to find an example that does not satisfy the Berry–Esséen inequality is to choose the function h in such a way that $h(X_i,X_j) = \eta_i + \eta_j + Z_iZ_j$, where $\mathbb{E}Z_i = 0$. Moreover, to make the situation simpler, we may assume that Z_i are symmetrically distributed. Then $\tau_n^{-1}U_n$ is distributed as $\eta + S_n$, where $S_n = \tau_n^{-1}\Sigma_{1\le i<j\le n}Z_iZ_j$. The Berry–Esséen inequality is violated if the sum S_n is relatively large, and this happens if the variables Z_i have few moments. This idea, which is formulated more explicitly in Lemmas 11 and 12, is behind the definition of the class $\mathcal{A}_\alpha(\mathbb{X},h)$.

For a given $\alpha > 0$, a measurable space $\mathbb{X}$ and a symmetric function $h\colon \mathbb{X}^2 \to \mathbb{R}$, let $\mathcal{A}_\alpha(\mathbb{X},h)$ stand for the class of all r.v.'s X assuming values in $\mathbb{X}$ and such that

$$\mathbb{E}g(X) = 0, \qquad \mathbb{E}g^2(X) = 1, \qquad \mathbb{E}|g(X)|^3 = 4,$$
$$\mathbb{E}\psi(x,X) \equiv 0, \qquad \mathbb{E}|\psi(X,X_1)|^\alpha = 1.$$

EXAMPLE 2. There exist a function $h = h_0$ and a measurable space $\mathbb{X}$ such that the class $\mathcal{A}_\alpha(\mathbb{X},h)$ is nonempty. Indeed, let $\mathbb{X} = \mathbb{R}^2$,

$$h_0(x,y) = x_1 + y_1 + x_2y_2, \qquad x = (x_1,x_2), \qquad y = (y_1,y_2), \qquad x,y \in \mathbb{R}^2. \tag{4}$$

Let η be a standard normal real r.v. with mean 0 and variance 1. Let Z be a symmetric real r.v. independent of η. We shall denote by $\mathcal{C}$ the class of all r.v.'s $X = (\eta, Z) \in \mathbb{R}^2$. If $X \in \mathcal{C}$ and $\mathbb{E}|Z|^\alpha = 1$, then $g_0(x) = x_1$, $\psi_0(x,y) = x_2 y_2$ and $X \in \mathcal{A}_\alpha(\mathbb{R}^2, h_0)$.

THEOREM 3. *Let $\varepsilon \le 1/3$. There exist a measurable space $\mathbb{X}$, a function h and a positive constant $c(\varepsilon) > 0$ depending only on ε such that*

$$\sup \sup_{x \in \mathbb{R}} \left| P\{\tau_n^{-1} U_n < x\} - \Phi(x) \right| \ge c(\varepsilon) n^{(3\varepsilon - 1)/2} \qquad \forall\, n \ge 2, \tag{5}$$

where the first sup *is taken over all r.v.'s X of the class $\mathcal{A}_{5/3 - \varepsilon}(\mathbb{X}, h)$.*

The space $\mathbb{X}$ and function h in Theorem 3 are given in a constructive way and one may choose $\mathbb{X} = \mathbb{R}^2$ and $h = h_0$ from Example 2. More details and a description of some minimal and simple subclasses of $\mathcal{A}_{5/3 - \varepsilon}$ for which (5) holds are given below. Let us note here only that it seems that there does not exist a r.v. X realizing a lower bound in Theorem 3.

Van Zwet (1984), Friedrich (1989), Götze (1991) and Bolthausen and Götze (1989) have obtained Berry–Esséen bounds for classes of statistics containing U-statistics as a subclass. It follows from Theorem 3 that moment conditions of these papers are final.

Lower bounds of order $O(n^{-1/2})$ for the rates of convergence have been obtained by Maesono (1988, 1991). In these papers the author solves slightly different problems assuming the existence of $\mathbb{E}h^2(X, X_1)$ and moments of higher order.

Von Mises statistics. For a symmetric function h such that $\mathbb{E}h(X, X_1) = 0$ the corresponding von Mises statistic is defined by

$$V_n = V_n(h) = \sum_{1 \le i, j \le n} h(X_i, X_j).$$

It is clear that

$$V_n = 2n \sum_{1 \le i \le n} g(X_i) + \sum_{1 \le i, j \le n} \psi(X_i, X_j).$$

It may be shown that

$$\sup_{x \in \mathbb{R}} \left| P\{2^{-1} n^{-3/2} V_n < x\} - \Phi(x) \right| = O\left(n^{-1/2}\right) \tag{6}$$

provided that

$$\mathbb{E}|g(X)|^3 < \infty, \qquad \mathbb{E}|\psi(X, X_1)|^{5/3} < \infty, \qquad \mathbb{E}|\psi(X, X)| < \infty. \tag{7}$$

The following two theorems show that the moment conditions given in (7) are final for (6).

For $\alpha > 0$ let $\mathcal{B}_\alpha(\mathbb{X}, h)$ stand for the class of all r.v.'s X assuming values in $\mathbb{X}$ and such that

$$\mathbb{E}g(X) = 0, \qquad \mathbb{E}g^2(X) = 1, \qquad \mathbb{E}|g(X)|^3 = 4,$$
$$\mathbb{E}\psi(x, X) \equiv 0, \qquad \mathbb{E}|\psi(X, X_1)|^\alpha = 1, \qquad \mathbb{E}|\psi(X, X)|^{\alpha/2} \le 1.$$

THEOREM 4. *Let $\varepsilon \le 2/3$. There exist a measurable space $\mathbb{X}$, a symmetric function h and a positive constant $c(\varepsilon) > 0$ depending only on ε such that*

$$\sup \sup_{x \in \mathbb{R}} \left|P\{2^{-1}n^{-3/2}V_n < x\} - \Phi(x)\right| \ge c(\varepsilon) n^{3\varepsilon/4 - 1/2} \qquad \forall\, n \ge 1, \tag{8}$$

where the first sup *is taken over all r.v.'s X of the class $\mathcal{B}_{2-\varepsilon}(\mathbb{X}, h)$.*

It follows from Theorem 4 that the last moment condition in (7) is final for (6). The second condition is final as well according to the following theorem.

THEOREM 5. *Let $\varepsilon \le 1/3$. There exist a measurable space $\mathbb{X}$, a symmetric function h and a positive constant $c(\varepsilon) > 0$ depending only on ε such that*

$$\sup \sup_{x \in \mathbb{R}} \left|P\{2^{-1}n^{-3/2}V_n < x\} - \Phi(x)\right| \ge c(\varepsilon) n^{(3\varepsilon - 1)/2} \qquad \forall\, n \ge 1,$$

where the first sup *is taken over all r.v.'s X of the class $\mathcal{B}_{5/3-\varepsilon}(\mathbb{X}, h)$ such that* $\mathbb{E}|\psi(X, X)| = 0$.

The previous theorems present results showing that bounds of type (6) are unimprovable. However, they do not imply the existence of a (independent of n) r.v. X and a function h realizing the rate in n. For von Mises statistics we are able to construct such X and h.

THEOREM 6. *Let $\varepsilon \le 2/3$. There exist a measurable space $\mathbb{X}$, a symmetric function h, a r.v. X of the class $\bigcup_{0<\delta<1} \mathcal{B}_{2-\varepsilon-\delta}(\mathbb{X}, h)$ and a positive constant $c(\varepsilon) > 0$ depending only on ε such that*

$$\sup_{x \in \mathbb{R}} \left|\mathbf{P}\{2^{-1}n^{-3/2}V_n < x\} - \Phi(x)\right| \ge c(\varepsilon) n^{3\varepsilon/4 - 1/2} \qquad \forall\, n \ge 1.$$

Some remarks and possible extensions.

REMARK 7. In Theorems 3 and 5 it is sufficient to take the first sup over all r.v.'s $X = (\eta, Z) \in \mathcal{C} \cap \mathcal{A}_{5/3-\varepsilon}(\mathbb{R}^2, h_0)$ with Z symmetric and assuming at most three values (we recall that $\mathcal{C}$ and h_0 are described in Example 2). In Theorem 4 it is sufficient to take the first sup over all r.v.'s $X = (\eta, Z) \in \mathcal{C} \cap \mathcal{B}_{2-\varepsilon}(\mathbb{R}^2, h_0)$ with Z symmetric and assuming at most three values.

REMARK 8. It is possible to replace $\mathbb{R}^2$ by $\mathbb{R}$. Of course, in this case the proofs and the structure of h are more complicated. In the higher order case when the main part of a statistic has a Gaussian limit we may show that for the convergence rate $O(n^{-1/2})$ it is necessary to have the condition $\mathbb{E}|g_r|^{(2r+1)/(2r-1)} < \infty$, $1 \le r \le k$, where $g_1, \dots, g_k$ are the well-known functions in the Hoeffding decomposition of a U-statistic (if $k = 2$, then $g_1 = g$, $g_2 = \psi$).

REMARK 9. In Theorem 6 one may choose the r.v. $X = (\eta, Z) \in \mathcal{C}$ and the function $h = h_0$ from Example 2 with Z being symmetric and $(2-\varepsilon)$-stable.

REMARK 10. It would be of considerable interest to improve and extend Theorems 3–6 in the following directions: to get an analogue of Theorem 6 for U-statistics [a natural candidate for r.v. X is $X = (\eta, Z)$ as in Theorem 6 (see Remark 9) with a stable Z] and to investigate the higher order case when the main part of a statistics has non-Gaussian limit as $n \to \infty$.

Auxiliary lemmas and proofs. We shall consider the functions h_0, g_0, ψ_0 and r.v. $X = (\eta, Z) \in \mathbb{R}^2$ of the class $\mathcal{C}$ described in Example 2. Let $Z_1, Z_2, \dots$ be independent copies of the r.v. Z. We may write

$$\tau_n^{-1} U_n(h_0) = \eta + S_n, \qquad S_n = \tau_n^{-1} \sum_{1 \le i < j \le n} Z_i Z_j, \qquad \tau_n^2 = n(n-1)^2, \tag{9}$$

$$\left(2n^{3/2}\right)^{-1} V_n(h_0) = \eta + M_n, \qquad M_n = \left(2n^{3/2}\right)^{-1} \sum_{1 \le i, j \le n} Z_i Z_j \tag{10}$$

because η is standard normal and independent of Z. Furthermore, we have

$$P\{\tau_n^{-1} U_n(h_0) < x\} = \mathbb{E}\Phi(x - S_n), \tag{11}$$

$$P\left\{\left(2n^{3/2}\right)^{-1} V_n(h_0) < x\right\} = \mathbb{E}\Phi(x - M_n). \tag{12}$$

LEMMA 11. *We have*

$$\mathbb{E}\Phi(x - S_n) = \Phi(x) + \frac{1}{4(n-1)} \Phi''(x) \left(\mathbb{E}Z^2\right)^2 + \Phi'''(x) \mathbb{E}S_n^3 + R,$$

where

$$|R| \le c n^{-4} \left(\mathbb{E}Z^4\right)^2 + c n^{-2} \left(\mathbb{E}Z^2\right)^4. \tag{13}$$

PROOF. The Taylor expansion yields

$$\mathbb{E}\Phi(x - S_n) = \Phi(x) - \mathbb{E}\Phi'(x) S_n + \tfrac{1}{2} \mathbb{E}\Phi''(x) S_n^2 - \tfrac{1}{6} \mathbb{E}\Phi'''(x) S_n^3 + R,$$

where $|R| \le c\mathbb{E}S_n^4$. Now the result follows because elementary calculations and symmetry of Z show that

$$\mathbb{E}S_n = 0, \qquad \mathbb{E}S_n^2 = \frac{1}{2(n-1)} \left(\mathbb{E}Z^2\right)^2$$

and that $\mathbb{E}S_n^4$ does not exceed the right-hand side of (13). For example,

$$\mathbb{E}S_n^2 = \tau_n^{-1} \sum_{1\le i<j\le n} Z_iZ_jS_n = \frac{1}{2}\tau_n^{-1}n(n-1)\mathbb{E}Z_1Z_2S_n = \frac{1}{2(n-1)}\left(\mathbb{E}Z^2\right)^2$$

because $Z, Z_1, Z_2, \dots$ are i.i.d. □

LEMMA 12. *Let $\alpha \ge 4/3$. Then there exists a constant $c(\alpha) > 0$ depending only on α such that*

$$\Delta_n = \sup \sup_{x\in\mathbb{R}} |\mathbb{E}\Phi(x - S_n) - \Phi(x)| \ge c(\alpha)n^{(4-3\alpha)/2} \qquad \forall\, n \ge 2,$$

where the first sup *is taken over all symmetric r.v.'s Z assuming at most three values and such that $\mathbb{E}|Z|^\alpha = 1$.*

PROOF. It is clear that $\Delta_n > 0$, for all $n \ge 2$. Therefore, it is sufficient to prove the estimate of the lemma for $n \ge C(\alpha)$ where $C(\alpha)$ is some (sufficiently large) positive constant to be specified later. Let Z be a symmetric r.v. such that

$$P\{Z = 0\} = 1 - 2p, \qquad P\{|Z| = a\} = 2p \tag{14}$$

for some $0 \le p \le 1$ and $a \ge 0$. Clearly,

$$\mathbb{E}|Z|^\alpha = 1 \iff 2pa^\alpha = 1. \tag{15}$$

Applying Lemma 11 with r.v. Z defined in (14) and using (15), we get

$$\Delta_n \ge c_1 n^{-1}a^{4-2\alpha}|\Phi''(1)| - |R|,$$

where

$$|R| \le c_2 n^{-1}a^{4-2\alpha}\{n^{-3}a^4 + n^{-1}a^{4-2\alpha}\}$$

and where c_1, c_2 are some positive absolute constants. Let us choose $a = \gamma n^{3/4}$ with some $0 < \gamma \le 1$ to be specified later. Then

$$\Delta_n \ge n^{(4-3\alpha)/2}\gamma^{4-2\alpha}\left(c_3 - c_2\gamma^4 - c_2\gamma^{4-2\alpha}n^{2-3\alpha/2}\right)$$

and the result follows if we choose $\gamma > 0$ sufficiently small [this is possible only for $n \ge c_4(\alpha)$ because of $p \le 1$]. The term $c_2\gamma^{4-2\alpha}n^{2-3\alpha/2}$ is small for $n \ge C(\alpha)$ provided $C(\alpha)$ is sufficiently large. □

LEMMA 13. *We have*

$$\mathbb{E}\Phi(x - M_n) = \Phi(x) - \frac{1}{2\sqrt{n}}\Phi'(x)\mathbb{E}Z^2 + R,$$

where

$$|R| \le cn^{-2}\mathbb{E}Z^4 + cn^{-1}\left(\mathbb{E}Z^2\right)^2.$$

PROOF. The proof is similar to that of Lemma 11. □

LEMMA 14. *Let $\alpha \geq 4/3$. Then there exists a constant $c(\alpha) > 0$ depending only on α such that*

$$\sup\left[\Phi(0) - \mathbb{E}\Phi(-M_n)\right] \geq c(\alpha)n^{1-3\alpha/4} \qquad \forall\, n \geq 1,$$

where the sup *is taken over all symmetric r.v.'s Z assuming at most three values and such that $\mathbb{E}|Z|^\alpha = 1$.*

PROOF. The proof is similar to that of Lemma 12. □

LEMMA 15. *Let Z be a stable symmetric r.v. having the characteristic function $\mathbb{E}\exp\{itZ\} = \exp\{-|t|^\alpha\}$ with an exponent $4/3 \leq \alpha < 2$. Then there exists a constant $c(\alpha) > 0$ depending only on α such that*

$$\Phi(0) - \mathbb{E}\Phi(-M_n) \geq c(\alpha)n^{1-3\alpha/4} \qquad \forall\, n \geq 1.$$

PROOF. Let us note that M_n has the distribution of $n^{-3/2+2/\alpha}Z^2 \geq 0$. Therefore,

$$\begin{aligned}\Phi(0) - \mathbb{E}\Phi(-M_n) &\geq \mathbb{E}I\{M_n \geq 1\}\big(\Phi(0) - \Phi(-M_n)\big)\\ &\geq \big(\Phi(0) - \Phi(-1)\big)P\big\{|Z| \geq n^{3/4-1/\alpha}\big\} \geq c(\alpha)n^{1-3\alpha/4}\end{aligned}$$

because $P\{|Z| \geq t\} \geq c(\alpha)t^{-\alpha}$ for $t \geq 1$. □

PROOF OF THEOREM 3. Combine Lemma 12 and representations (9) and (11). □

PROOF OF THEOREM 4. Combine Lemma 14 and representations (10) and (12). □

PROOF OF THEOREM 5. Let h be the function from Theorem 3. Define $h_1(x,y) = h(x,y)$ if $x \neq y$ and $h_1(x,x) = 0$. Then Theorem 3 with $h = h_1$ implies Theorem 5 if we note that $\mathbb{E}h_1(x,X) = \mathbb{E}h(x,X)$ for X and $h = h_0$ from our construction. □

PROOF OF THEOREM 6. Combine Lemma 15 and representations (10) and (12). □

Acknowledgment. The authors would like to thank R. Helmers and B. A. Zalesskii for suggestions and helpful discussions, and the referee for comments and suggestions that improved the clarity of the paper.

REFERENCES

BHATTACHARYA, R. N. and DENKER, M. (1990). *Asymptotic Statistics.* Birkhäuser, Boston.
BICKEL, P. J. (1974). Edgeworth expansions in nonparametric statistics. *Ann. Statist.* **2** 1–20.
BICKEL, P. J., GÖTZE, F. and VAN ZWET, W. R. (1986). The Edgeworth expansion for U-statistics of degree two. *Ann. Statist.* **14** 1463–1484.

BOLTHAUSEN and GÖTZE, F. (1989). The rate of convergence for multivariate sampling statistics. Preprint 89-007 of SFB 343, Diskrete Strukturen in der mathematik, Univ. Bielefeld.

CALLERT, H. and JANSSEN, P. (1980). The Berry–Esséen theorem for U-statistics. *Ann. Statist.* **8** 417–421.

CALLERT, H., JANSSEN, P. and VERAVERBEKE, N. (1980). An Edgeworth expansion for U-statistics. *Ann. Statist.* **8** 299–312.

CHAN, Y. K. and WIERMAN, J. (1977). On the Berry–Esséen theorem for U-statistics. *Ann. Probab.* **5** 136–139.

FRIEDRICH, K. O. (1989). A Berry–Esséen bound for functions of independent random variables. *Ann. Statist.* **17** 170–183.

GÖTZE, F. (1984). Expansions for von Mises functionals. *Z. Wahrsch. Verw. Gebiete* **65** 599–625.

GÖTZE, F. (1987). Approximations for multivariate U-statistics. *J. Multivariate Anal.* **22** 212–229.

GÖTZE, F. (1991). On the rate of convergence in the multivariate CLT. *Ann. Probab.* **19** 724–739.

GÖTZE, F. and VAN ZWET, W. (1991). Edgeworth expansions for asymptotically linear statistics. Preprint 91-034 of SFB 343, Diskrete Strukturen in der mathematik, Univ. Bielefeld.

GRAMS, W. F. and SERFLING, R. J. (1973). Convergence rates for U-statistics and related statistics. *Ann. Statist.* **1** 153–160.

HELMERS, R. (1985). The Berry–Esséen bound for studentized U-statistics. *Canad. J. Statist.* **13** 79–82.

HELMERS, R. and VAN ZWET, W. R. (1982). The Berry–Esséen bound for U-statistics. In *Statistical Decision Theory and Related Topics III* (S. S. Gupta and J. O. Berger, eds.) **1** 497–512. Academic Press, New York.

HOEFFDING, W. (1948). A class of statistics with asymptotically normal distribution. *Ann. Math. Statist.* **19** 293–325.

KOROLIUK, V. S. and BOROVSKIH, YU. V. (1989). *Theory of U-Statistics.* Naukova Dumka, Kiev (in Russian).

LIN, Z. Y. (1983). On generalizations of the Berry–Esséen inequality for U-statistics. *Acta Math. Appl. Sinica* **6** 468–475.

MAESONO, Y. (1988). A lower bound for the normal approximation of U-statistic. *Metrika* **35** 255–274.

MAESONO, Y. (1991). On the normal approximation of U-statistics of degree two. *J. Statist. Plann. Inference* **27** 37–50.

SERFLING, R. J. (1980). *Approximation Theorems of Mathematical Statistics.* Wiley, New York.

VAN ZWET, W. R. (1977). Asymptotic expansions for the distribution functions of linear combinations of order statistics. In *Statistical Decision Theory and Related Topics II* (S. S. Gupta and D. S. Moore, eds.) 421–437. Academic Press, New York.

VAN ZWET, W. R. (1984). A Berry–Esséen bound for symmetric statistics. *Z. Wahrsch. Verw. Gebiete* **66** 425–440.

VON MISES, R. (1947). On the asymptotic distribution of differentiable statistical functions. *Ann. Math. Statist.* **18** 309–348.

ZHAO, L. C. (1983). The rate of the normal approximation for a studentized U-statistic. *Sci. Exploration* **3** 45–52.

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