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STOCHASTIC INTEGRATION OF PROCESSES WITH FINITE GENERALIZED VARIATIONS. I

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In this paper the L^1-stochastic integral and the mixed stochastic integral of a process Y with respect to a process X is defined in a way that extends Riemann–Stieltjes integration of deterministic functions with respect to X. The L^1-integral will include the classical Itô integral. However, the concepts of "filtration" and adaptability do not play any role; instead, the p-variation of Dolean functions of the processes X and Y is the determining factor.

1. Introduction. This is the first of two papers on stochastic integration. In this paper, we consider the integrals of processes with parameters in [0,1]. In the second paper, we consider stochastic integrals with parameters in several variables. Here we introduce the concept of the L^1-stochastic integral of a process Y with respect to a process X. Let $\Omega = (\Omega, \mathscr{A}, P)$ be a probability space and let

$$X = \{X(t,\omega)\colon \ \omega \in \Omega, \ t \in T\}$$

be a stochastic process. The problem of stochastic integration is this: Given processes X and Y, how can we define the random variable

$$Z(\omega) = \int_T Y(t,\omega)\, dX(t,\omega)?$$

The difficulty is that many processes, for example, the Brownian motion process [Billingsley (1986)], have paths $t \to X(t,\omega)$ with unbounded variation, in which case $dX(t,\omega)$ (for fixed $\omega \in \Omega$) does not define a measure. To overcome this difficulty, traditionally one follows the approach which we now describe. We will assume throughout the rest of this section that $T = [0,1]$. Start with a simple function on $T \times \Omega$ (simple process),

$$(1) \qquad Y(t,\omega) = \sum_i a_i I_{A_i \times [s_i,t_i]}(t,\omega), \qquad 0 \le s_i \le t_i \le 1, \ A_i \in \mathscr{F},$$

where I_E is the indicator function of the set E. The stochastic integral of Y with respect to X is then defined by

$$(2) \qquad \int_0^1 Y\, dX \equiv \sum_i a_i I_{A_i}(X_{t_i} - X_{s_i}),$$

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The stochastic integral $\int_0^1 Y\,dX$ is a linear mapping. The problem is to extend the mapping $Y \to \int_0^1 Y\,dX$ to a class of processes larger than the simple processes. Traditionally an extension is obtained in case that X is a right-continuous L^2-bounded local martingale or it is a *semimartingale* (=: martingale + a finite variation process) while the simple processes are the so-called predictable processes. This method allows the integration of processes which are adapted to X. For a detailed treatment of this approach, we refer the reader to Itô (1961), Metivier and Pellaumail (1980) and Chung and Williams (1986).

For integrals involving predictable processes and semimartingales, assume the existence of a "filtration," that is, an increasing net of σ-fields. In a general setting "filtration" does not occur naturally, for example, processes indexed by spatial parameters. In a series of papers, Blei (1985), (1988a), (1989), used multi-linear integration theory based on Grothendieck's inequality and factorization theorem [Grothendieck (1956)] to construct general stochastic integrals of deterministic functions with respect to stochastic processes. However, filtration and the notion of adaptability do not have preassigned roles in Blei's work. His approach is based on a natural identification of a stochastic process with a finitely additive set function defined on the class of measurable rectangles in $\Omega \times [0,1]$. More precisely, given a stochastic process X on $[0,1] \times \Omega$, define the scalar-valued function

$$\lambda_X(A \times (s,t)) = \int_A [X(t,\omega) - X(s,\omega)]P(d\omega). \tag{3}$$

Here $0 \le s \le t \le 1$ and $A \in \mathscr{A}$. We call λ_X the *Dolean function* of X. Note that usually the Dolean function is the function defined on the so-called predictable rectangles of $\Omega \times [0,1]$. In his work, Blei exploited the fact that the set function λ_X associated with X often gives rise to a *bimeasure*. A bimeasure is a set function which is a finite complex measure in each coordinate (see Section 2). Blei (1985, 1988a) developed an integration theory with respect to bimeasures using Grothendieck's fundamental inequality and the machinery of vector-valued measures. Bimeasures in general need not have finite variations, that is, they may not be extendible to a measure. For example, the bimeasure associated with Brownian motion fails to have finite variation. The fact that bimeasures need not have finite total variation tells us that the theory of bimeasures is distinct from the theory of measures. In his work Blei considered the higher variations of bimeasures. By Littlewood's inequality, he obtains that if M is a bimeasure, then the total *p-variation* (see Section 2) of M is finite whenever $p \ge 4/3$.

We develop a general measure-theoretic approach to stochastic integration. We will develop an integration theory with respect to processes whose Dolean functions give rise to bimeasures. The p-variation of Dolean functions will play the role of total variation in some sense. The idea of integrating with respect to functions of bounded p-variation was originated by Young (1936). In his paper, Young gave sufficient conditions for the existence of Riemann–Stieltjes integrals $\int_0^1 f\,dg$, where both f and g are functions of finite higher variations.

Our work is a "stochastic" analogue of Young's result. That is, we will give sufficient conditions for the existence of the stochastic integral $\int_0^1 Y\,dX$, where X and Y are processes whose Dolean function has finite generalized variation. The techniques we use in estimating various sums are two-dimensional versions of Young's techniques.

We define the stochastic integral of a process Y with respect to a process X in a way that extends Riemann–Stieltjes integration of deterministic functions with respect to X. More precisely, let $\omega \in \Omega$ and $\tau =: \{t_j\}_{j=0}^n$ be a partition of [0,1]. Let

$$L(\tau, \omega, X, Y) = L(\tau, \omega) = \sum_{j=1}^{n} Y(t_{j-1}, \omega)\Delta_j X(\omega).$$

We say *the L_1-integral of Y with respect to X* exists, if there exists a random variable $J_{(X,Y)} \in L_1(\Omega, P)$ such that

$$\lim_{\|\tau\|\to 0} \|L(\tau,\cdot) - Y(0,\cdot)(X(1,\cdot) - X(0,\cdot)) - J_{(X,Y)}(\cdot)\|_1 = 0. \tag{4}$$

Here $\|\cdot\|_1$ refers to the usual L^1-norm. If (4) holds we define the the L_1-integral of Y with respect to X to be

$$J_{(X,Y)}(\omega) + Y(0,\omega)(X(1,\omega) - X(0,\omega))$$

and denote it by

$$(L_1)\int_{[0,1]} Y\,dX.$$

We will show that the following theorem is valid.

THEOREM 1.1. *If X and Y are two processes such that λ_X and λ_Y can be extended to bimeasures, and X and Y are independent on disjoint intervals, then $(L_1)\int_{[0,1]} Y\,dX$ exists. [Two processes X and Y are said to be independent on disjoint intervals if for each pair of disjoint intervals (s_1,t_1) and (s_2,t_2), the random variables $X(t_1,\cdot) - X(s_1,\cdot)$ and $Y(s_2,\cdot) - Y(s_2,\cdot)$ are independent of each other.]*

We will in fact prove a stronger result (Theorem 3.2) which will imply Theorem 1.1. We remark that if $X = Y$ is the Brownian motion process, then λ_X can be extended to a bimeasure [Blei (1989)].

The usual martingale methods deal with semimartingales as an integrator and an adapted process as an integrand. Now if X and Y are mutually independent and hence independent on disjoint intervals, then certainly Y is not adapted to the filtration of X. This of course is the extreme case. At any rate, the main point of the approach of this paper is that a martingale structure and the subsequent "adapted integrand" approach is not required to be able to integrate stochastically. Thus if X is a process whose Dolean function is extendible to a bimeasure and if Y is an independent copy of X, then X can integrate Y even if X is not a martingale or a semimartingale. This approach also works in higher dimensions where there is no natural filtration. An example of a process which acts as a stochastic integrator for a class of processes is the *Rademacher process*.

Rademacher random process. For each $n = 0, 1, 2, \ldots$, let e_n denote the function $x \to \sqrt{2}\cos(2\pi n x)$ on the interval $T = [-1, 1]$. Let $\mathscr{Y}$ be the closed linear span in $L^2(T, dx)$ of $\{e_n\colon n = 0, 1, 2, \ldots\}$. Let $\Omega = [0, 1]$ and let P be the Lebesgue measure on the Borel σ-field of [0,1]. For $n = 0, 1, 2, \ldots$, let r_n denote the nth Rademacher function on Ω, that is,

$$r_n(\omega) = 1 - 2\varepsilon_n,$$

where $\omega = \sum_{k=1}^{\infty} \varepsilon_k/(2^k)$ is the binary expansion of $\omega \in [0, 1]$. Let $\mathscr{X}$ be a closed linear span in $L^2([0, 1], dx)$ of $\{r_n\colon n = 0, 1, 2, \ldots\}$. Thus $\mathscr{X}$ and $\mathscr{Y}$ are unitarily equivalent Hilbert spaces. Let $U\colon \mathscr{Y} \to \mathscr{X}$ be a unitary operator which realizes this equivalence and so that $U(e_n) = r_n$. For each $t \in [0, 1]$, let

$$X_t = U(I_{[-t,t]})$$

be the image under U of the indicator function of the interval $[-t, t]$. Then $X = \{X_t\}$ is a process on $[0, 1] \times \Omega$ which we call the Rademacher random process. By observing that the orthogonal expansion of $I_{[0,t]}$ with respect to $\{e_n\}$ is given by

$$I_{[0,t]} \sim \sum_{n=1}^{\infty} \frac{\sqrt{2}\sin(2\pi n t)}{\pi n} e_n,$$

we see that the orthogonal expansion of X_t with respect to $\{r_n\}$ is given by

$$X_t \sim \sum_{n=1}^{\infty} \frac{\sqrt{2}\sin(2\pi n t)}{\pi n} r_n.$$

For each t, the above expansion converges almost surely, as well as in L^2-norm. Furthermore, for almost all ω (P), the expansion represents a continuous function of t [Kahane (1968)]. As a process, the Rademacher process is an instance of α-chaos [Blei (1988b)]. It can also be viewed as a random Fourier series [Kahane (1968)]. The Rademacher process is not independent on disjoint intervals with itself. As far as we can tell it is not a martingale or a semimartingale. However, the Dolean function of the Rademacher process can be extended to a bimeasure on measurable rectangles of $[0, 1] \times [0, 1]$ [Blei (1988b)]. Therefore, the Rademacher process acts as an integrator for the processes whose Dolean functions are extendible to bimeasures and are independent of the Rademacher process. More generally, the Rademacher process acts as an integrator for the processes whose Dolean functions are extendible to bimeasures and are independent on disjoint intervals with respect to the Rademacher process.

Organization of the paper. In Section 2 we introduce the Fréchet pseudomeasures and define the Fréchet variation and p-variation of these objects. We will state a generalization of an important result of Littlewood known as Littlewood's 4/3 inequality. We will give examples of processes whose Dolean functions give rise to bimeasures. Such processes are said to have finite expectations. The Brownian motion process and, more generally, any L^2-bounded process with orthogonal increments are examples of processes with finite expectations. In Section 3 we define the L^1-integral of Y with respect X. This integral (if it exists) is an L^1-limit of Riemann–Stieltjes type sums. We will prove that if X and Y are processes with finite expectations and are independent on disjoint intervals (see Definition 3.1), then the L^1-integral of Y with respect to X exists and belongs to $L^1(\Omega)$. In Section 4 we will introduce the mixed stochastic integral $[=: (M)\int_T Y\,dX]$ of a process Y with respect to a process X. This integral (if it exists) is the $L^1(\Omega \times \Omega, P \otimes P)$-limit of Riemann–Stieltjes type sums. In other words, $(M)\int_T Y\,dX$ will be a random variable in $L^1(\Omega \times \Omega, P \otimes P)$. We will show that if X and Y are processes with finite expectations, then $(M)\int_T Y\,dX$ exists and belongs to $L^1(\Omega \times \Omega, P \otimes P)$.

2. Fréchet pseudomeasures. Let $(E_1, \mathscr{B}_1), \ldots, (E_n, \mathscr{B}_n)$ be measurable spaces. A *measurable rectangle* in the n-fold Cartesian product $\prod_{j=1}^n E_j$ will be a set of the form $A_1 \times \cdots \times A_n$, $A_1 \in \mathscr{B}_1, \ldots, A_n \in \mathscr{B}_n$. As usual, $\prod_{j=1}^n \mathscr{B}_j$ will denote the *product σ-algebra* generated by the measurable rectangles and $(\prod_{j=1}^n E_j, \prod_{j=1}^n \mathscr{B}_j)$ will designate the corresponding measurable product space. Also, $\mathscr{A}(\mathscr{R}_n)$ will denote the algebra generated by the measurable rectangles of $\prod_{j=1}^n E_j$. A *partition* of a measurable set A will mean here a countable collection of mutually disjoint measurable sets whose union is A.

A scalar-valued function μ defined on the measurable rectangles in $\prod_{j=1}^n E_j$ is an *n-dimensional Fréchet pseudomeasure*, or an *$\mathscr{F}_n$-pseudomeasure*, if for every $A_1 \in \mathscr{B}_1, \ldots, A_n \in \mathscr{B}_n$ and each $j \in \{1, \ldots, n\}$ the set function

$$(\mu_j)(F) = \mu(A_1 \times \cdots \times A_{j-1} \times F \times A_{j+1} \times \cdots \times A_n) \qquad F \in \mathscr{B}_j,$$

is a complex measure on $(E_j, \mathscr{B}_j)$. The space of n-dimensional pseudomeasures on $\prod_{j=1}^n E_j$ is denoted by $\mathscr{F}_n(\prod_{j=1}^n E_j)$. Pseudomeasures will also be called Fréchet measures. When $n = 2$, pseudomeasures are referred to as bimeasures. We refer the reader to Blei (1985), Fréchet (1915) and Morse and Transue (1949), (1950) for a detailed review of bimeasures.

We define a norm on $\mathscr{F}_n(\prod_{j=1}^n E_j)$, which we denote by $\|\cdot\|_{\mathscr{F}_n}$, and refer to it as the Frechet variation norm. For $\mu \in \mathscr{F}_n(\prod_{j=1}^n E_j)$, the *$\mathscr{F}_n$-variation of μ* over the measurable rectangle $A_1 \times \cdots \times A_n$ is given by

$$|\mu|_{\mathscr{F}_n}(A_1 \times \cdots \times A_n) = \sup\left\{\left\| \sum_{i_1=1}^M \cdots \sum_{i_n=1}^M \mu(F_1(i_1) \times \cdots \times F_n(i_n)) r_{i_1} \otimes \cdots \otimes r_{i_n} \right\|_\infty\right\},$$

where the sup is taken over

$$\{(F_j(i))_{i \in \mathbf{N}} \text{ partition of } E_j, \text{ for } j = 1, \ldots, n, \text{ and } M \text{ a positve integer}\}.$$

Here $\{r_i\}$ is the usual *Rademacher system* realized as a sequence of functions on $[0,1]$. The following result [Blei (1985)] shows that the the Fréchet norm of a pseudomeasure is always finite.

THEOREM 2.1. *If* $\mu \in \mathscr{F}_n(\prod_{j=1}^n E_j)$, *then* $\|\mu\|_{\mathscr{F}_n} < \infty$.

We will now define another variation related to μ, called the φ-variation of μ. Given an *Orlicz function* φ [i.e., φ is a convex, nondecreasing continuous function on $[0,\infty)$, $\varphi(0) = 0$ and $\lim_{t\to\infty}\varphi(t) = \infty$], we define the *Orlicz norm with respect to* φ (in short, φ*-norm*) of a countable set of scalars $\{c_\ell\}_{\ell\in\Lambda}$, to be

$$\|\{c_\ell\}_{\ell\in\Lambda}\|_\varphi = \inf\left\{\rho > 0\colon \sum_{\ell\in\Lambda}\varphi\left(\frac{|c_\ell|}{\rho}\right) \le 1\right\}.$$

A *grid* τ will be a Cartesian product of n measurable partitions of $E_1,\ldots,E_n$. We define the φ*-variation of* μ over $\prod_{j=1}^n E_j$ by

$$|\mu|_\varphi = \sup_\tau\{\|\{\mu(c)\}_{c\in A_\tau}\|_\varphi\}.$$

When $\varphi(t) = t^p$, we write $|\mu|_p$ instead of $|\mu|_\varphi$.

We now state the multilinear measure-theoretic version of Littlewood's results. Let $(E_1,\mathscr{B}_1),\ldots,(E_n,\mathscr{B}_n)$ be measurable spaces.

2.2. MULTILINEAR LITTLEWOOD INEQUALITY [Blei (1985)]. *Let* $\mu \in \mathscr{F}_n(\prod_{j=1}^n E_j)$ *and* $\varphi(t) = t^p$. *Then*

$$|\mu|_\varphi\left(\prod_{j=1}^n E_j\right) \le \lambda_n\|\mu\|_{\mathscr{F}_n} \quad \text{whenever } p \ge \frac{2n}{n+1}.$$

Here $\lambda_n > 0$ *is a constant depending only on n. Furthermore there exists* $\mu \in \mathscr{F}_n(\prod_{j=1}^n E_j)$ *such that* $|\mu|_p = \infty$ *for* $p < 2n/(n+1)$.

Let $\mu \in \mathscr{F}_n(\prod_{j=1}^n E_j)$. We define the *Littlewood exponent of* μ as

$$\ell_\mu = \inf\left\{p\colon |\mu|_p\left(\prod_{j=1}^n E_j\right) < \infty\right\}.$$

We now consider the bimeasures which arise in connection with stochastic processes. Recall for a given process X that λ_X is the Dolean function of X defined by (3).

DEFINITION 2.3. We say that a process X has *finite expectation* if λ_X is uniquely extendible to an element of $\mathscr{F}_2 = \mathscr{F}_2(\Omega \times [0,1])$.

In other words X has finite expectations if λ_X is extendible to a bimeasure. By Theorem 2.1, a stochastic process has finite expectations if and only if

$$\|X\|_{\mathscr{F}_2} = \sup\left\{\left\|\int_\Omega \sum_{k=1}^n r_k(X(t_k,\omega) - X(t_{k-1},\omega))P(d\omega)\right\|_\infty\right\} < \infty, \tag{5}$$

where the sup is taken over finite partitions $\{0 \le t_0 \le t_1 \le \cdots \le t_n = 1\}$ of the interval [0,1].

In Blei (1989) it is shown that processes with finite expectation can be identified as elements of a closed subpace of $\mathscr{F}_2$ denoted by $\mathscr{F}_2(P \times [0,1])$ and consisting of those μ in $\mathscr{F}_2$ for which

$$\mu(\cdot, B) \ll P \text{ for all Borel subsets } B \text{ of } [0,1].$$

Archetypical examples of stochastic processes with finite expectations are [Blei (1989)]:

1. L^2-bounded processes with orthogonal increments; in particular, L^2-bounded martingales.
2. Additive symmetric L^1-bounded processes.

Given a process X, we now consider the higher variations of its Dolean function λ_X. Let $\mu \in \mathscr{F}_2(P \times [0,1])$, let τ: $\{0 = t_0 < t_1 < \cdots < t_n = 1\}$ be a partition of [0,1] and let $\mathscr{A} = \{A_i\}_{i=1}^k$ be a partition of Ω. For p_1, $p_2 > 0$ let

$$L(p_1, p_2, \tau, \mathscr{A}) = \left[\sum_{j=1}^{k}\left(\sum_{i=1}^{n}|\mu(A_i \times (t_{j-1}, t_j))|^{p_1}\right)^{p_2/p_1}\right]^{1/p_2}. \tag{6}$$

We define the mixed (p_1, p_2)-variation of μ to be

$$|\mu|_{(p_1,p_2)} = \sup\{|L(p_1, p_2, \tau, \mathscr{A})|: \ \tau, \mathscr{A}\}. \tag{7}$$

Note $|\mu|_{(p,p)} = |\mu|_p$ as defined previously. Now Littlewood's 4/3 inequality [Littlewood (1930)] states that for $\lambda_X \in \mathscr{F}_2(P \times [0,1])$,

$$|\mu|_{4/3} + |\mu|_{(2,1)} + |\mu|_{(1,2)} \le c\|\mu\|_{\mathscr{F}_2}, \tag{8}$$

where c is a fixed universal constant. Recall that the Littlewood exponent of μ is

$$\ell_\mu = \inf\{p: \ |\mu|_p < \infty\}. \tag{9}$$

If $\mu \in \mathscr{F}_2(P \times [0,1])$, we will call its Littlewood exponent the *inner Littlewood exponent of* μ. We define the *outer Littlewood exponent of* μ to be

$$\ell_\mu^{(2)} = \inf\{p: \ |\mu|_{(1,p)} < \infty\}.$$

Littlewood's result tells us

$$\ell_\mu \le \frac{4}{3} \quad \text{and} \quad \ell_\mu^{(2)} \le 2. \tag{10}$$

In Blei (1988b), it is shown that for all α-chaos processes X, which include the Wiener process, $\ell_X = 1$. In other words, for all α-chaos processes X, $|\lambda_X|_p < \infty$, whenever $p > 1$. However, $|\lambda_X|_1 = \infty$ for any α-chaos processes X. In fact, Blei and Kahane (1988) have obtained sharper resolution of the Littlewood exponent for such processes. They have shown, for example, that if X is the Wiener process, then $|\lambda_X|_{\theta_{(\gamma)}} < \infty$, for all $\gamma > 1$, where

$$\theta_{(\gamma)}(x) = |x|\left(\ln^+\left(\frac{1}{|x|}\right)\right)^{-\gamma/2}.$$

On the other hand, the outer Littlewood exponent of a square-integrable martingale is at most 2. To see this, let $\{C_i\}_{i=1}^n$ be a partition of Ω and let $\tau =: \{t_j\}_{j+1}^m$ be a partition of [0,1]. Then for a square-integrable martingale M,

$$\begin{aligned}\sum_{j=1}^m\left(\sum_{i=1}^n \Big| \int_{C_i} \Delta_j M(\omega)P(d\omega)\right)^2 &\le \sum_{j=1}^m\left[\int_\Omega |\Delta_j M(\omega)|P(d\omega)\right]^2 \\ &\le \sum_{j=1}^m \int_\Omega |\Delta_j M(\omega)|^2 P(d\omega) \\ &\le ||M(1,\cdot) - M(0,\cdot)||_2^2.\end{aligned}$$

This shows that the outer Littlewood exponent of a square-integrable martingale is at most 2. If X is a $\Lambda(q)$ process for some $q > 0$, then the Littlewood exponent of X is at most $(q+2)/(q+1)$ [Blei (1990)]. We refer the reader to Blei (1990) for the definition and examples of $\Lambda(q)$ processes.

3. Stochastic integration. In this section we take up the problem of the stochastic integral of processes whose Dolean function has finite generalized variation. We need to introduce the concept of L^p-continuity of stochastic processes.

A process X is right/left L^p-continuous at $s \in [0,1]$ if the function $t \to X_t$ is right/left continuous at $s \in [0,1]$ in $L^p(\Omega)$-norm. For instance, X is right p-continuous at $s \in [0,1]$ if for each ε there exists a δ such that

$$\int_\Omega |X(s,\omega) - X(t,\omega)|^p P(d\omega) < \varepsilon \quad \text{whenever } 0 < t - s < \delta.$$

Let

$$C_p^+ = \{X\colon\ X \text{ is right } L^p\text{-continuous for each } s \in [0,1]\},$$
$$C_p^- = \{X\colon\ X \text{ is left } L^p\text{-continuous for each } s \in [0,1]\}$$

and $C_p = C_p^+ \cap C_p^-$. We note that if X is an L_p-bounded process and it is right continuous, left continuous or continuous, then X belongs to C_p^+, C_p^- or C_p, respectively. To see this, let $\{t_n\}$ be a sequence in [0,1] converging to s. Continuity of X implies $X_{t_n} \to X_s$ almost surely. If X is uniformly bounded in L_p-norm, then a standard result in measure theory implies

$$\lim_{n\to\infty} \int_\Omega |X(s,\omega) - X(t_n,\omega)|^p P(d\omega) = 0.$$

This shows $X \in C_p$.

We now define the stochastic integral of a process Y with respect to a process X in a way that extends Riemann–Stieltjes integration of deterministic functions with respect to X. Let $\omega \in \Omega$ and $\tau =: \{t_j\}_{j=0}^{n}$ be a partition of [0,1]. Let

$$L(\tau, \omega, X, Y) = L(\tau, \omega) = \sum_{j=1}^{n} Y(t_{j-1}, \omega)\Delta_j X(\omega).$$

We say *the L_1-integral of Y with respect to X* exists if there exists a random variable $J_{(X,Y)} \in L_1(\Omega, P)$ such that

$$\lim_{||\tau||\to 0} ||L(\tau, \cdot) - Y(0, \cdot)(X(1, \cdot) - X(0, \cdot)) - J_{(X,Y)}(\cdot)||_1 = 0. \tag{11}$$

If (11) holds we define the the L_1-integral of Y with respect to X to be

$$J_{(X,Y)}(\omega) + Y(0, \omega)(X(1, \omega) + X(0, \omega))$$

and denote it by

$$(L_1) \int_{[0,1]} Y\, dX.$$

First we note that if X is right 1-continuous and Y is a step process, that is,

$$Y(t, \omega) = \sum_{i=1}^{n} a_i I_{(t_{i-1}, t_i] \times F_i}(t, \omega),$$

where $0 = t_0 < t_1 < \cdots < t_n = 1$ and F_i's are measurable subsets of Ω, then

$$(L_1) \int_{[0,1]} Y\, dX(\omega) = \sum_{i=1}^{n} a_i I_{F_i} \Delta_i X(\omega).$$

To see this let $\tau =: \{s_j\}_{j=0}^{k}$ be a partition of [0,1]. We may assume $Y(0, \omega) = 0$ and $||\tau|| < \min_i\{|t_i - t_{i-1}|\}$. For each $i < n$ let

$$m_i = \min\{j\colon\ s_j > t_i\}.$$

Then

$$\begin{aligned} L(\tau, \omega) - J_{(X,Y)}(\omega) &= \sum_{i=1}^{n} a_i I_{F_i}(\omega)(X(s_{m_i}, \omega) - X(t_i, \omega)) \\ &\quad + \sum_{i=1}^{n} a_i I_{F_i}(\omega)(X(s_{m_{i-1}}, \omega) - X(t_{i-1}, \omega)). \end{aligned}$$

Right 1-continuity of X implies

$$\lim_{||\tau||\to 0} \int_{\Omega} |X(s_{m_i}, \omega) - X(t_i, \omega)| P(d\omega) = 0$$

and, similarly

$$\lim_{||\tau||\to 0} \int_{\Omega} |X(s_{m_{i-1}}, \omega) - X(t_{i-1}, \omega)| P(d\omega) = 0.$$

This shows

$$J_{(X,Y)}(\omega)=\sum_{i=1}^{n}a_iI_{F_i}\Delta_iX(\omega).$$

Since convergence in L^2-norm implies convergence in L^1-norm, the L_1-integral effectively generalizes the classical Itô integral. That is, if X is an L^2-bounded martingale and Y is a right-continuous process adapted to X, then the L_1-integral of Y with respect to X exists and is the same as the Itô integral of Y with respect to X. We will show that the L_1-integral of Y with respect to X exists under much more general conditions.

DEFINITION 3.1. Two processes X and Y are *independent on disjoint intervals* if for any pair of disjoint intervals (s_1,t_1) and (s_2,t_2), the random variables $X(\cdot,t_1)-X(\cdot,s_1)$ and $Y(\cdot,t_2)-Y(\cdot,s_2)$ are independent.

We now state the main result of this section.

THEOREM 3.2. *Let X and Y be stochastic processes which are independent on disjoint intervals. Suppose either X is L^1-continuous or Y is L^1-continuous. Let q be the outer Littlewood exponent of λ_Y and let p be the inner Littlewood exponent of λ_X. If $(1/p)+(1/q)>1$, then the (L_1)-integral of Y with respect to X exists and*

$$||(L_1)\int_{[0,1]}Y\,dX||_1\le C(p,q)|\lambda_X|_{p_1,p_1}|\lambda_Y|_{(q_1,1)}. \tag{12}$$

Here $C(p,q)>0$ is a constant and p_1 and q_1 are any two numbers such that $1/p_1+1/q_1>1$, $p_1>p$ and $q_1>q$.

Before proving the theorem we note that if X and Y are processes with finite expectations, then the conditions imposed on their Dolean functions λ_X and λ_Y are satisfied. The outer Littlewood exponent of a bimeasure is no more than 2 and the inner Littlewood exponent of a bimeasure is at most 4/3, and of course $3/4+1/2>1$. Thus Theorem 1.1 is proved once we prove Theorem 3.2. To prove Theorem 3.2 we need the following lemmas.

Recall that for random variables $X_1,X_2,\dots,X_N$, $\sigma(X_1,X_2,\dots,X_N)$ is the smallest σ-algebra for which $X_1,X_2,\dots,X_N$ are measurable.

The proof of the following lemma is an exercise in measure theory. For the convenience of the reader, we prove it in the Appendix.

LEMMA 3.3. *Let $X_1,X_2,\dots,X_N,Y_1,\dots,Y_N$ be random variables. Suppose for each i,j, $X_i,Y_j\in L^1(\Omega)$ and X_i and Y_j are independent of each other. Let $A^+=\{\omega\colon \sum_{i=1}^N Y_i(\omega)X_i(\omega)>0\}$ and $A^-=\{\omega\colon \sum_{i=1}^N Y_i(\omega)X_i(\omega)<0\}$. Let A be either of the sets A^+ or A^-. Then there exists a sequence of measurable subsets $\{A_m\}_{m=1}^\infty$ such that $A_m\uparrow A$ as $m\uparrow\infty$ and for each m,*

$$A_m=\bigcup_{j=1}^{n_m}(E_{j,m}\cap G_{j,m}), \tag{13}$$

where for each fixed m, $E_{j,m}$'s are mutually disjoint and

$$\int_{A_m} X(\omega)Y(\omega)P(d\omega) = \sum_{j=1}^{n_m} \int_{E_{j,m}} X(\omega)P(d\omega) \int_{G_{j,m}} Y(\omega)P(d\omega), \tag{14}$$

where X is any of the random variables X_i and Y is any of the random variables Y_j, and

$$\lim_{m\to\infty} \int_{A_m} X(\omega)Y(\omega)P(d\omega) = \int_A X(\omega)Y(\omega)P(d\omega). \tag{15}$$

The next lemma is a standard result in set theory.

LEMMA 3.4. *Suppose $\{B_j\}_{j=1}^{K}$ is a sequence of measurable subsets of Ω. Then there exists a sequence $\{C_l\}_{l=1}^{L_k}$ of disjoint measurable subsets of Ω such that*

$$\bigcup_{j=1}^{K} B_j = \bigcup_{l=1}^{L_K} C_l$$

and for each j and each l, either C_l is a subset of B_j or $C_l \cap B_j = \varnothing$.

The next lemma is used to obtain the L^1-bound on $(L_1)\int_{[0,1]} Y\,dX$.

LEMMA 3.5. *Let X and Y be stochastic processes which are independent on disjoint intervals. Let q be the outer Littlewood exponent of λ_Y and let p be the inner Littlewood exponent of λ_X. Suppose $(1/p)+(1/q) > 1$ and p_1, q_1 are positive scalars such that $q_1 > q$, $p_1 > p$ and $(1/p_1)+(1/q_1) > 1$. Then, for any partition $\tau := \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ of $[0,1]$,*

$$\int_\Omega |L(\tau,\omega) - Y(0,\omega)(X(1,\omega) - X(0,\omega))|P(d\omega)$$
$$\le C(q_1,p_1)|\lambda_Y|_{(q_1,1)}|\lambda_X|_{(p_1,p_1)},$$

where $C(q_1,p_1)$ is a constant depending on q_1 and p_1 and

$$L(\tau,\omega) = \sum_{k=1}^{n} Y(t_{k-1},\omega)(X(t_k,\omega) - X(t_{k-1},\omega)).$$

PROOF. Let $\Delta_k Y(\omega) = Y(t_k,\omega) - Y(t_{k-1},\omega)$ and for each ω let

$$D(\omega) = L(\tau,\omega) - Y(0,\omega)[X(1,\omega) - X(0,\omega)].$$

Summation by parts gives us

$$D(\omega) = \sum_{j=2}^{n}\sum_{r=1}^{j-1} \Delta_r Y(\omega)\Delta_j X(\omega). \tag{16}$$

For each $1 \le j \le n$, let

$$A_j^+ = \left\{\omega\colon \sum_{r=1}^{j-1} \Delta_r Y(\omega)\Delta_j X_k(\omega) > 0\right\}$$

and

$$A_j^- = \left\{\omega\colon \sum_{r=1}^{j-1} \Delta_r Y(\omega)\Delta_j X(\omega) < 0\right\}$$

Let

$$L^+ = \sum_{j=1}^{n} \int_{A_j^+} \sum_{r=1}^{j-1} \Delta_r Y(\omega)\Delta_j X(\omega) P(d\omega) \tag{17}$$

and

$$L^- = \left| \sum_{j=1}^{n} \int_{A_j^-} \sum_{r=1}^{j-1} \Delta_r Y(\omega)\Delta_j X(\omega) P(d\omega)\right|. \tag{18}$$

Thus

$$\int_\Omega |D(\omega)| P(d\omega) \le L^+ + L^-. \tag{19}$$

We now estimate L^+. For each $j \le n$ let

$$F_j(\omega) = \sum_{r=1}^{j-1} \Delta_r Y(\omega)\Delta_j X(\omega). \tag{20}$$

Thus

$$|L^+| \le \left| \sum_{j=2}^{n} \int_{A_j^+} F_j(\omega) P(d\omega)\right|. \tag{21}$$

Since X and Y are independent on disjoint intervals, for each fixed j the random variable $\Delta_j X(\cdot)$ is independent of the random variable $\Delta_r Y(\cdot)$ whenever $r \le j$. Therefore by Lemma 3.3, we may assume for each $j \le n$,

$$A_j^+ = \bigcup_{l=1}^{L_j} (A_{l,j} \cap B_{l,j}),$$

where $A_{l,j}$'s are disjoint and

$$\int_{A_j^+} F_j(\omega)P(d\omega) = \sum_{r=1}^{j-1}\sum_{l=1}^{L_j} \int_{A_{l,j}} \Delta_j X(\omega)P(d\omega) \int_{B_{l,j}} \Delta_r Y(\omega)P(d\omega).$$

For $1 \le r, j \le n$, let

$$R(0,r,j) = \sum_{l=1}^{L_j} \int_{B_{l,j}} \Delta_r Y(\omega)P(d\omega) \int_{A_{l,j}} \Delta_j X(\omega)P(d\omega). \tag{22}$$

Choose r_0 with $1 \le r_0 \le n-1$ so that for each $j \le n-1$ the following holds:

$$|R(0, r_0+1, r_0)| \le |R(0, j+1, j)|. \tag{23}$$

Let

$$S(0,\tau) = \sum_{j=2}^{n} \sum_{r=1}^{j-1} \sum_{l=1}^{L_j} \int_{A_{l,j}} \Delta_j X(\omega) P(d\omega) \int_{B_{l,j}} \Delta_r Y(\omega) P(d\omega). \tag{24}$$

For $0 \le i \le n-1$ let

$$c_i = \begin{cases} i, & \text{if } i < r_0, \\ i+1, & \text{if } r_0 \le i \le n-1, \end{cases} \tag{25}$$

$$Y_1(i,\omega) = Y(t_{c_i}, \omega) \tag{26}$$

and

$$X_1(i,\omega) = X(t_{c_i}, \omega). \tag{27}$$

For $1 \le i, j \le n-1$, let

$$\Delta_j Y_1(\omega) = Y_1(j,\omega) - Y_1(j-1,\omega)$$

and

$$\Delta_i X_1(\omega) = X_1(i,\omega) - X_1(i-1,\omega).$$

Let

$$\begin{aligned} S(1,\tau) = \sum_{j=2}^{n-1} \sum_{r=1}^{j-1} \sum_{l=1}^{L_{c_j}} &\int_{B_{l,c_j}} \Delta_r Y_1(\omega) P(d\omega) \\ &\times \int_{A_{l,c_j}} \Delta_j X_1(\omega) P(d\omega). \end{aligned} \tag{28}$$

The following equation is easily verified:

$$S(0,\tau) = S(1,\tau) - R(0, r_0+1, r_0). \tag{29}$$

Therefore, by Minkowski's inequality,

$$|S(0,\tau)| \le |S(1,\tau)| + |R(0, r_0+1, r_0)|. \tag{30}$$

We now estimate $|R(0, r_0+1, r_0)|$. Let $1 < \alpha < 1/p_1 + 1/q_1$. By (23),

$$|R(0, r_0+1, r_0)| \le \left[\prod_{j=2}^{n-1} |R(0, j+1, j)| \right]^{1/(n-2)}. \tag{31}$$

An application of geometric-arithmetic mean inequality gives us

$$|R(0, r_0+1, r_0)| \le \left(\frac{1}{n-2} \right)^{\alpha} \left(\sum_{j=2}^{n-1} |R(0, j+1, j)|^{1/\alpha} \right)^{\alpha}. \tag{32}$$

Note

$$|R(0,j+1,j)| \le \sum_{l=1}^{L_j} \left| \int_{B_{l,j+1}} \Delta_{j+1} Y(\omega) P(d\omega) \int_{A_{l,j}} \Delta_j X(\omega) P(d\omega) \right|.$$

By Lemma 3.4 there exists a sequence $\{C_i\}_{i=1}^K$ of disjoint measurable subsets of Ω such that

$$\bigcup_{j=1}^{n} \bigcup_{l=1}^{L_j} A_{l,j} = \bigcup_{i=1}^{K} C_i$$

and for each l and each i, either C_i is a subset of $A_{l,j}$ or $C_i \cap A_{l,j} = \varnothing$. Note for each j, $K \ge L_j$. Similarly there exists a sequence $\{E_i\}_{i=1}^M$ of disjoint measurable subsets of Ω such that

$$\bigcup_{j=1}^{n} \bigcup_{l=1}^{L_j} B_{l,j} = \bigcup_{i=1}^{M} E_i$$

and for each l and each i, either E_i is a subset of $B_{l,j}$ or $E_i \cap B_{l,j} = \varnothing$. For each $l \in \{1, 2, \ldots, K\}$, let

$$a_l = \min\left\{ m\colon \bigcup_{i=1}^{m} E_i = \bigcup_{j=1}^{n} \bigcup_{i=1}^{l} B_{i,j} \right\}. \tag{33}$$

Since for each l, i and j either C_i is a subset of $A_{l,j}$ or $C_i \cap A_{l,j} = \varnothing$, and similarly, for each l, i and j either E_i is a subset of $B_{l,j}$ or $E_i \cap B_{l,j} = \varnothing$, we obtain

$$|R(0,j,j+1)| \le \sum_{l=1}^{K} \sum_{i=1}^{a_l} \left| \int_{E_i} \Delta_j Y_k(\omega) P(d\omega) \right| \left| \int_{C_l} \Delta_j X(\omega) P(d\omega) \right|. \tag{34}$$

Let

$$Q(K,j) = \sum_{l=1}^{K} \sum_{i=1}^{a_l} \left| \int_{E_i} \Delta_{j+1} Y(\omega) P(d\omega) \int_{C_l} \Delta_j X(\omega) P(d\omega) \right|. \tag{35}$$

If we set

$$\sum_{i=1}^{a_0} \left| \int_{C_i} \Delta_{j+1} Y(\omega) P(d\omega) \right| = 0,$$

then summation by parts gives us the following equality:

$$\begin{aligned} Q(K,j) &= \sum_{l=1}^{K} \sum_{s=1}^{l} \Bigg\{ \Bigg(\sum_{i=1}^{a_s} \left| \int_{E_i} \Delta_{j+1} Y(\omega) P(d\omega) \right| \\ &\qquad - \sum_{i=1}^{a_{s-1}} \left| \int_{E_i} \Delta_{j+1} Y(\omega) P(d\omega) \right| \Bigg) \left| \int_{C_l} \Delta_j X(\omega) P(d\omega) \right| \Bigg\} \\ &= \sum_{l=1}^{K} \sum_{s=1}^{l} \Bigg(\sum_{i=a_{s-1}+1}^{a_s} \left| \int_{E_i} \Delta_{j+1} Y(\omega) P(d\omega) \right| \Bigg) \left| \int_{A_l} \Delta_j X(\omega) P(d\omega) \right|. \end{aligned} \tag{36}$$

For $1 \le j \le n-1$ and $1 \le l \le K$, let

$$V(0,l,j) = \left(\sum_{i=a_{l-1}+1}^{a_l} \left| \int_{E_i} \Delta_{j+1} Y(\omega) P(d\omega) \right| \right) \left| \int_{C_l} \Delta_j X(\omega) P(d\omega) \right|. \tag{37}$$

For $1 \le l \le L$, let

$$W(Y, q_1, l) = \left(\sum_{j=2}^{n-1} \left[\sum_{i=a_{l-1}+1}^{a_l} \left| \int_{E_i} \Delta_{j+1} Y(\omega) P(d\omega) \right| \right]^{q_1} \right)^{1/\alpha q_1},$$

$$W(X, p_1, l) = \left(\sum_{j=2}^{n-1} \left| \int_{C_l} \Delta_j X(\omega) P(d\omega) \right|^{p_1} \right)^{1/\alpha p_1}$$

and

$$\tilde{V}(0,l) = W(Y, q_1, l) W(X, p_1, l). \tag{38}$$

Choose l_0 with $1 \le l_0 \le K-1$ so that for each $l \le K-1$ the following holds:

$$\tilde{V}(0, l_0) \le \tilde{V}(0, l). \tag{39}$$

For $0 \le l \le L-1$, let

$$d_l = \begin{cases} l, & \text{if } l < l_0, \\ l+1, & \text{if } l_0 \le i \le L-1, \end{cases} \tag{40}$$

$$\tilde{W}(Y, q_1, l, j) = \sum_{i=1}^{a_{d_l}} \left| \int_{E_i} \Delta_{j+1} Y(\omega) P(d\omega) \right| - \sum_{i=1}^{a_{d_{l-1}}} \left| \int_{E_i} \Delta_{j+1} Y(\omega) P(d\omega) \right|$$

and

$$\tilde{W}(X, p_1, l, j) = \left| \int_{C_{d_l}} \Delta_j X(\omega) P(d\omega) \right|.$$

Let

$$Q(K-1, j) = \sum_{l=1}^{K-1} \sum_{s=1}^{l} \tilde{W}(Y, q_1, l, j) \tilde{W}(X, p_1, s, j). \tag{41}$$

The following equation can be verified:

$$Q(K, j) = Q(K-1, j) + V(0, l_0, j). \tag{42}$$

Therefore by Minkowski's inequality and the fact that $\alpha > 1$, we obtain

$$\sum_{j=2}^{n-1} Q(K,j)^{1/\alpha} \le \sum_{j=2}^{n-1} Q(K-1,j)^{1/\alpha} + \sum_{j=2}^{n-1} |V(0, l_0, j)|^{1/\alpha}. \tag{43}$$

Recall $1/\alpha p_1 + 1/\alpha q_1 > 1$. Applying Hölder's inequality with exponents αp_1 and αq_1, we obtain

$$\begin{aligned}\sum_{j=2}^{n-1} |V(0,l_0,j)|^{1/\alpha} &\leq \left[\sum_{j=2}^{n-1}\left(\sum_{i=a_{l_0-1}+1}^{a_{l_0}}\left|\int_{E_i}\Delta_{j+1}Y(\omega)P(d\omega)\right|\right)^{q_1}\right]^{1/\alpha q_1}\\ &\quad\times\left[\sum_{j=2}^{n-1}\left|\int_{C_{l_0}}\Delta_j X(\omega)P(d\omega)\right|^{p_1}\right]^{1/\alpha p_1}\\ &= \tilde{V}(0,l_0).\end{aligned}$$

Thus

$$\sum_{j=2}^{n-1} |V(0,l_0,j)|^{1/\alpha} \leq \tilde{V}(0,l_0). \tag{44}$$

By our choice of l_0 [inequality (39)],

$$\tilde{V}(0,l_0) \leq \left(\prod_{l\neq l_0}^{K}\tilde{V}(0,l)\right)^{1/(K-1)} = \left(\prod_{l=1}^{K-1}\tilde{V}(0,d_l)\right)^{1/(K-1)}. \tag{45}$$

Now

$$\begin{aligned}&\left(\prod_{l=1}^{K-1}\tilde{V}(0,d_l)\right)^{1/(K-1)}\\ &\quad=\left\{\prod_{l=1}^{K-1}\left(\sum_{j=2}^{n-1}\left(\sum_{i=a_{d_{l-1}}+1}^{a_{d_l}}\left|\int_{E_i}\Delta_{j+1}Y(\omega)P(d\omega)\right|\right)^{q_1}\right)^{1/\alpha q_1}\right\}^{1/(K-1)}\\ &\quad\quad\times\left\{\prod_{l=1}^{K-1}\left(\sum_{j=2}^{n-1}\left|\int_{C_{d_l}}\Delta_j X(\omega)P(d\omega)\right|^{p_1}\right)^{1/\alpha p_1}\right\}^{1/(K-1)}.\end{aligned}$$

Applying the geometric-arithmetic mean inequality to the right side of the previous inequality, we obtain

$$\tilde{V}(0,l_0) \leq \left(\frac{1}{K-1}\right)^{(1/\alpha p_1+1/\alpha q_1)}(\tilde{Y}(q_1)\tilde{X}(p_1))^{1/\alpha}, \tag{46}$$

where

$$\tilde{Y}(q_1) = \left\{\sum_{l=1}^{K-1}\sum_{j=2}^{n-1}\left(\sum_{i=a_{d_{l-1}}+1}^{a_{d_l}}\left|\int_{E_i}\Delta_{j+1}Y(\omega)P(d\omega)\right|\right)^{q_1}\right\}^{1/q_1}, \tag{47}$$

and

$$\tilde{X}(p_1) = \left(\sum_{l=1}^{K-1}\sum_{j=2}^{n-1}\left|\int_{C_{d_l}}\Delta_j X(\omega)P(d\omega)\right|^{p_1}\right)^{1/p_1}. \tag{48}$$

Since $q_1 \geq 1$, the right-hand side of (47) is majorized by

$$\left\{\sum_{j=2}^{n-1}\left(\sum_{l=1}^{K-1}\left[\sum_{i=a_{d_{l-1}}+1}^{a_l}\left|\int_{E_i}\Delta_{j+1}Y(\omega)P(d\omega)\right|\right]\right)^{q_1}\right\}^{1/q_1}.$$

Since E_i's are disjoint, the above expression is equal to

$$\left\{\sum_{j=2}^{n-1}\left(\sum_{i=1}^{M}\left|\int_{E_i}\Delta_{j+1}Y(\omega)P(d\omega)\right|\right)^{q_1}\right\}^{1/q_1}.$$

The above expression is certainly bounded by $|\lambda_Y|_{(q_1,1)}$. Therefore,

$$\tilde{Y}(q_1) \leq |\lambda_Y|_{(q_1,1)}. \tag{49}$$

Since C_l's are disjoint,

$$\left\{\sum_{l=1}^{K-1}\sum_{j=2}^{n-1}\left|\int_{C_{d_l}}\Delta_j X(\omega)P(d\omega)\right|^{p_1}\right\}^{1/p_1} \leq |\lambda_X|_{(p_1,p_1)}.$$

Thus

$$\tilde{X}(p_1) \leq |\lambda_X|_{(p_1,p_1)}. \tag{50}$$

Combining inequalities (43), (44), (46), (49) and (50), we obtain

$$\begin{aligned}\sum_{j=2}^{n-1} Q(K,j)^{1/\alpha} &\leq \sum_{j=2}^{n-1} Q(K-1,j)^{1/\alpha}\\ &\quad + \left(\frac{1}{K-1}\right)^{1/\alpha q_1 + 1/\alpha p_1}(|\lambda_Y|_{(q_1,1)}|\lambda_X|_{(p_1,p_1)})^{1/\alpha}.\end{aligned} \tag{51}$$

By a similar argument we break up $Q(K-1,j)$ as the sum of two quantities; that is,

$$Q(K \dot{-} 1, j) = Q(K-2,j) + V(1,l_1,j), \tag{52}$$

and [compare with inequalities (42) and (43)]

$$\sum_{j=2}^{n-1} Q(K-1,j)^{1/\alpha} \leq \sum_{j=2}^{n-1} Q(K-2,j)^{1/\alpha} + \sum_{j=2}^{n-1}|V(1,l_1,j)|^{1/\alpha}. \tag{53}$$

We obtain $Q(K-2,j)$ and $V(1,l_1,j)$ in the same manner as $Q(K-1,j)$ and $V(0,l_0,j)$ were obtained from $Q(K,j)$. Furthermore, $\sum_{j=2}^{n-1} V(1,l_1,j)$ satisfies the following inequality [compare with (44) and (45)]:

$$\sum_{j=2}^{n-1} V(1,l_1,j) \leq \left(\prod_{l\neq l_1}^{K-1}\tilde{V}(1,l)\right)^{1/(K-2)}, \tag{54}$$

where

$$\tilde{V}(1,l) = \left(\sum_{j=2}^{n-1}\left(\sum_{i=a_{d_{l-1}}+1}^{a_{d_l}}\left|\int_{E_i}\Delta_{j+1}Y(\omega)P(d\omega)\right|\right)^{q_1}\right)^{1/\alpha q_1}$$
$$\times\left\{\sum_{j=2}^{n-1}\left|\int_{C_{d_l}}\Delta_j X(\omega)P(d\omega)\right|^{p_1}\right\}^{1/\alpha p_1}.$$

Arguing as before we obtain

$$\sum_{j=2}^{n-1}|V(1,l_1,j)|^{1/\alpha} \le \left(\frac{1}{K-2}\right)^{1/\alpha q_1+1/\alpha p_1}(|\lambda_Y|_{(q_1,1)}|\lambda_X|_{(p_1,p_1)})^{1/\alpha}. \tag{55}$$

Combining inequalities (51), (52), (53) and (55), we obtain

$$\begin{aligned}\sum_{j=2}^{n-1}Q(K,j)^{1/\alpha} &\le \sum_{j=2}^{n-1}Q(K-2,j)^{1/\alpha}\\ &\quad+\left(\frac{1}{K-1}\right)^{1/\alpha q_1+1/\alpha p_1}(|\lambda_Y|_{(q_1,1)}|\lambda_X|_{(p_1,p_1)})^{1/\alpha}\\ &\quad+\left(\frac{1}{K-2}\right)^{1/\alpha q_1+1/\alpha p_1}(|\lambda_Y|_{(q_1,1)}|\lambda_X|_{(p_1,p_1)})^{1/\alpha}.\end{aligned} \tag{56}$$

Continuing the above process by writing $Q(K-2,j)$ as the sum of $Q(K-3,j)$ and $V(2,l_2,j)$ and estimating as before, we obtain

$$\sum_{j=2}^{n-1}Q(K,j)^{1/\alpha} \le \left(1+\zeta\left(\frac{1}{\alpha q_1}+\frac{1}{\alpha p_1}\right)\right)(|\lambda_Y|_{(q_1,1)}|\lambda_X|_{(p_1,p_1)})^{1/\alpha}, \tag{57}$$

where $\zeta(s)=\sum_{n=1}^{\infty}(1/n^s)$. Thus by (32), (34), (35) and (57) we obtain

$$|R(0,r_0+1,r_0)| \le \left(\frac{1}{n-2}\right)^{\alpha}\left(1+\zeta\left(\frac{1}{\alpha q_1}+\frac{1}{\alpha p_1}\right)\right)^{\alpha}|\lambda_Y|_{(q_1,1)}|\lambda_X|_{(p_1,p_1)}.$$

Therefore, by (30), we obtain

$$\begin{aligned}|S(0,\tau)| &\le |S(1,\tau)|\\ &\quad+\left(\frac{1}{n-1}\right)^{\alpha}\left(1+\zeta\left(\frac{1}{\alpha q_1}+\frac{1}{\alpha p_1}\right)\right)^{\alpha}|\lambda_Y|_{(q_1,1)}|\lambda_X|_{(p_1,p_1)}.\end{aligned} \tag{58}$$

Since $S(1,\tau)$ is an expression similar to $S(0,\tau)$, we can deal with it in a similar manner. More precisely, we break up $S(1,\tau)$ as the difference of two quantities; that is [compare with (29)],

$$S(1,\tau) = S(2,\tau,k) - R(1,r_1+1,r_1), \tag{59}$$

where for $r,j\le n-1$, we obtain [compare with (22)]

$$R(1,r,j) = \sum_{l=1}^{K}\int_{B_{l,j}}\Delta_r Y_1(\omega)P(d\omega)\int_{A_{l,j}}\Delta_j X_1(\omega)P(d\omega)$$

and for each $j \le n-2$ [compare with inequality (23)],

$$|R(1, r_1+1, r_1)| \le |R(1, j+1, j)|.$$

Estimating $|R(1, k, r_1+1, r_1)|$ by an argument similar to that above which gave us the estimate on $|R(0, k, r_0+1, r_0)|$, we obtain

$$|R(1, r_1+1, r_1)| \le C(n-3)|\lambda_Y|_{(q_1,1)}|\lambda_X|_{(p_1,p_1)}, \tag{60}$$

where

$$C(n-3) = \left(\frac{1}{n-3}\right)^{\alpha}\left(1+\zeta\left(\frac{1}{\alpha q_1}+\frac{1}{\alpha p_1}\right)\right)^{\alpha}. \tag{61}$$

By (58) through (61) and Minkowski's inequality we obtain

$$\begin{aligned} |S(0,\tau)| &\le |S(2,\tau)| \\ &\quad + \left(\frac{1}{n-2}\right)^{\alpha}\left(1+\zeta\left(\frac{1}{\alpha q_1}+\frac{1}{\alpha p_1}\right)\right)^{\alpha}|\lambda_Y|_{(q_1,1)}|\lambda_X|_{(p_1,p_1)} \\ &\quad + \left(\frac{1}{n-3}\right)^{\alpha}\left(1+\zeta\left(\frac{1}{\alpha q_1}+\frac{1}{\alpha p_1}\right)\right)^{\alpha}|\lambda_Y|_{(q_1,1)}|\lambda_X|_{(p_1,p_1)}. \end{aligned} \tag{62}$$

Continuing the above process by writing $S(2,\tau)$ as the difference of $S(3,\tau)$ and $R(2, r_2+1, r_2)$ and estimating as before, we obtain

$$|S(0,\tau)| \le (1+\zeta(\alpha))\left(1+\zeta\left(\frac{1}{\alpha q_1}+\frac{1}{\alpha p_1}\right)\right)^{\alpha}|\lambda_Y|_{(q_1,1)}|\lambda_X|_{(p_1,p_1)}. \tag{63}$$

Therefore,

$$|L^+| \le (1+\zeta(\alpha))\left(1+\zeta\left(\frac{1}{\alpha q_1}+\frac{1}{\alpha p_1}\right)\right)^{\alpha}|\lambda_Y|_{(q_1,1)}|\lambda_X|_{(p_1,p_1)}. \tag{64}$$

A similar inequality is obtained for $|L^-|$. Thus, by (19) we obtain

$$\int_{\Omega}|D(\omega)|P(d\omega) \le c(p_1,q_1,\alpha)|\lambda_Y|_{(q_1,1)}|\lambda_X|_{(p_1,p_1)}, \tag{65}$$

where

$$c(p_1,q_1,\alpha) = 2(1+\zeta(\alpha))\left(1+\zeta\left(\frac{1}{\alpha q_1}+\frac{1}{\alpha p_1}\right)\right)^{\alpha}. \tag{66}$$

This completes the proof of the lemma. □

To prove Theorem 3.2, we need to prove a generalized version of the previous lemma, the proof of which runs along the same line as the proof of the lemma. We need to introduce more notation. For any partition $\tau := \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ of [0,1], $\theta(\tau)$ is another partition of [0,1] which refines τ. For a given partition τ of [0,1], $\theta(\tau)$ can be viewed as a two-dimensional sequence $\{x_{i,j}\}_{i=0,j=0}^{i=n,j=m_i}$, where for each fixed $1 \le i \le n$, $\tau_i =: \{x_{i,j}\}_{j=0}^{m_i}$ is a partition of

$[t_{i-1}, t_i]$. Let X be a process and $\mathscr{A} =: \{A_l\}_{l=1}^{L}$ be a partition of Ω. Let τ and $\theta(\tau)$ be as above. Let

$$T_1(X, q, p, \tau, \theta(\tau), \mathscr{A}) = \left[\sum_{k=1}^{n}\sum_{j=1}^{m_k}\left(\sum_{l=1}^{L}\left|\int_{A_l}\Delta_j X_i(\omega)P(d\omega)\right|^p\right)^{q/p}\right]^{1/q}, \tag{67}$$

where

$$\Delta_j X_i(\omega) = X(x_{i,j}, \omega) - X(x_{i,j-1}, \omega).$$

Let

$$|\lambda_X|_{(q,p)}^{\tau} = \sup\{T_1(X, q, p, \tau, \theta(\tau), \mathscr{A}) \colon \theta(\tau), \mathscr{A}\}. \tag{68}$$

LEMMA 3.6. *Let X, Y, p_1 and q_1 satisfy the hypothesis of Lemma 3.4. Then for any partitions $\tau := \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ and $\theta(\tau) := \{x_{i,k}\}_{i=0,j=0}^{i=n,j=m_i}$ of $[0,1]$,*

$$\int_{\Omega}|L(\tau, \theta(\tau), \omega) - L(\tau, \omega)|P(d\omega) \le C(q_1, p_1)|\lambda_Y|_{(q_1,1)}^{\tau}|\lambda_X|_{(p_1,p_1)}^{\tau},$$

where $C(q_1, p_1)$ is constant depending on q_1 and p_1,

$$L(\tau, \theta(\tau), \omega) = \sum_{k=1}^{n}\sum_{j=1}^{m_k} Y(x_{k,j-1}, \omega)\Delta_j X_k(\omega)$$

and

$$L(\tau, \omega) = \sum_{k=1}^{n} Y(t_{k-1}, \omega)(X(t_k, \omega) - X(t_{k-1}, \omega)).$$

[We should point out that for any partition θ which refines τ, the two quantities $L(\tau, \theta(\tau), \omega)$ and $L(\theta, \omega)$ are exactly the same.]

PROOF. Let

$$D(\omega) = L(\tau, \theta(\tau), \omega) - L(\tau, \omega).$$

First we note that for each fixed $1 \le k \le n$, summation by parts gives us

$$\sum_{j=1}^{m_k} Y(x_{k,j-1}, \omega)\Delta_j X_k(\omega)$$
$$= \sum_{j=1}^{m_k}\sum_{r=1}^{j-1}\Delta_r Y_k(\omega)\Delta_j X_k(\omega) + Y(t_{k-1}, \omega)(X(t_k, \omega) - X(t_{k-1}, \omega)).$$

Therefore,

$$D(\omega) = \sum_{k=1}^{n}\sum_{j=2}^{m_k}\sum_{r=1}^{j-1}\Delta_r Y_k(\omega)\Delta_j X_k(\omega). \tag{69}$$

Let $M = \max_{1 \le i \le n}\{m_i\}$ and for each $1 \le k \le n$, let

$$x_{k,j} = x_{k,m_k}, \quad \text{if } m_k \le j \le M.$$

Thus

$$D(\omega) = \sum_{k=1}^{n} \sum_{j=2}^{M} \sum_{r=1}^{j-1} \Delta_r Y_k(\omega) \Delta_j X_k(\omega). \tag{70}$$

For each $1 \le j \le M$, let

$$A_j^+ = \left\{\omega\colon \sum_{k=1}^{n} \sum_{r=1}^{j-1} \Delta_r Y_k(\omega) \Delta_j X_k(\omega) > 0\right\}$$

and

$$A_j^- = \left\{\omega\colon \sum_{k=1}^{n} \sum_{r=1}^{j-1} \Delta_r Y_k(\omega) \Delta_j X_k(\omega) < 0\right\}.$$

Thus

$$\int_\Omega |D(\omega)| P(d\omega) \le L^+ + L^-, \tag{71}$$

where

$$L^+ = \sum_{j=2}^{M} \int_{A_j^+} \sum_{k=1}^{n} \sum_{r=1}^{j-1} \Delta_r Y_k(\omega) \Delta_j X_k(\omega) P(d\omega) \tag{72}$$

and

$$L^- = \left| \sum_{j=2}^{M} \int_{A_j^-} \sum_{k=1}^{n} \sum_{r=1}^{j-1} \Delta_r Y_k(\omega) \Delta_j X_k(\omega) P(d\omega) \right|. \tag{73}$$

We now estimate L^+. For each $j \le M$ let

$$F_j(\omega) = \sum_{k=1}^{n} \sum_{r=1}^{j-1} \Delta_r Y_k(\omega) \Delta_j X_k(\omega). \tag{74}$$

Clearly

$$|L^+| \le \left| \sum_{j=2}^{M} \int_{A_j^+} F_j(\omega) P(d\omega) \right|. \tag{75}$$

Since X and Y are independent on disjoint intervals, for each fixed j the random variable $\Delta_j X_k(\cdot)$ is independent of the random variable $\Delta_i Y_k(\cdot)$ whenever $i \le j$. Therefore by Lemma 3.3, for each $j \le M$,

$$A_j^+ = \bigcup_{l=1}^{L_j} (A_{l,j} \cap B_{l,j}),$$

where $A_{l,j}$'s are disjoint and

$$\int_{A_j^+} F_j(\omega) P(d\omega) = \sum_{k=1}^{n} \sum_{r=1}^{j-1} \sum_{l=1}^{L_j} \int_{A_{l,j}} \Delta_j X_k(\omega) P(d\omega) \int_{B_{l,j}} \Delta_r Y_k(\omega) P(d\omega).$$

For $1 \le r, j \le M$ and $1 \le k \le n$, let

$$R(0,k,r,j) = \sum_{l=1}^{L_j} \int_{B_{l,j}} \Delta_r Y_k(\omega) P(d\omega) \int_{A_{l,j}} \Delta_j X_k(\omega) P(d\omega). \tag{76}$$

Choose r_0 with $1 \le r_0 \le M-1$ so that for each $j \le M-1$ the following holds:

$$\sum_{k=1}^{n} |R(0,k,r_0+1,r_0)| \le \sum_{k=1}^{n} |R(0,k,j+1,j)|. \tag{77}$$

For $1 \le k \le n$, let

$$S(0,\tau,k) = \sum_{j=2}^{M} \sum_{r=1}^{j-1} \sum_{l=1}^{L_j} \int_{A_{l,j}} \Delta_j X_k(\omega) P(d\omega) \int_{B_{l,j}} \Delta_r Y_k(\omega) P(d\omega). \tag{78}$$

For $0 \le i \le M-1$, let

$$c_i = \begin{cases} i, & \text{if } i < r_0, \\ i+1, & \text{if } r_0 \le i \le M-1, \end{cases} \tag{79}$$

$$Y_{(1,k)}(\omega, i) = Y(x_{k,c_i}, \omega) \tag{80}$$

and

$$X_{(1,k)}(\omega, i) = X(x_{k,c_i}, \omega). \tag{81}$$

For $1 \le i, j \le M-1$, let

$$\Delta_j Y_{(1,k)}(\omega) = Y_{(1,k)}(\omega, j) - Y_{(1,k)}(\omega, j-1)$$

and

$$\Delta_i X_{(1,k)}(\omega) = X_{(1,k)}(\omega, i) - X_{(1,k)}(\omega, i-1).$$

Let

$$\begin{aligned} S(1,\tau,k) = \sum_{j=2}^{M-1} \sum_{r=1}^{j} \sum_{l=1}^{L_{c_j}} \int_{B_{l,c_j}} \Delta_r Y_{(1,k)}(\omega) P(d\omega) \\ \times \int_{A_{l,c_j}} \Delta_j X_{(1,k)}(\omega) P(d\omega). \end{aligned} \tag{82}$$

The following equation is easily verified:

$$S(0,\tau,k) = S(1,\tau,k) - R(0,k,r_0+1,r_0). \tag{83}$$

Recall $\sum_{k=1}^{n} S(0,\tau,k) = L^+$. Therefore, by Minkowski's inequality,

$$|L^+| \le \sum_{k=1}^{n} |S(1,\tau,k)| + \sum_{k=1}^{n} |R(0,k,r_0+1,r_0)|. \tag{84}$$

For $1 \le j \le M-1$, let

$$\tilde{Q}(j) = \sum_{k=1}^{n} |R(0,k,j+1,j)|. \tag{85}$$

We now estimate

$$\tilde{Q}(r_0) = \sum_{k=1}^{n} |R(0, k, r_0 + 1, r_0)|.$$

Let $1 < \alpha < 1/p_1 + 1/q_1$. By (77),

$$\tilde{Q}(r_0) \leq \left[\prod_{j=2}^{M-1} \tilde{Q}(j) \right]^{1/(M-1)}. \tag{86}$$

An application of the geometric-arithmetic mean inequality gives us

$$\tilde{Q}(r_0) \leq \left(\frac{1}{M-2} \right)^{\alpha} \left(\sum_{j=2}^{M-1} \tilde{Q}(j)^{1/\alpha} \right)^{\alpha}. \tag{87}$$

Now

$$\tilde{Q}(j) \leq \sum_{k=1}^{n} \sum_{l=1}^{L_j} \left| \int_{B_{l,j+1}} \Delta_{j+1} Y_k(\omega) P(d\omega) \int_{A_{l,j}} \Delta_j X_k(\omega) P(d\omega) \right|.$$

By Lemma 3.4 there exists a sequence $\{C_i\}_{i=1}^{K}$ of disjoint measurable subsets of Ω such that

$$\bigcup_{j=1}^{M} \bigcup_{l=1}^{L_j} A_{j,l} = \bigcup_{i=1}^{K} C_i,$$

and for each l and each i, either C_i is a subset of $A_{l,j}$ or $C_i \cap A_{l,j} = \varnothing$ (note for each j, $K \geq L_j$). Similarly there exists a sequence $\{E_i\}_{i=1}^{N}$ of disjoint measurable subsets of Ω such that

$$\bigcup_{j=1}^{M} \bigcup_{l=1}^{L_j} B_{j,l} = \bigcup_{i=1}^{N} E_i$$

and for each l and each i, either E_i is a subset of $B_{l,j}$ or $E_i \cap B_{l,j} = \varnothing$. For each $l \in \{1, 2, \ldots, K\}$, let

$$a_l = \min \left\{ m \colon \bigcup_{i=1}^{m} D_i = \bigcup_{j=1}^{M} \bigcup_{i=1}^{l} B_{i,j} \right\}. \tag{88}$$

Since for each l, i and j either C_i is a subset of $A_{l,j}$ or $C_i \cap A_{l,j} = \varnothing$ and, similarly, for each l, i and j either E_i is a subset of $B_{l,j}$ or $E_i \cap B_{l,j} = \varnothing$, we obtain

$$\tilde{Q}(j) \leq \sum_{k=1}^{n} \sum_{l=1}^{K} \sum_{i=1}^{a_l} \left| \int_{E_i} \Delta_{j+1} Y_k(\omega) P(d\omega) \right| \left| \int_{C_l} \Delta_j X_k(\omega) P(d\omega) \right|. \tag{89}$$

Let,

$$Q(K, j) = \sum_{k=1}^{n} \sum_{l=1}^{K} \sum_{i=1}^{a_l} \left| \int_{E_i} \Delta_{j+1} Y_k(\omega) P(d\omega) \int_{C_l} \Delta_j X_k(\omega) P(d\omega) \right|. \tag{90}$$

Now if we set

$$\sum_{i=1}^{a_0}\left|\int_{E_i}\Delta_{j+1}Y_k(\omega)P(d\omega)\right|=0,$$

then summation by parts gives us the following equality:

$$\begin{aligned}
Q(K,j)&=\sum_{k=1}^{n}\sum_{l=1}^{K}\sum_{s=1}^{l}\Bigg\{\bigg(\sum_{i=1}^{a_l}\left|\int_{E_i}\Delta_{j+1}Y_k(\omega)P(d\omega)\right|\\
&\qquad-\sum_{i=1}^{a_{l-1}}\left|\int_{C_i}\Delta_{j+1}Y_k(\omega)P(d\omega)\right|\bigg)\\
&\qquad\times\left|\int_{C_s}\Delta_jX_k(\omega)P(d\omega)\right|\Bigg\}. \qquad (91)\\
&=\sum_{k=1}^{n}\sum_{l=1}^{K}\sum_{s=1}^{l}\bigg(\sum_{i=a_{l-1}+1}^{a_l}\left|\int_{C_i}\Delta_{j+1}Y_k(\omega)P(d\omega)\right|\bigg)\\
&\qquad\times\left|\int_{A_s}\Delta_jX_k(\omega)P(d\omega)\right|.
\end{aligned}$$

For $1\le j\le M-1$ and $1\le l\le K$, let

$$V(0,l,j)=\sum_{k=1}^{n}\Bigg[\sum_{i=a_{l-1}+1}^{a_l}\left|\int_{E_i}\Delta_{j+1}Y_k(\omega)P(d\omega)\right|\left|\int_{C_l}\Delta_jX_k(\omega)P(d\omega)\right|\Bigg]. \qquad (92)$$

For $1\le l\le L$, let

$$W(Y,q_1,l)=\left(\sum_{k=1}^{n}\sum_{j=2}^{M-1}\Bigg[\sum_{i=a_{l-1}+1}^{a_l}\left|\int_{E_i}\Delta_{j+1}Y_k(\omega)P(d\omega)\right|\Bigg]^{q_1}\right)^{1/\alpha q_1},$$

$$W(X,p_1,l)=\left(\sum_{k=1}^{n}\sum_{j=2}^{M-1}\left|\int_{C_l}\Delta_jX_k(\omega)P(d\omega)\right|^{p_1}\right)^{1/\alpha p_1}$$

and

$$\tilde V(0,l)=W(Y,q_1,l)W(X,p_1,l). \qquad (93)$$

Choose l_0 with $1\le l_0\le K-1$ so that for each $l\le K-1$ the following holds:

$$\tilde V(0,l_0)\le\tilde V(0,l). \qquad (94)$$

For $0\le l\le K-1$, let

$$d_l=\begin{cases} l, & \text{if } l<l_0,\\ l+1, & \text{if } l_0\le i\le L-1,\end{cases} \qquad (95)$$

$$\begin{aligned}
\tilde W(Y,q_1,l,j)&=\sum_{k=1}^{n}\sum_{i=1}^{a_{d_l}}\left|\int_{E_i}\Delta_{j+1}Y_k(\omega)P(d\omega)\right|\\
&\qquad-\sum_{i=1}^{a_{d_{l-1}}}\left|\int_{E_i}\Delta_{j+1}Y_k(\omega)P(d\omega)\right|
\end{aligned}$$

and

$$\tilde{W}(X, p_1, l, j) = \sum_{k=1}^{n} \left| \int_{C_{d_l}} \Delta_j X_k(\omega) P(d\omega) \right|.$$

Let

$$Q(K-1, j) = \sum_{l=1}^{K-1} \sum_{s=1}^{l} \tilde{W}(Y, q_1, l, j) \tilde{W}(X, p_1, s, j). \tag{96}$$

The following equation can be verified:

$$Q(K, j) = Q(K-1, j) + V(0, l_0, j). \tag{97}$$

Therefore, by Minkowski's inequality and the fact that $\alpha > 1$, we obtain

$$\sum_{j=2}^{M-1} Q(K, j)^{1/\alpha} \leq \sum_{j=2}^{M-1} Q(K-1, j)^{1/\alpha} + \sum_{j=2}^{M-1} |V(0, l_0, j)|^{1/\alpha}. \tag{98}$$

Recall $1/\alpha p_1 + 1/\alpha q_1 > 1$. Applying Hölder's inequality with exponents αp_1 and αq_1, we obtain

$$\begin{aligned}
\sum_{j=2}^{M-1} |V(0, l_0, j)|^{1/\alpha} &\leq \left(\sum_{k=1}^{n} \sum_{j=2}^{M-1} \left[\sum_{i=a_{l_0-1}+1}^{a_{l_0}} \left| \int_{E_i} \Delta_{j+1} Y_k(\omega) P(d\omega) \right| \right]^{q_1} \right)^{1/\alpha q_1} \\
&\quad \times \left(\sum_{k=1}^{n} \sum_{j=2}^{M-1} \left| \int_{C_{l_0}} \Delta_j X_k(\omega) P(d\omega) \right|^{p_1} \right)^{1/\alpha p_1} \\
&= \tilde{V}(0, l_0).
\end{aligned}$$

Thus

$$\sum_{j=2}^{M-1} |V(0, l_0, j)|^{1/\alpha} \leq \tilde{V}(0, l_0). \tag{99}$$

By our choice of l_0 [inequality (94)],

$$\tilde{V}(0, l_0) \leq \left(\prod_{l \neq l_0}^{K} \tilde{V}(0, l) \right)^{1/(K-1)} = \left(\prod_{l=1}^{K-1} \tilde{V}(0, d_l) \right)^{1/(K-1)} \tag{100}$$

Now

$$\begin{aligned}
&\left(\prod_{l=1}^{K-1} \tilde{V}(0, d_l) \right)^{1/(K-1)} \\
&\quad = \left\{ \prod_{l=1}^{K-1} \left(\sum_{k=1}^{n} \sum_{j=2}^{M-1} \left(\sum_{i=a_{d_{l-1}}+1}^{a_{d_l}} \left| \int_{E_i} \Delta_{j+1} Y_k(\omega) P(d\omega) \right| \right)^{q_1} \right)^{1/\alpha q_1} \right\}^{1/(K-1)} \\
&\quad \times \left\{ \prod_{l=1}^{L-1} \left(\sum_{k=1}^{n} \sum_{j=2}^{M-1} \left| \int_{E_{d_l}} \Delta_j X_k(\omega) P(d\omega) \right|^{p_1} \right)^{1/\alpha p_1} \right\}^{1(K-1)}
\end{aligned}$$

Applying the geometric-arithmetic mean inequality to the right side of the previous inequality, we obtain

$$\tilde{V}(0, l_0) \le \left(\frac{1}{K-1}\right)^{(1/\alpha p_1 + 1/\alpha q_1)} (\tilde{Y}(q_1)\tilde{X}(p_1))^{1/\alpha}, \tag{101}$$

where

$$\tilde{Y}(q_1) = \left\{\sum_{k=1}^{n}\sum_{l=1}^{K-1}\sum_{j=2}^{M-1}\left(\sum_{i=a_{d_{l-1}}+1}^{a_{d_l}}\left|\int_{E_i} \Delta_{j+1}Y_k(\omega)P(d\omega)\right|\right)^{q_1}\right\}^{1/q_1}, \tag{102}$$

and

$$\tilde{X}(p_1) = \left\{\sum_{k=1}^{n}\sum_{l=1}^{K-1}\sum_{j=2}^{M-1}\left|\int_{C_{d_l}} \Delta_j X_k(\omega)P(d\omega)\right|^{p_1}\right\}^{1/p_1}. \tag{103}$$

Since $q_1 \ge 1$ the right-hand side of (102) is majorized by

$$\left\{\sum_{k=1}^{n}\sum_{j=2}^{M-1}\left(\sum_{l=1}^{K-1}\left[\sum_{i=a_{d_{l-1}}+1}^{a_l}\left|\int_{E_i} \Delta_{j+1}Y_k(\omega)P(d\omega)\right|\right]\right)^{q_1}\right\}^{1/q_1}.$$

Since E_i's are disjoint, the above expression is equal to

$$\left\{\sum_{k=1}^{n}\sum_{j=2}^{M-1}\left(\sum_{i=1}^{N}\left|\int_{E_i} \Delta_{j+1}Y_k(\omega)P(d\omega)\right|\right)^{q_1}\right\}^{1/q_1}.$$

The above expression is certainly bounded by $|\lambda_Y|^{\tau}_{(q_1,1)}$. Therefore,

$$\tilde{Y}(q_1) \le |\lambda_Y|^{\tau}_{(q_1,1)}. \tag{104}$$

Since C_l's are disjoint,

$$\left\{\sum_{k=1}^{n}\sum_{l=1}^{K-1}\sum_{j=2}^{M-1}\left|\int_{A_{d_l}} \Delta_j X_k(\omega)P(d\omega)\right|^{p_1}\right\}^{1/p_1} \le |\lambda_X|^{\tau}_{(p_1,p_1)}.$$

Thus

$$\tilde{X}(p_1) \le |\lambda_X|^{\tau}_{(p_1,p_1)}. \tag{105}$$

Combining inequalities (98), (101), (104) and (125), we obtain

$$\begin{aligned}\sum_{j=2}^{M-1} Q(K,j)^{1/\alpha} &\le \sum_{j=2}^{M-1} Q(K-1,j)^{1/\alpha} \\ &\quad + \left(\frac{1}{K-1}\right)^{1/\alpha q_1 + 1/\alpha p_1} (|\lambda_Y|^{\tau}_{(q_1,1)}|\lambda_X|^{\tau}_{(p_1,p_1)})^{1/\alpha}.\end{aligned} \tag{106}$$

By a similar argument we break up $Q(K-1,j)$ as the sum of two quantities, that is,

$$Q(K-1,j) = Q(K-2,j) + V(1,l_1,j) \tag{107}$$

and [compare with inequalities (97) and (98)]

$$\sum_{j=2}^{M-1} Q(K-1,j)^{1/\alpha} \le \sum_{j=2}^{M-1} Q(K-2,j)^{1/\alpha} + \sum_{j=2}^{M-1} |V(1,l_1,j)|^{1/\alpha}. \tag{108}$$

The quantities $Q(K-2,j)$ and $V(1,l_1,j)$ are obtained in the same manner as $Q(K-1,j)$ and $V(0,l_0,j)$ were obtained from $Q(K,j)$. Furthermore $\sum_{j=2}^{M-1} V(1,l_1,j)$ satisfies the following inequality [compare with (99) and (100)]:

$$\sum_{j=2}^{M-1} |V(1,l_1,j)|^{1/\alpha} \le \left(\prod_{l\neq l_1}^{K-1} \tilde{V}(1,l)\right)^{1/(K-2)}, \tag{109}$$

where

$$\tilde{V}(1,l) = \left(\sum_{k=1}^{n}\left(\sum_{j=2}^{M-1}\sum_{i=d_{l-1}+1}^{m_{d_l}} \left|\int_{E_i} \Delta_{j+1}Y_k(\omega)P(d\omega)\right|\right)^{q_1}\right)^{1/\alpha q_1}$$
$$\times\left(\sum_{k=1}^{n}\left(\sum_{j=2}^{M-1}\left|\int_{C_{d_l}} \Delta_j X_k(\omega)P(d\omega)\right|^{p_1}\right)\right)^{1/\alpha p_1}.$$

Arguing as before we obtain

$$\sum_{j=2}^{M-1} |V(1,l_1,j)|^{1/\alpha} \le \left(\frac{1}{K-2}\right)^{1/\alpha q_1 + 1/\alpha p_1} (|\lambda_Y|^{\tau}_{(q_1,1)}|\lambda_X|^{\tau}_{(p_1,p_1)})^{1/\alpha}. \tag{110}$$

Combining inequalities (106), (107), (108) and (110), we obtain

$$\begin{aligned}\sum_{j=2}^{M-1} Q(K,j)^{1/\alpha} &\le \sum_{j=2}^{M-1} Q(K-2,j)^{1/\alpha}\\ &\quad + \left(\frac{1}{K-1}\right)^{1/\alpha q_1 + 1/\alpha p_1} (|\lambda_Y|^{\tau}_{(q_1,1)}|\lambda_X|^{\tau}_{(p_1,p_1)})^{1/\alpha}\\ &\quad + \left(\frac{1}{K-2}\right)^{1/\alpha q_1 + 1/\alpha p_1} (|\lambda_Y|^{\tau}_{(q_1,1)}|\lambda_X|^{\tau}_{(p_1,p_1)})^{1/\alpha}.\end{aligned} \tag{111}$$

Continuing the above process by writing $Q(K-2,j)$ as the sum of $Q(K-3,j)$ and $V(2,l_2,j)$ and estimating as before, we obtain

$$\sum_{j=2}^{M-1} Q(K,j)^{1/\alpha} \le \left(1+\zeta\left(\frac{1}{\alpha q_1}+\frac{1}{\alpha p_1}\right)\right)(|\lambda_Y|^{\tau}_{(q_1,1)}|\lambda_X|^{\tau}_{(p_1,p_1)})^{1/\alpha}. \tag{112}$$

Statements (87), (89), (91) and (112) imply

$$\tilde{Q}(r_0) \le \left(\frac{1}{M-2}\right)^{\alpha}\left(1+\zeta\left(\frac{1}{\alpha q_1}+\frac{1}{\alpha p_1}\right)\right)^{\alpha} |\lambda_Y|^{\tau}_{(q_1,1)}|\lambda_X|^{\tau}_{(p_1,p_1)}. \tag{113}$$

Recall

$$\tilde{Q}(r_0) = \sum_{k=1}^{n} |R(0,k,r_0+1,r_0)|.$$

Therefore, by (84), (85) and (113),

$$|L^+| \le \sum_{k=1}^{n} |S(1,\tau,k)| + \left(\frac{1}{M-2}\right)^{\alpha}\left(1+\zeta\left(\frac{1}{\alpha q_1}+\frac{1}{\alpha p_1}\right)\right)^{\alpha} |\lambda_Y|^{\tau}_{(q_1,1)} |\lambda_X|^{\tau}_{(p_1,p_1)}. \tag{114}$$

Now $S(1,\tau,k)$ is an expression similar to $S(0,\tau,k)$. Thus we can deal with it in a similar manner. More precisely, we break up $S(1,\tau,k)$ as the difference of two quantities, that is [compare with (83)],

$$S(1,\tau,k) = S(2,\tau,k) - R(1,k,r_1+1,r_1), \tag{115}$$

where for $r, j \le M-1$ and $k \le n$ we have [compare with (76)]

$$R(1,k,r,j) = \sum_{l=1}^{L_{c_j}} \int_{B_{l,c_j}} \Delta_r Y_{(1,k)}(\omega) P(d\omega) \int_{A_{l,c_j}} \Delta_j X_{(1,k)}(\omega) P(d\omega),$$

and for each $j \le M-2$ [compare with inequality (77)],

$$\sum_{k=1}^{n} |R(1,k,r_1+1,r_1)| \le \sum_{k=1}^{n} |R(1,k,j+1,j)|.$$

Estimating $\sum_{k=1}^{n} |R(1,k,r_1+1,r_1)|$ by an argument similar to the above which gave us the estimate on $\sum_{k=1}^{n} |R(0,k,r_0+1,r_0)|$, we obtain

$$\sum_{k=1}^{n} |R(1,k,r_1+1,r_1)| \le C(M-3) |\lambda_Y|^{\tau}_{(q_1,1)} |\lambda_X|^{\tau}_{(p_1,p_1)}, \tag{116}$$

where

$$C(M-3) = \left(\frac{1}{M-3}\right)^{\alpha}\left(1+\zeta\left(\frac{1}{\alpha q_1}+\frac{1}{\alpha p_1}\right)\right)^{\alpha}. \tag{117}$$

By (114), (115), (116) and the Minkowski's inequality, we obtain

$$\begin{aligned} |L^+| \le{}& \sum_{k=1}^{n} |S(2,\tau,k)| \\ &+ \left(\frac{1}{M-2}\right)^{\alpha}(1+\zeta\left(\frac{1}{\alpha q_1}+\frac{1}{\alpha p_1}\right)\Big)^{\alpha} |\lambda_Y|^{\tau}_{(q_1,1)} |\lambda_X|^{\tau}_{(p_1,p_1)} \\ &+ \left(\frac{1}{M-3}\right)^{\alpha}\left(1+\zeta\left(\frac{1}{\alpha q_1}+\frac{1}{\alpha p_1}\right)\right)^{\alpha} |\lambda_Y|^{\tau}_{(q_1,1)} |\lambda_X|^{\tau}_{(p_1,p_1)}. \end{aligned} \tag{118}$$

Continuing the above process by writing $S(2,\tau,k)$ as the difference of $S(3,\tau,k)$ and $R(2,k,r_2+1,r_2)$ and estimating as before, we obtain

$$|L^+| \le (1+\zeta(\alpha))\left(1+\zeta\left(\frac{1}{\alpha q_1}+\frac{1}{\alpha p_1}\right)\right)^{\alpha} |\lambda_Y|^{\tau}_{(q_1,1)} |\lambda_X|^{\tau}_{(p_1,p_1)}. \tag{119}$$

A similar inequality is obtained for $|L^-|$. Thus by (71) we obtain

$$\int_\Omega |D(\omega)|P(d\omega) \le c(p_1, q_1, \alpha)|\lambda_Y|^\tau_{(q_1,1)}|\lambda_X|^\tau_{(p_1,p_1)}, \tag{120}$$

where

$$c(p_1, q_1, \alpha) = 2(1+\zeta(\alpha))\left(1+\zeta\left(\frac{1}{\alpha q_1}+\frac{1}{\alpha p_1}\right)\right)^\alpha. \tag{121}$$

This completes the proof of the lemma. □

To state the next proposition we need to define the *integral-oscillation* (IO) of a process X. For $\delta > 0$, let

$$\mathrm{IO}\,sc(\lambda_X, \delta) = \sup\left\{\int_\Omega |X(x,\omega) - X(y,\omega)|P(d\omega)\colon\ x, y \in [0,1],\ |x-y| < \delta\right\}.$$

It is easy to see that, if X is 1-continuous, then

$$\lim_{\delta\to 0}(\lambda_X, \delta) = 0.$$

PROPOSITION 3.7. *Let X be a process and τ a partition of* $[0,1]$ *such that* $||\tau|| < \delta$. *Then for* $p > q > 0$,

$$|\lambda_X|^\tau_{(p,p)} \le [|\lambda_X|^\tau_{(q,q)}]^{q/p}(\mathrm{IO}sc(\lambda_X, \delta))^{(p-q)/p} \tag{122}$$

and

$$|\lambda_X|^\tau_{(p,1)} \le [|\lambda_X|^\tau_{(q,1)}]^{q/p}(\mathrm{IO}sc(\lambda_X, I))^{(p-q)/p}. \tag{123}$$

PROOF. Let $\tau =: \{0 = s_0 < s_1 < \cdots < s_n = 1\}$ be the partition of $[0,1]$ and let $\mathscr{A} = \{A_k\}_{k=1}^L$ be a partition of Ω. Let θ be a partition of [0,1] which refines τ. Then θ can be viewed as a two-dimensional sequence, that is, $\theta =: \{x_{i,j}\}_{i=0,j=0}^{i=n,j=m_i}$. Let $\Delta_j X_i(\omega) = X(x_{i,j},\omega) - X(x_{i,j-1},\omega)$. Now

$$\begin{aligned}\left|\int_{A_k}\Delta_j X_i(\omega)P(d\omega)\right|^p &= \left|\int_{A_k}\Delta_j X_i(\omega)P(d\omega)\right|^q\left|\int_{A_k}\Delta_j X_i(\omega)P(d\omega)\right|^{p-q}\\ &\le \left|\int_{A_k}\Delta_j X_i(\omega)P(d\omega)\right|^q \int_\Omega |\Delta_j X_i(\omega)|P(d\omega)^{p-q}\\ &\le \left|\int_{A_k}\Delta_j X_i(\omega)P(d\omega)\right|^q(\mathrm{IO}\,sc(\lambda_X,\delta))^{p-q}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sum_{i=1}^n\left(\sum_{j=1}^{m_i}\sum_{k=1}^L\left|\int_{A_k}\Delta_j X_i(\omega)P(d\omega)\right|^p\right)\\ &\qquad\le\left(\sum_{i=1}^n\left(\sum_{j=1}^{m_i}\sum_{k=1}^L\left|\int_{A_k}\Delta_j X_i(\omega)P(d\omega)\right|\right)^q\right)(\mathrm{IO}\,sc(\lambda_X,\delta))^{p-q}.\end{aligned}$$

This proves (122). To prove (123) we note for fixed i and fixed j,

$$\left(\sum_{k=1}^{L}\left|\int_{A_k}\Delta_j X_i(\omega)P(d\omega)\right|\right)^p$$

$$= \left(\sum_{k=1}^{L}\left|\int_{A_k}\Delta_j X_i(\omega)P(d\omega)\right|\right)^q \left(\sum_{k=1}^{L}\left|\int_{A_k}\Delta_j X_i(\omega)P(d\omega)\right|\right)^{p-q}$$

$$\leq \left(\sum_{k=1}^{L}\left|\int_{A_k}\Delta_j X_i(\omega)P(d\omega)\right|\right)^q \left(\int_{\Omega}|\Delta_j X_i(\omega)|P(d\omega)\right)^{p-q}$$

$$\leq \left(\sum_{K=1}^{L}\left|\int_{A_K}\Delta_j X_i(\omega)P(d\omega)\right|\right)^q (\mathrm{IO}\, sc(\lambda_X,\delta))^{p-q}.$$

Therefore

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{m_i}\sum_{k=1}^{L}\left|\int_{A_k}\Delta_j X_i(\omega)P(d\omega)\right|\right)^p$$

$$\leq \sum_{i=1}^{m}\left(\sum_{j=1}^{m_i}\sum_{k=1}^{L}\left|\int_{A_k}\Delta_j X_i(\omega)P(d\omega)\right|\right)^q (\mathrm{IO}\, sc(\lambda_X,I))^{p-q}.$$

This proves (123). □

We now complete the proof of Theorem 3.2. For any partition τ of $[0,1]$, we define $L(\tau,\omega)$ as in Lemma 3.6. To show the existence $(L_1)\int_{[0.1]} Y\,dX$ it suffices to show that for each $\varepsilon > 0$ there exists a $\delta(\varepsilon)$ such that

$$||L(\tau,\omega) - L(\tau',\omega)||_1 < \varepsilon \quad \text{whenever } ||\tau||, ||\tau'|| < \delta. \tag{124}$$

Let $\varepsilon > 0$ and let $\tau =: \{t_i\}_{i=0}^{n}$ and $\tau' =: \{s_i\}_{i=0}^{m}$ be partitions of [0,1] such that $||\tau||, ||\tau'|| < \delta$ (value of δ yet to be determined). Now let θ be a partition of [0,1] which refines both τ and τ'. Now

$$||L(\tau,\omega) - L(\tau',\omega)||_1 \leq ||L(\tau,\omega) - L(\theta(\tau),\omega)||_1 + ||L(\tau',\omega) - L(\theta(\tau'),\omega)||_1.$$

By Lemma 3.6,

$$||L(\tau,\omega) - L(\theta(\tau),\omega)||_1 \leq c(p_1,q_1,\alpha)|\lambda_Y|^{\tau}_{(q_1,1)}|\lambda_X|^{\tau}_{(p_1,p_1)}. \tag{125}$$

Now suppose X is 1-continuous. Choose δ such that $\mathrm{IO}\, sc(\lambda_X,\delta) < \varepsilon$. Choose $p_2 > p_1$ such that $1/p_2 + 1/q_1 \geq 1$. Then by (122) of Proposition 3.7 we obtain

$$|\lambda_Y|^{\tau}_{(q_1,1)}|\lambda_X|^{\tau}_{(p_1,p_1)} \leq \varepsilon^{(p_2-p_1)/p_1}[|\lambda_X|^{\tau}_{(p_1,p_1)}]^{p_1/p_2}|\lambda_Y|^{\tau}_{(q_1,1)}. \tag{126}$$

It is easy to see that

$$|\lambda_Y|^{\tau}_{(q_1,1)}| \leq 2|\lambda_Y|_{(q_1,1)} \tag{127}$$

and

$$|\lambda_X|^{\tau}_{(p_1,p_1)} \leq 2|\lambda_X|_{p_1}. \tag{128}$$

Combining the previous inequalities we obtain

$$||L(\tau,\omega)-L(\theta(\tau),\omega)||_1 \le 4c(p_1,q_1,\alpha)\varepsilon^{(p_2-p_1)/p_1}|\lambda_X|_{p_1}^{p_1/p_2}|\lambda_Y|_{(q_1,1)}.$$

A similar bound is obtained for

$$||L(\tau,\omega)-L(\theta(\tau),\omega)||_1.$$

This shows that if X is 1-continuous, then (124) is satisfied. If Y is 1-continuous, then by applying inequality (123) of Proposition 3.7 and arguing as above, we show that (124) is satisfied. This completes the proof of Theorem 3.2. □

We remark that the 1-continuity condition on X or Y in Theorem 3.2 can be replaced by the weaker condition of right 1-continuity.

4. Mixed stochastic integration. In this section we want to give another version of the stochastic integral of Y with respect to X. We first give the following motivation for this new definition of the stochastic integral. If for each $\omega \in \Omega$, $X(\omega,\cdot)$ is a function of bounded variation on $[0,1]$ and for each $\omega \in \Omega$, $Y(\omega,\cdot)$ is a continuous function on $[0,1]$, then for $\omega,\omega' \in \Omega$,

$$Z(\omega,\omega') = \int_0^1 Y(t,\omega)\,dX(t,\omega'),$$

the Riemann–Stieltjes integral of $Y(\omega,\cdot)$ with respect to $X(\omega',\cdot)$ exists. We call $Z(\omega,\omega')$ the mixed integral of Y with respect to X. Furthermore, if

$$\sup\{||Y(t,\cdot)||_1\colon t\in[0,1]\} < \infty$$

and the functions $\{X(\cdot,\omega)\}_{\omega\in\Omega}$ are uniformly bounded in variation on [0,1], then $Z(\omega,\omega')$ belongs to $L_1(\Omega\times\Omega,\ P\otimes P)$. Here $P\otimes P$ is the product measure on $\Omega\times\Omega$. An application of Fubini's theorem implies that for almost every $\omega\in\Omega$, $Z(\omega,\cdot)$ belongs to $L_1(\omega,P)$. Similarly, for almost every $\omega\in\Omega$, $Z(\cdot,\omega)$ belongs to $L_1(\Omega,P)$. We call $Z(\omega,\omega')$ the mixed integral of Y with respect to X.

Given any two processes X and Y on $(\Omega,\mathscr{A})\times[0,1]$, we would still like to define the *mixed stochastic integral of* Y *with respect to* X. Let $(\omega,\omega')\in\Omega\times\Omega$ and $\tau =: \{t_j\}_{j=0}^n$ be a partition of $[0,1]$. Let

$$L(\tau,\omega,\omega',Y,X) = L(\tau,\omega,\omega') = \sum_{j=1}^n Y(t_{j-1},\omega)\Delta_j X(\omega').$$

We say *the* (M)*-integral of* Y *with respect to* X exists if there exists a r.v. $J_{(X,Y)} \in L_1(\Omega\times\Omega,\ P\otimes P)$ such that

$$\lim_{||\tau||\to 0} ||L(\tau,\cdot,\cdot) - J_{(X,Y)}||_1 = 0. \tag{129}$$

If (129) holds, we say that $J_{(X,Y)}$ is the mixed integral of Y with respect to X and denote it by

$$(M)\int_{[0,1]} Y\,dX.$$

The proof of the following theorem is similar to the proof of Theorem 3.2. To prove it one first proves a lemma analogous to Lemma 3.3 by using Lemma 4.1 below. We omit the proofs.

LEMMA 4.1. *Let X, Y be L^1-bounded random variables defined on probability space (Ω, P). Suppose A is a measurable subset of $\Omega \times \Omega$. Then there exists a sequence of measurable subsets*

$$F_m = \bigcup_{j=1}^{n_m} A_{(j,m)} \times B_{(j,m)}, \tag{130}$$

where $\{A_{(j,m)}\}_{j=1}^{n_m}$ and $\{B_{(j,m)}\}_{j=1}^{n_m}$ are subsets of Ω and for each fixed m, $A_{(i,m)}$'s are mutually disjoint, such that $F_m \uparrow A$ as $m \uparrow \infty$ and

$$\lim_{m\to\infty} \iint_{F_m} X(\omega)Y(\omega')P \otimes P(d\omega \times d\omega') = \iint_A X(\omega)Y(\omega')P \otimes P(d\omega \times d\omega').$$

THEOREM 4.2. *Let X and Y be stochastic processes. Let q be the outer Littlewood exponent of λ_Y and let p be the inner Littlewood exponent of λ_X. Suppose either Y is 1-continuous or X is 1-continuous. If $(1/p) + (1/q) > 1$, then an (M)-integral of Y with respect to X exists and*

$$\left\| (M)\int_{[0,1]} Y\,dX \right\|_1 \le C(p,q)|\lambda_X|_{p_1}|\lambda_Y|_{(q_1,1)}. \tag{131}$$

Here, $C(p,q) > 0$ is a constant and p_1 and q_1 are any two numbers such that $1/p_1 + 1/q_1 > 1$, $p_1 > p$ and $q_1 > q$.

APPENDIX

In this section we prove Lemma 3.3 by proving a sequence of sublemmas.

LEMMA A.1. *Let $X_1, X_2, \ldots, X_N$ be random variables and let $A = \{\omega\colon \sum_{i=1}^N X_i < 0\}$. Then A has the following representation:*

$$A = \bigcup_{n=1}^{\infty}\left[\bigcap_{i=1}^{N} F_i^n\right], \qquad F_i^n \in \sigma(X_i), \tag{132}$$

where each F_i^n is a set of the type $\{\omega\colon X_i < q\}$ for some rational number q.

PROOF. We first assume that $N = 2$. Let $A = \{\omega\colon X_1(\omega) + X_2(\omega) < 0\}$. Thus $A \in \sigma(\sum_{i=1}^{2} X_i)$. We will now construct sets $\{F_1^{(n)}\}_{n=1}^{\infty}$ and $\{F_2^{(n)}\}_{n=1}^{\infty}$ such that $F_i^{(n)} \in \sigma(X_i)$ and

$$A = \bigcup_{n=1}^{\infty} (F_1^{(n)} \cap F_2^{(n)}).$$

Let $\{q_n\}_{n=1}^{\infty}$ be a listing of the rationals. For each $n \in \mathbf{N}$, let

$$F_1^n = \{\omega\colon X_1(\omega) < q_n\},$$
$$F_2^n = \{\omega\colon X_2(\omega) < -q_n\}.$$

We note that for each n, $F_1^n \in \sigma(X_1)$ and $F_2^n \in \sigma(X_2)$. It is clear that

$$\bigcup_{n=1}^{\infty} (F_1^n \cap F_2^n) \subset A.$$

Suppose $\omega \in A$ and $X_1(\omega) = r$. Then $X_2(\omega) < -r$. We can find a subsequence $\{q_{n_j}\} \downarrow r$ as $j \uparrow \infty$. Therefore, for each j, $X_1(\omega) < q_{n_j}$. Since $X_2(\omega) < -r$ and $\{q_{n_j}\} \downarrow r$, if j is sufficiently large, then $X_2(\omega) < -q_{n_j}$. Thus there exists a j such that

$$X_1(\omega) < q_{n_j} \tag{133}$$

and

$$X_2(\omega) < -q_{n_j}. \tag{134}$$

Inequalities (133) and (134) imply

$$\omega \in F_1^{n_j} \cap F_2^{n_j}.$$

This shows that $A \subset \bigcup_{n=1}^{\infty} (F_1^n \cap F_2^n)$. Therefore,

$$A = \bigcup_{n=1}^{\infty} (F_1^n \cap F_2^n). \tag{135}$$

Now suppose the statement of the lemma holds for $N - 1$ random variables. Then

$$A = \bigcup_{n=1}^{\infty} \left[\left(\bigcap_{i=1}^{N-2} F_i^n \right) \cap G_{N-1}^n \right], \tag{136}$$

where for each i, F_i^n are sets of type $\{\omega\colon X_i(\omega) < q\}$ for some rational number q and G_{N-1}^n are the sets of type $\{\omega\colon X_{N-1}(\omega) + X_N(\omega) < q\}$ for some rational number q. An argument similar to the above shows that for each n,

$$G_{N-1}^n = \bigcup_{k=1}^{\infty} (F_{N-1}^{(n,k)} \cap F_N^{(n,k)}) \tag{137}$$

and for $i = N-1, N$, the sets $F_i^{(n,k)}$ are of the type $\{\omega\colon\ X_i(\omega) < q\}$ for some rational number q. If we let

$$F_i^{(n,k)} = F_i^n, \qquad n,k \geq 1,\ 1 \leq i \leq N-2,$$

then

$$A = \bigcup_{n,k=1}^{\infty} \left(\bigcap_{i=1}^{N} F_i^{(n,k)} \right). \tag{138}$$

This proves the lemma. □

LEMMA A.2. *Let $X_1, X_2, \dots, X_N, Y_1, \dots, Y_N$ be random variables. Let $A^+ = \{\omega\colon\ \sum_{i=1}^N Y_i(\omega)X_i(\omega) > 0\}$ and $A^- = \{\omega\colon\ \sum_{i=1}^N Y_i(\omega)X_i(\omega) < 0\}$. Then both A^+ and A^- are sets of type*

$$\bigcup_{n=1}^{\infty} F_x^n \cap F_y^n, \qquad F_x^n \in \bigcup_{i=1}^{N} \sigma(X_i), \qquad F_y^n \in \bigcup_{i=1}^{N} \sigma(Y_i).$$

PROOF. By Lemma A.1,

$$A^- = \bigcup_{n=1}^{\infty} \left[\bigcap_{i=1}^{N} F_i^n \right], \qquad F_i^n \in \sigma(Y_i X_i),$$

where $F_i^n = \{\omega\colon\ Y_i X_i < q\}$ for some $q \in \mathbf{Q}$. First we show

$$F_i^n = \bigcup_{k=1}^{\infty} \left(F_{(x,i)}^{(n,k)} \cap F_{(y,i)}^{(n,k)} \right),$$

where for each i, n and k

$$F_{(x,i)}^{(n,k)} \in \sigma(X_i), \qquad F_{(y,i)}^{(n,k)} \in \sigma(Y_i).$$

Let $\{q_n\}_{n=1}^{\infty}$ be a listing of the rational numbers. Henceforth if $q_n = 0$ and $q > 0$, then $(q/(-|q_n|)) = -\infty$ and $(q/|q_n|) = \infty$ and if $q_n = 0$ and $q < 0$, then $(q/(-|q_n|)) = \infty$ and $(q/|q_n|) = -\infty$. Let

$$\begin{aligned} D_{n,1} &= \{\omega\colon\ 0 \leq X_i(\omega) < |q_n|\}, \\ E_{n,1} &= \left\{\omega\colon\ Y_i(\omega) < \frac{q}{|q_n|}\right\}, \\ D_{n,2} &= \{\omega\colon\ -|q_n| < X_i(\omega) \leq 0\}, \\ E_{n,2} &= \left\{\omega\colon\ \frac{q}{-|q_n|} < Y_i(\omega)\right\}, \end{aligned}$$

$$\begin{aligned} G(n,1) &= D_{n,1} \cap E_{n,1}, \\ G(n,2) &= D_{n,1} \cap E_{n,2}, \\ G(n,3) &= D_{n,2} \cap E_{n,1}, \\ G(n,4) &= D_{n,2} \cap E_{n,2}. \end{aligned}$$

First suppose $q \geq 0$. Let

$$B = (Y_i^{-1}(0) \cup X_i^{-1}(0)) \cup \left[\bigcup_{n=1}^{\infty} (G(n,1) \cup G(n,4)) \right]. \tag{139}$$

Clearly $B \subset F_i^n$. Now suppose $\omega \in F_i^n$. If $q = 0$, then either $X_i(\omega) = 0$ or $Y_i(\omega) = 0$. Since $Y_i^{-1}(0) \cup X_i^{-1}(0) \subset B$, therefore $B = F_i^n$. So we assume $q > 0$.

If $X_i(\omega) = 0$, then $\omega \in X_i^{-1}(0)$. This shows that $\omega \in B$. If $X_i(\omega) = r > 0$, then $Y_i(\omega) < q/r$. There is a subsequence $\{q_{n_k}\}$ of rationals such that $q_{n_k} \downarrow r$ as $k \uparrow \infty$. This means $(q/q_{n_k}) \downarrow (q/r)$. Thus for each k, $\omega \in D_{n_k,1}$. On the other hand, $Y_i(\omega) < q/r$ and $(q/q_{n_k}) \downarrow (q/r)$. Therefore, $\omega \in E_{n_k,1}$ when k is sufficiently large, which implies $\omega \in B$. Now suppose $X_i(\omega) = r < 0$. This implies $Y_i(\omega) > q/r$. As before we can find a subsequence $\{q_{n_k}\}$ of rationals such that $-|q_{n_k}| \uparrow r$ as $k \uparrow \infty$. Thus $Y_i(\omega) > q/-|q_{n_k}|$ for k sufficiently large. On the other hand, $-|q_{n_k}| \uparrow r$ implies $-|q_{n_k}| < X_i(\omega)$ for each k. Therefore $\omega \in G(n_k, 4)$ for some k and thus $\omega \in B$. If $q < 0$, then let

$$B = \bigcup_{n=1}^{\infty} G(n,2) \cup G(n,3). \tag{140}$$

An argument similar to that above will show that $B = F_i^n$. Note $\Omega \in \sigma(Y_i)$ and $X_i^{-1}(0) = X_i^{-1}(0) \cap \Omega$. Similarly, $Y_i^{-1}(0) = Y_i^{-1}(0) \cap \Omega$. By reindexing the sets obtained in expressions for B; that is, the sets on the right sides of (139) and (140), we can express F_i^n as

$$F_i^n = \bigcup_{k=1}^{\infty} (F_{(x,i)}^{(n,k)} \cap F_{(y,i)}^{(n,k)}),$$

where for each i, n and k

$$F_{(x,i)}^{(n,k)} \in \sigma(X_i), \qquad F_{(y,i)}^{(n,k)} \in \sigma(Y_i).$$

This means

$$A^- = \bigcup_{n=1}^{\infty} \left[\bigcap_{i=1}^{N} \left(\bigcup_{k=1}^{\infty} (F_{(x,i)}^{(n,k)} \cap F_{(y,i)}^{(n,k)}) \right) \right].$$

By elementary set theory and an induction argument on N (the argument is similar to the proof of Lemma A.1), we can express A^- as

$$A^- = \bigcup_{n=1}^{\infty} \left[\bigcap_{i=1}^{N} (F_{(x,i)}^n \cap F_{(y,i)}^n) \right], \tag{141}$$

where for each i, n,

$$F_{(x,i)}^n \in \sigma(X_i), \qquad F_{(y,i)}^n \in \sigma(Y_i).$$

Now for each m let

$$F_x^m = \bigcap_{i=1}^{N} F_{(x,i)}^m$$

and

$$F_y^m = \bigcap_{i=1}^{N} F_{(y,i)}^m.$$

Then

$$A^- = \bigcup_{n=1}^{\infty} [F_x^n \cap F_y^n] \tag{142}$$

and

$$F_x^n \in \bigcup_{i=1}^{N} \sigma(X_i), \qquad F_y^n \in \bigcup_{i=1}^{N} \sigma(Y_i). \tag{143}$$

A similar expression can be obtained for A^+ by noting that $A^+ = \{\omega: -\sum_{i=1}^N Y_i(\omega)X_i(\omega) < 0\}$. This completes the proof. □

The following lemma is standard. A proof is given in Peterson (1977).

LEMMA A.3. *If X and Y are independent r.v. on a probability space (Ω, P), A is a measurable subset and Y is independent of the σ-algebra generated by A and $\sigma(X)$, then*

$$\int_A X(\omega)Y(\omega)P(d\omega) = \int_{\omega' \in A} \int_{\omega \in \Omega} X(\omega')Y(\omega)P(d\omega') \otimes P(d\omega).$$

We now prove Lemma 3.3. For the convenience of the reader we restate the lemma.

LEMMA 3.3. *Let $X_1, X_2, \dots, X_N, Y_1, \dots, Y_N$ be random variables. Suppose for each i, j, $X_i, Y_j \in L^1(\Omega)$ and X_i and Y_j are independent of each other. Let $A^+ = \{\omega: \sum_{i=1}^N Y_i(\omega)X_i(\omega) > 0\}$ and $A^- = \{\omega: \sum_{i=1}^N Y_i(\omega)X_i(\omega) < 0\}$. Let A be either of the sets A^+ or A^-. Then there exists a sequence of measurable subsets $\{A_m\}_{m=1}^\infty$ such that $A_m \uparrow A$ as $m \uparrow \infty$ and for each m,*

$$A_m = \bigcup_{j=1}^{n_m} (E_{j,m} \cap G_{j,m}), \tag{144}$$

where for each fixed m, $E_{j,m}$'s are mutually disjoint and

$$\int_{A_m} X(\omega)Y(\omega)P(d\omega) = \sum_{j=1}^{n_m} \int_{E_{j,m}} X(\omega)P(d\omega) \int_{G_{j,m}} Y(\omega)P(d\omega), \tag{145}$$

where X is any of the random variables X_i and Y is any of the random variables Y_j, and

$$\lim_{m\to\infty} \int_{A_m} X(\omega)Y(\omega)P(d\omega) = \int_A X(\omega)Y(\omega)P(d\omega). \tag{146}$$

PROOF. By Lemma A.2,

$$A = \bigcup_{n=1}^{\infty} [F_x^n \cap F_y^n],$$

where

$$F_x^n \in \bigcup_{i=1}^{N} \sigma(X_i), \qquad F_y^n \in \bigcup_{i=1}^{N} \sigma(Y_i).$$

For each positive integer m, let

$$A_m = \bigcup_{n=1}^{m} [F_x^n \cap F_y^n].$$

We show by induction on m that

$$A_m = \bigcup_{n=1}^{n_m} (E_n \cap G_n), \tag{147}$$

where E_j's are disjoint and for each j,

$$E_j \in \bigcup_{i=1}^{N} \sigma(X_i), \qquad G_j \in \bigcup_{i=1}^{N} \sigma(Y_i).$$

When $m = 1$ trivially A_m has the desired representation. Now suppose A_{m-1} has the above type representation. That is,

$$A_{m-1} = \bigcup_{n=1}^{n_{(m-1)}} (H_n \cap C_n), \tag{148}$$

where H_j's are disjoint and for each j,

$$H_j \in \bigcup_{i=1}^{N} \sigma(X_i), \qquad C_j \in \bigcup_{i=1}^{N} \sigma(Y_i).$$

Now $A_m = A_{m-1} \cup (F_x^m \cap F_y^m)$. Let

$$E_j = \begin{cases} F_x^m - (\bigcup_{l=1}^{n_{(m-1)}} H_l), & \text{if } j = 2n_{(m-1)} + 1, \\ F_x^m \bigcap H_j, & \text{if } 1 \le j \le n_{(m-1)}, \\ H_j - F_x^m, & \text{if } n_{(m-1)} + 1 \le j \le 2n_{(m-1)}. \end{cases}$$

Clearly E_j's are disjoint and for each j, $E_j \in \bigcup_{i=1}^{N} \sigma(X_i)$. Let

$$G_j = \begin{cases} F_y^m, & \text{if } j = 2n_{(m-1)} + 1, \\ F_y^m \bigcup C_j, & \text{if } 1 \le j \le n_{(m-1)}, \\ C_j, & \text{if } n_{(m-1)} + 1 \le j \le 2n_{(m-1)}. \end{cases}$$

We note for each j, $G_j \in \bigcup_{i=1}^N \sigma(Y_i)$ and $A_m = \bigcup_{j=1}^{2n_{m-1}+1}(E_j \cap G_j)$. Thus A_m has the desired representation. Let $n_m = 2n_{(m-1)} + 1$. Then since E_j's are disjoint, we have

$$\int_{A_m} X(\omega)Y(\omega)P(d\omega) = \sum_{j=1}^{n_m} \int_{E_j \cap G_j} XYP(d\omega). \tag{149}$$

Now

$$\int_{E_j \cap G_j} X(\omega)Y(\omega)P(d\omega) = \int_{E_j} X(\omega)Y(\omega)I_{G_j}P(d\omega).$$

By assumptions on X and Y we note for each j, YI_{G_j} is independent of σ-algebra generated by E_j and X. Therefore, by Lemma A.3,

$$\sum_{j=1}^{n_m} \int_{E_j \cap G_j} X(\omega)Y(\omega)P(d\omega) = \sum_{j=1}^{n_m} \int_{E_j} X(\omega)P(d\omega) \int_{G_j} Y(\omega)P(d\omega).$$

Now $A_m \uparrow A$ as $m \uparrow \infty$ and

$$\begin{aligned} \left| \int_{A_m} X(\omega)Y(\omega)P(d\omega) \right| &\leq \int_\Omega |X(\omega)Y(\omega)|P(d\omega) \\ &= \int_\Omega |Y(\omega)|P(d\omega) \int_\Omega |X(\omega)|P(d\omega) \\ &= ||X||_1||Y||_1. \end{aligned}$$

Therefore, by Lebesgue's dominated convergence theorem,

$$\lim_{m\to\infty} \int_{A_m} X(\omega)Y(\omega)P(d\omega) = \int_A X(\omega)Y(\omega)P(d\omega). \tag{150}$$

This completes the proof. □

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