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# A STOCHASTIC REPRESENTATION THEOREM WITH APPLICATIONS TO OPTIMIZATION AND OBSTACLE PROBLEMS

BY PETER BANK[1] AND NICOLE EL KAROUI

*Humboldt University, and École Polytechnique and Université de Paris VI*

We study a new type of representation problem for optional processes with connections to singular control, optimal stopping and dynamic allocation problems. As an application, we show how to solve a variant of Skorohod's obstacle problem in the context of backward stochastic differential equations.

**1. Introduction.** In this paper, we study a new type of representation problem for optional processes. Specifically, given such a process $X = (X(t), 0 \le t \le \hat{T})$, our aim is to construct a progressively measurable process $L$ such that $X$ can be written as an optional projection of the form

$$X(s) = \mathbb{E}\left[\int_s^{\hat{T}} f\left(t, \sup_{s \le v \le t} L(v)\right) dt \,\middle|\, \mathcal{F}_s\right], \qquad 0 \le s \le \hat{T},$$

where $f = f(t, l)$ is a prescribed function which strictly decreases in $l$.

This representation problem has some surprising connections to a variety of stochastic optimization problems.

Indeed, our original interest in this problem comes from a singular stochastic control problem arising in economics, namely the problem of optimal consumption choice when consumption preferences are given through a Hindy–Huang–Kreps utility functional; see Hindy, Huang and Kreps (1992) and Hindy and Huang (1993). In a general semimartingale setting, Bank and Riedel (2001) show how to reduce this optimization problem to a representation problem of the above type. The process $X$ is given in terms of a stochastic price process, the function $f$ is determined by consumption preferences, and it turns out that a solution $L$ to the corresponding representation problem yields an explicit description of the optimal consumption plan. Hence, in this context, a representation problem of the above type serves as a substitute for the Hamilton–Jacobi–Bellman equation which extends beyond the Markovian framework.

There is also a close connection to dynamic allocation problems where one has to spend a limited amount of effort to a number of different projects. Each of these projects accrues a specific reward proportional to the effort spent on it.

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Of course, one wishes to allocate the available effort to the given projects so as to maximize the total expected reward. It is well known that solutions to such problems can be described in terms of so-called Gittins indices which provide a dynamic performance measure for each single project; see, for example, El Karoui and Karatzas (1994). It turns out that a solution to our representation problem coincides with such a Gittins index process when $X$ describes the cumulative discounted rewards from a given project and $f$ yields the discount factor.

Moreover, for the special case where $f$ takes the separable form $f(t,l) = -g(t)l$, we shall see that a solution $L$ of the representation problem provides the value process for the nonstandard optimal stopping problems

$$L(s) = \operatorname*{ess\,inf}_{T} \frac{\mathbb{E}[X(T) - X(s)|\mathcal{F}_s]}{\mathbb{E}[\int_s^T g(t)\,dt|\mathcal{F}_s]}, \qquad 0 \le s < \hat{T},$$

where the ess inf is taken over all stopping times $T > s$. For special choices of $X$, such problems also occur in Gittins index theory; see, for example, Karatzas (1994). We also refer to Morimoto (1991) for further discussion.

Finally, a solution to our representation problem provides a solution to a certain obstacle problem of Skorohod type. In this problem, the process $X$ describes some randomly fluctuating obstacle and one seeks to construct a semimartingale $Y$ with Doob–Meyer decomposition

$$dY(t) = f\big(t, A(t)\big)\,dt + dM(t) \quad \text{and} \quad Y(\hat{T}) = 0$$

such that $Y \le X$ where $A$ is an adapted, right continuous and increasing process satisfying the flat-off condition

$$\mathbb{E}\int_0^{\hat{T}} |X(t) - Y(t)|\,dA(t) = 0.$$

We will show that, for a large class of optional processes $X$, there is a unique process $A$ with these properties, namely the running supremum $A(t) \triangleq \sup_{0\le v\le t+} L(v)$ of a solution $L$ to our representation problem for $X$.

Let us now describe our results and techniques in greater detail.

We start with a general uniqueness result and show in Theorem 1 that, up to optional sections, there can be at most one upper-right continuous, progressively measurable solution $L$ to the above representation problem.

For the question of existence, we first focus on the case when $X$ is given by a deterministic function $x : [0, \hat{T}] \to \mathbb{R}$, that is, we look for a deterministic function $l$ such that

$$x(s) = \int_s^{\hat{T}} f\Big(t, \sup_{s\le v\le t} l(v)\Big)\,dt \qquad \text{for all } 0 \le s \le \hat{T}.$$

Our construction of such a function $l$ is based on an inhomogeneous notion of convexity which allows us to account for the time-inhomogeneity introduced by the function $f$. We develop analogues to the basic properties of usual convexity. In particular, we introduce the inhomogeneously convex envelope of a given function. In terms of these envelopes, we explicitly construct the solution $l$ to the above problem if $x$ is lower-semicontinuous. More precisely, Theorem 2 reveals that precisely the lower-semicontinuous functions $x$ with $x(\hat{T}) = 0$ can be represented in the above form when $l$ varies over the deterministic upper-semicontinuous functions.

Existence of a solution in the general stochastic case is established in Theorem 3. The proof of this theorem uses techniques developed by El Karoui and Karatzas (1994) in their investigation of Gittins' problem of optimal dynamic scheduling. The main idea is to consider a family of auxiliary optimal stopping problems of Gittins type whose value functions in the end allow us to describe the solution to our original representation problem. These auxiliary Gittins problems are analyzed by means of the "théorie generale" of Snell envelopes as it is developed in El Karoui (1981).

The paper is organized as follows: in Section 2 we give a precise formulation of our representation problem and present the main results. Section 3 explains the connections between the representation problem and the mentioned optimization and obstacle problems in more detail. Proofs and some supplementary results are contained in Section 4; the more technical arguments are relegated to the Appendix.

**2. Formulation of the problem and main results.** Let $X = (X(t), 0 \leq t \leq \hat{T})$ be a real-valued optional process on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual conditions of right continuity and completeness. The time horizon for our setting is $\hat{T} \in [0, +\infty]$, and we assume that $X$ is of class (D), that is, the family of random variables $(X(T), T \leq \hat{T}$ a stopping time) is uniformly $\mathbb{P}$-integrable with $X(\hat{T}) = 0$, $\mathbb{P}$-a.s. Furthermore we consider a function $f$ satisfying the following assumption.

ASSUMPTION 1. The mapping $f : \Omega \times [0, \hat{T}] \times \mathbb{R} \to \mathbb{R}$ satisfies:

(i) For each $\omega \in \Omega$ and any $t \in [0, \hat{T}]$, the function $f(\omega, t, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous and strictly decreasing from $+\infty$ to $-\infty$.

(ii) For any $l \in \mathbb{R}$, the stochastic process $f(\cdot, \cdot, l) : \Omega \times [0, \hat{T}] \to \mathbb{R}$ is progressively measurable with

$$\mathbb{E} \int_0^{\hat{T}} |f(t, l)| \, dt < +\infty.$$

We ask under which conditions there exists a progressively measurable process $L = (L(t), 0 \leq t < \hat{T})$ such that $X$ coincides with the optional projection

$$X = {}^{O}\left( \int_s^{\hat{T}} f\left(t, \sup_{s \leq v \leq t} L(v)\right) dt, 0 \leq s \leq \hat{T} \right). \tag{1}$$

To which extent is such a process $L$ uniquely determined?

We can state (1) in the equivalent form of the stochastic backward equation

$$X(S) = \mathbb{E}\left[\int_S^{\hat{T}} f\left(t, \sup_{S \le v \le t} L(v)\right) dt \,\middle|\, \mathcal{F}_S\right], \qquad \mathbb{P}\text{-a.s. for all stopping times } S \le \hat{T}.$$

In order to ensure that the right-hand side in the above expresion makes sense, we shall follow the convention that whenever we say that a progressively measurable process $L$ is a solution to our representation problem (1) this implies in particular that $L$ satisfies the integrability condition

$$(2) \qquad f\left(t, \sup_{S \le v \le t} L(v)\right) \mathbb{1}_{[S,\hat{T})}(t) \in L^1(\mathbb{P} \otimes dt) \qquad \text{for any stopping time } S \le \hat{T}.$$

NOTATION. For the sake of notational simplicity, let us introduce the following sets of stopping times:

$$\mathcal{S} \triangleq \{T : \Omega \to [0, \hat{T}] \mid T \text{ is a stopping time}\} \quad \text{and} \quad \hat{\mathcal{S}} \triangleq \{T \in \mathcal{S} \mid T < \hat{T}, \mathbb{P}\text{-a.s.}\}.$$

Given a stopping time $S \in \mathcal{S}$, we shall furthermore make frequent use of

$$\mathcal{S}(S) \triangleq \{T \in \mathcal{S} \mid T \ge S, \mathbb{P}\text{-a.s.}\}$$

and

$$\mathcal{S}^{>}(S) \triangleq \{T \in \mathcal{S} \mid T > S, \mathbb{P}\text{-a.s. on } \{S < \hat{T}\}\}.$$

Our first result concerning representation problem (1) is the following uniqueness theorem:

THEOREM 1. *Under Assumption* 1, *any progressively measurable*, *upper-right continuous solution L to our representation problem* (1) *satisfies*

$$(3) \qquad L(S) = \operatorname*{ess\,inf}_{T \in \mathcal{S}^{>}(S)} l_{S,T} \qquad \textit{for every stopping time } S \in \hat{\mathcal{S}}$$

*where*, *for* $S \in \hat{\mathcal{S}}$ *and* $T \in \mathcal{S}^{>}(S)$, $l_{S,T}$ *is the unique* (*up to a* $\mathbb{P}$-*null set*) $\mathcal{F}_S$-*measurable random variable satisfying*

$$\mathbb{E}[X(S) - X(T) | \mathcal{F}_S] = \mathbb{E}\left[\int_S^T f(t, l_{S,T})\, dt \,\middle|\, \mathcal{F}_S\right].$$

*In particular, the solution to* (1) *is uniquely determined on* $[0, \hat{T})$ *up to optional sections*.

The proof of this result will be given in Section 4.1. Lemma 4.1 in the same section shows that with any solution to our representation problem also its upper-right continuous modification satisfies (1). Thus, the assumption of upper-right continuity of a solution $L$ made in the above theorem comes without loss of generality.

In Section 4.2 we treat the deterministic case, where the process $X$ can be identified with a function $x:[0,\hat{T}]\to\mathbb{R}$ and where Assumption 1 reduces to:

ASSUMPTION 1′. The mapping $f=f(t,l):[0,\hat{T}]\times\mathbb{R}\to\mathbb{R}$ is Lebesgue integrable in $t\in[0,\hat{T}]$ for any fixed $l\in\mathbb{R}$, and continuous and strictly decreasing from $+\infty$ to $-\infty$ in $l\in\mathbb{R}$ for any fixed $t\in[0,\hat{T}]$.

In this framework, we shall show that the characterization (3) of Theorem 1 naturally leads to a generalized notion of convexity which accounts for the problem's time-inhomogeneity due to the function $f$. Specifically, we shall call a function $x:[0,\hat{T}]\to\mathbb{R}$ $(-f)$-convex if for any $s<t<u$ in $[0,\hat{T}]$ we have

$$x(t)\le x(s)+\int_s^t\bigl(-f(v,l_{s,u})\bigr)\,dv$$

where $l_{s,u}\in\mathbb{R}$ is the unique constant satisfying

$$x(u)=x(s)+\int_s^u\bigl(-f(v,l_{s,u})\bigr)\,dv.$$

In the homogeneous case, where $f(t,l)=-l$ this constant $l_{s,u}$ turns into the usual difference quotient and the above inequality corresponds to the usual condition that a secant in the graph of a convex function always stays above this function. Section 4.2.1 shows how the usual properties of convexity carry over to this generalized notion of convexity. In particular, this section proves absolute continuity of $(-f)$-convex functions and establishes existence of $(-f)$-convex envelopes. These concepts will be used in Section 4.2.2 in order to obtain:

THEOREM 2. *Under Assumption* 1′, *any lower-semicontinuous function* $x:[0,\hat{T}]\to\mathbb{R}$ *with* $x(\hat{T})=0$ *admits a representation*

$$x(s)=\int_s^{\hat{T}}f\Bigl(t,\sup_{s\le v\le t}l(v)\Bigr)\,dt,\qquad 0\le s\le\hat{T},\tag{4}$$

*where* $l:[0,\hat{T})\to\mathbb{R}\cup\{-\infty\}$ *is a uniquely determined upper-semicontinuous function such that, for each* $s\in[0,\hat{T}]$, *the above integrand* $f(\cdot,\sup_{s\le v\le\cdot}l(v))$ *is Lebesgue-integrable over* $[s,\hat{T}]$. *This function* $l$ *is given by*

$$-f(s,l(s))=(\partial^+\breve{x}^s)(s),\qquad 0\le s<\hat{T},\tag{5}$$

*where* $\partial^+\breve{x}^s$ *denotes the density for the* $(-f)$*-convex envelope* $\breve{x}^s$ *of the restriction* $x|_{[s,\hat{T}]}$ *introduced in Convention* 4.7.

*Conversely, any function* $x:[0,\hat{T}]\to\mathbb{R}$ *which admits such a representation is lower-semicontinuous.*

In Section 4.3, we deal with the general stochastic case and study the representation problem for optional processes $X$. Here, results from the "théorie générale" of Snell envelopes [El Karoui (1981)] and techniques from the theory of Gittins indices [El Karoui and Karatzas (1994)] allow us to prove the following existence theorem:

THEOREM 3. *Under Assumption* 1, *every optional process* $X$ *of class* (D) *which is lower-semicontinuous in expectation with* $X(\hat{T}) = 0$ *admits a representation of the form* (1), *that is*, *it can be written as*

$$X(S) = \mathbb{E}\left[\int_S^{\hat{T}} f\left(t, \sup_{S \le v \le t} L(v)\right) dt \middle| \mathcal{F}_S\right], \qquad S \in \mathcal{S},$$

*for some suitable optional process* $L$ *taking values in* $\mathbb{R} \cup \{-\infty\}$ *which satisfies the integrability condition* $f(t, \sup_{S \le v \le t} L(v)) \mathbb{1}_{[S,\hat{T})}(t) \in L^1(\mathbb{P} \otimes dt)$ *for any stopping time* $S \le \hat{T}$.

In fact, in Section 4.3 we shall construct such a process $L$ explicitly in terms of certain Snell envelopes.

REMARK 2.1. All the above theorems remain valid when passing from a deterministic to a random time horizon $\hat{T}$, provided this is a predictable stopping time. Moreover, Lebesgue measure $dt$ can always be replaced by any nonnegative random Borel measure $\mu(\omega, dt)$ on $[0, \hat{T}]$, $\omega \in \Omega$, whose distribution function $(\mu([0,t]), 0 \le t \le \hat{T})$ defines an adapted process with strictly increasing, continuous paths. Indeed, using the time change induced by this strictly increasing, continuous process, one can reduce this more general case to the one considered above.

**3. Relation to optimization and obstacle problems.** Let us now give a more detailed discussion of the various relations between our representation problem and the optimization or obstacle problems mentioned in the Introduction.

3.1. *A singular control problem arising in economics.* The original interest in our representation problem (1) stems from a singular control problem arising in the microeconomic theory of intertemporal consumption choice. In this problem, one considers an economic agent whose preferences on the set of cumulative consumption plans

$$\mathcal{C} \triangleq \{C \ge 0 \mid C \text{ is a right-continuous, increasing and adapted process}\}$$

are given through an expected utility functional of the nontime additive form

$$\mathbb{E}U(C) \triangleq \mathbb{E} \int_0^{\hat{T}} u\big(t, Y^C(t)\big)\, dt$$

with

$$Y^C(t) \triangleq \eta e^{-\beta t} + \int_0^t \beta e^{-\beta(t-s)}\, dC(s).$$

Such preferences have been proposed by Hindy, Huang and Kreps (1992). The exponential average of past consumption $Y^C(t)$ is interpreted as the level of satisfaction which the agent derives from his consumption up to time $t$; $\eta \geq 0$ is the initial level of satisfaction, $\beta > 0$ a discount rate. The felicity function $u = u(t, y)$ is assumed to be strictly concave and increasing in $y \in [0, +\infty)$ for fixed $t \in [0, \hat{T}]$ with continuous partial derivative $\partial_y u(t, y) \in L^1(dt)$ for any $y > 0$ and $\partial_y u(t, 0) \equiv +\infty$, $\partial_y u(t, +\infty) \equiv 0$.

Given some wealth $w > 0$ and an optional discount process $\psi > 0$, the agent's problem is then to find his most preferred consumption plan in his budget-feasible set

$$\mathcal{A}(w) \triangleq \left\{ C \in \mathcal{C} \,\middle|\, \mathbb{E} \int_0^{\hat{T}} \psi(t)\, dC(t) \leq w \right\},$$

that is, he aims to

$$\text{Maximize } \mathbb{E}U(C) \qquad \text{subject to } \mathbb{E} \int_0^{\hat{T}} \psi(t)\, dC(t) \leq w.$$

Bank and Riedel (2001) use a Kuhn–Tucker characterization of optimal plans in order to show that the optimal consumption plan $C^M$ with Lagrange multiplier $M > 0$ is to always consume "just enough" to keep the induced level of satisfaction $Y^{C^M}$ above the solution $L^M$ of the representation problem

$$M\psi(S) = \mathbb{E}\left[ \int_S^{\hat{T}} \partial_y u\left(t, \sup_{S \leq v \leq t} \{L^M(v) e^{\beta(v-t)}\}\right) \beta e^{-\beta(t-S)}\, dt \,\middle|\, \mathcal{F}_S \right], \qquad S \in \hat{\mathcal{S}}.$$

With

$$X(t) \triangleq \frac{M}{\beta} \psi(t) e^{-\beta t} \mathbb{1}_{[0,\hat{T})}(t), \qquad f(t, l) \triangleq \begin{cases} \partial_y u(t, -e^{-\beta t}/l) e^{-\beta t}, & l < 0, \\ -l, & l \geq 0, \end{cases}$$

and

$$L(v) \triangleq -\frac{e^{-\beta v}}{L^M(v)},$$

this problem takes the form (1) of the representation problem studied in the present paper.

3.2. *Gittins' problem of optimal dynamic scheduling.* The Gittins problem amounts to finding an optimal allocation rule for a certain number of independent projects. When worked on, each of these projects accrues a specific stochastic reward proportional to the effort spent on the project. The aim is to allocate the available effort on the given projects so as to maximize the total expected reward.

Gittins' celebrated idea to solve this typically high-dimensional optimization problem was to introduce a family of simpler benchmark problems which allowed him to define a dynamic performance measure—later called Gittins index—for each of the original projects separately. An optimal schedule could then be given in form of an index rule: "always spent your effort on the projects with currently maximal Gittins index."

To describe the connection between the Gittins index and our representation problem (1), let us review some of the results on Gittins' auxiliary benchmark problems which can be found in El Karoui and Karatzas (1994). These authors consider a project whose reward at time $t$ is given by some stochastic rate $h(t) > 0$. With this project, they associate the family of optimal stopping problems

$$V(s,m) \triangleq \operatorname*{ess\,sup}_{T \in \mathcal{S}(s)} \mathbb{E}\left[\int_s^T e^{-\alpha(t-s)} h(t)\,dt + m e^{-\alpha(T-s)} \middle| \mathcal{F}_s\right], \qquad s,m \geq 0. \tag{6}$$

The constant $m$ is interpreted as a reward upon stopping, the optimization starts at time $s$ and $\alpha > 0$ is a constant discount rate.

El Karoui and Karatzas (1994) show that, under appropriate conditions, the Gittins index $M(s)$ of this project at time $s$ can, loosely speaking, be described as the minimal reward-upon-stopping such that immediate termination of the project is optimal in the auxiliary stopping problem (6):

$$M(s) = \inf\{m \geq 0 \mid V(s,m) = m\}, \qquad s \geq 0. \tag{7}$$

Without making further use of it, they also establish the alternative representation

$$M(s) = \operatorname*{ess\,sup}_{T \in \mathcal{S}^{>}(s)} \frac{\mathbb{E}[\int_s^T e^{-\alpha t} h(t)\,dt \,|\, \mathcal{F}_s]}{\mathbb{E}[\int_s^T \alpha e^{-\alpha t}\,dt \,|\, \mathcal{F}_s]}, \qquad s \geq 0, \tag{8}$$

which is provided as equation (3.11) in their Proposition 3.4. Note that this identity becomes precisely our equation (10) which characterizes the solution $L$ to the representation problem (1) in the special case where $\hat{T} \triangleq +\infty$ and where

$$f(t,l) \triangleq -\alpha e^{-\alpha t} l, \qquad X(t) \triangleq -\mathbb{E}\left[\int_t^{+\infty} e^{-\alpha s} h(s)\,ds \middle| \mathcal{F}_t\right], \qquad t \geq 0, l \in \mathbb{R}.$$

Moreover, in their equation (3.7), El Karoui and Karatzas (1994) note the identity

$$\begin{aligned} &\mathbb{E}\left[\int_s^{+\infty} e^{-\alpha t} h(t)\,dt \middle| \mathcal{F}_s\right] \\ &\quad = \mathbb{E}\left[\int_s^{+\infty} \alpha e^{-\alpha t} \sup_{s \leq v \leq t} M(v)\,dt \middle| \mathcal{F}_s\right], \qquad s \geq 0. \end{aligned} \tag{9}$$

For the above choices of $\hat{T}$, $f$ and $X$, this transforms into our backward formulation

$$X(s) = \mathbb{E}\left[\int_s^{\hat{T}} f\left(t, \sup_{s \le v \le t} M(v)\right) dt \,\middle|\, \mathcal{F}_s\right], \qquad s \ge 0,$$

of the representation problem. Thus, in this special case, the Gittins index $M$ for the project with rewards $(h(t), t \ge 0)$ coincides with the solution $L$ to our representation problem (1). Observe, however, that El Karoui and Karatzas consider identity (9) merely as a property of the Gittins index $M$ and not as a characterization of $M$ as the solution to a representation problem.

3.3. *Nonstandard optimal stopping problems.* Our representation problem (1) is also related to some nonstandard optimal stopping problems. Indeed, in the special case where $f(t,l) = -g(t)l$ for some constant strictly positive $dt$-integrable function $g > 0$, the characterization given by Theorem 1 takes the form of a value function for an optimal stopping problem in which one optimizes a difference quotient criterion:

$$L(S) = \operatorname*{ess\,inf}_{T \in \mathcal{S}^{>}(S)} \frac{\mathbb{E}[X(T) - X(S)|\mathcal{F}_S]}{\mathbb{E}[\int_S^T g(t)\,dt|\mathcal{F}_S]}, \qquad S \in \hat{\mathcal{S}}. \tag{10}$$

Recall that, for special choices of $X$ and $g$, such a representation has occurred in our discussion of the connection to Gittins indices in the previous section; see (8). In fact, it was this similarity which motivated the approach taken in Section 4.3 to prove existence of a solution to our representation problem (1).

Note furthermore that the above optimal stopping problem is not directly amenable to a solution following the standard approach via the Snell envelope. For a discussion of optimal stopping problems similar to (10), we refer the reader also to Morimoto (1991).

3.4. *A variant of Skorohod's obstacle problem.* Let us now view the optional process $X$ as a randomly fluctuating obstacle on the real line. We then may consider the set of class (D) processes $Y$ which never exceed the obstacle $X$ and which follow a backward semimartingale dynamics of the form

$$dY(t) = -f\big(t, A(t)\big)\,dt + dM(t) \quad \text{and} \quad Y(\hat{T}) = 0 \tag{11}$$

for some uniformly integrable martingale $M$ and for some increasing, right-continuous and adapted process $A$ satisfying $f(t, A(t))\mathbb{1}_{[0,\hat{T}]}(t) \in L^1(\mathbb{P} \otimes dt)$. Rewriting the above equation in integral form and taking conditional expectations, we see that any such process is of the form

$$Y(s) = \mathbb{E}\left[\int_s^{\hat{T}} f\big(t, A(t)\big)\,dt \,\middle|\, \mathcal{F}_s\right], \qquad 0 \le s \le \hat{T}. \tag{12}$$

In particular, the martingale $M$ is uniquely determined by $A$ and the terminal condition $Y(\hat{T}) = 0$ as

$$M(s) = \mathbb{E}\left[\int_0^{\hat{T}} f(t, A(t))\,dt \,\middle|\, \mathcal{F}_s\right], \qquad 0 \le s \le \hat{T}.$$

Since $f$ is decreasing to $-\infty$ in its second argument, it is easy to see that many increasing, right-continuous adapted processes $A$ induce a process $Y$ with dynamics (11) [or (12)] which stays below the obstacle, that is, which satisfies $Y \le X$. However, one may wonder whether there is any such process $A$ which does so in the minimal sense that it only increases at points in time when its associated process $Y$ hits the obstacle $X$, that is, so that it satisfies the flat-off condition

$$\mathbb{E}\int_0^{\hat{T}} |X(s) - Y(s)|\,dA(s) = 0. \tag{13}$$

This problem may be viewed as a variant of the classic Skorohod problem to construct an adapted increasing process $A$ such that $X + A \ge 0$ under the restriction that $A$ only increases in times $s$ when $X(s) + A(s) = 0$.

REMARK 3.1. For a nondecreasing function $A : [0, \hat{T}) \to \mathbb{R} \cup \{-\infty\}$, we call $t \in [0, \hat{T})$ a point of increase if, for any $\varepsilon > 0$, we have $A(t-) < A(t + \varepsilon)$. By convention $A(0-) = -\infty$ and, thus, 0 is a time of increase unless $A(t) \equiv -\infty$ on a nondegenerate interval containing 0, a singular case which is ruled out in the sequel by Lebesgue integrability of $f(t, A(t))$.

This definition entails in particular that if $A$ is given as a running supremum $A(t) = \sup_{0 \le v \le t+} L(v)$ over some function $L$ then $A(t) = \limsup_{s \searrow t} L(s)$ for any point of increase $t$ of $A$. Moreover, for a right-continuous function $Y$ and a lower right-semicontinuous function $X$ with $X \ge Y$ and $\int_0^{\hat{T}} |X(s) - Y(s)|\,dA(s) = 0$, we have $X(t) = Y(t)$ for any point of increase $t$ of $A$. In particular, $X(0) = Y(0)$ unless $A(t) \equiv -\infty$ on a nondegenerate interval containing 0.

Given a solution $L$ to our representation problem for $X$, existence and uniqueness of a solution to our variant of the Skorohod problem can be obtained easily:

PROPOSITION 3.2. *If $L$ solves the representation problem for the obstacle process $X$ then the right-continuous version of its running supremum*

$$A(t) \triangleq \sup_{0 \le v \le t+} L(v), \qquad 0 \le t < \hat{T}, \tag{14}$$

*is the unique adapted, increasing and right-continuous process such that $Y$ with* (12) *stays below $X$ and which is minimal in the sense of the flat-off condition* (13).

PROOF. First note that by the integrability assumption (2), the process $A$ defined by (14) satisfies $\mathbb{E}\int_0^{\hat{T}} |f(t, A(t))|\, dt < +\infty$ and the associated process $Y$ with (12) is of class (D). Thus, to verify that $A$ has the desired properties, only the flat-off condition (13) remains to be checked. By the properties of optional projections and by definition of $L$ and $A$, we have

$$\begin{aligned}
&\mathbb{E}\int_0^{\hat{T}} \{X(s) - Y(s)\}\, dA(s) \\
&\quad = \mathbb{E}\int_0^{\hat{T}} \left\{\int_s^{\hat{T}} \left[f\left(t, \sup_{s \le v \le t} L(v)\right) - f\left(t, \sup_{0 \le v \le t+} L(v)\right)\right] dt\right\} dA(s) \\
&\quad = \mathbb{E}\int_0^{\hat{T}} \left\{\int_0^{t} \left[f\left(t, \sup_{s \le v \le t+} L(v)\right) - f\left(t, \sup_{0 \le v \le t+} L(v)\right)\right] dA(s)\right\} dt.
\end{aligned}$$

Now observe that, if $s$ is a point of increase for $A(\omega, \cdot)$, then $\sup_{s \le v \le t+} L(\omega, v) = \sup_{0 \le v \le t+} L(\omega, v)$. Thus the $dA(s)$-integral in the above expression vanishes, and we obtain the desired flat-off condition.

In order to prove uniqueness, let $A'$ be another adapted, increasing and right-continuous process with

$$\mathbb{E}\int_0^{\hat{T}} \left|f\left(t, A'(t)\right)\right| dt < +\infty$$

such that the right-continuous version of the class (D) process

$$Y'(s) \triangleq \mathbb{E}\left[\int_s^{\hat{T}} f\left(t, A'(t)\right) dt \,\middle|\, \mathcal{F}_s\right], \qquad 0 \le s \le \hat{T},$$

satisfies $Y' \le X$ and such that the flat-off condition

$$\mathbb{E}\int_0^{\hat{T}} |X(s) - Y'(s)|\, dA'(s) = 0$$

holds true. Fix $\varepsilon > 0$ and consider the stopping times

$$S^\varepsilon \triangleq \inf\left\{t \in [0, \hat{T}) \mid A(t) > A'(t) + \varepsilon\right\} \wedge \hat{T} \in \mathcal{S}$$

and

$$T^\varepsilon \triangleq \inf\left\{t \in [S^\varepsilon, \hat{T}) \mid A'(t) > A(t) - \varepsilon/2\right\} \wedge \hat{T} \in \mathcal{S}(S^\varepsilon).$$

Then, by right continuity of $A$ and $A'$, we have $T^\varepsilon > S^\varepsilon$ almost surely on $\{S^\varepsilon < \hat{T}\}$. Moreover, since $S^\varepsilon$ is a point of increase for $A$ on $\{S^\varepsilon < \hat{T}\}$, the flat-off condition entails

$$X(S^\varepsilon) = Y(S^\varepsilon) = \mathbb{E}\left[\int_{S^\varepsilon}^{T^\varepsilon} f\left(t, A(t)\right) dt \,\middle|\, \mathcal{F}_{S^\varepsilon}\right] + \mathbb{E}\left[Y(T^\varepsilon) | \mathcal{F}_{S^\varepsilon}\right].$$

By definition of $T^\varepsilon$ and monotonicity of $f$, the first of these conditional expectations is $< \mathbb{E}[\int_{S^\varepsilon}^{T^\varepsilon} f(t, A'(t))\,dt|\mathcal{F}_{S^\varepsilon}]$ on $\{T^\varepsilon > S^\varepsilon\} \supseteq \{S^\varepsilon < \hat{T}\}$ while the second is always $\leq \mathbb{E}[X(T^\varepsilon)|\mathcal{F}_{S^\varepsilon}]$ since $Y \leq X$ by assumption. Hence, on $\{S^\varepsilon < \hat{T}\}$ we obtain the contradiction that almost surely

$$\begin{aligned} X(S^\varepsilon) &< \mathbb{E}\biggl[\int_{S^\varepsilon}^{T^\varepsilon} f\bigl(t, A'(t)\bigr)\,dt\bigg|\mathcal{F}_{S^\varepsilon}\biggr] + \mathbb{E}\bigl[X(T^\varepsilon)|\mathcal{F}_{S^\varepsilon}\bigr] \\ &= \mathbb{E}\biggl[\int_{S^\varepsilon}^{T^\varepsilon} f\bigl(t, A'(t)\bigr)\,dt\bigg|\mathcal{F}_{S^\varepsilon}\biggr] + \mathbb{E}\bigl[Y'(T^\varepsilon)|\mathcal{F}_{S^\varepsilon}\bigr] \\ &= Y'(S^\varepsilon) \leq X(S^\varepsilon), \end{aligned}$$

where for the first equality we used $Y'(T^\varepsilon) = X(T^\varepsilon)$ a.s. This holds true trivially on $\{T^\varepsilon = \hat{T}\}$ as $X(\hat{T}) = 0$ by assumption, and it holds true also on $\{T^\varepsilon < \hat{T}\}$ since on this set $T^\varepsilon$ is a point of increase for $A'$. It follows that $\mathbb{P}[S^\varepsilon < \hat{T}] = 0$, that is, $A \leq A' + \varepsilon$ on $[0, \hat{T})$ almost surely. Since $\varepsilon$ was arbitrary, this entails $A \leq A'$ on $[0, \hat{T})$, $\mathbb{P}$-a.s. Reversing the roles of $A$ and $A'$ in the above argument finally yields the assertion. □

**4. Proofs and supplementary results.** In this section we will give the proofs for our main results. Section 4.1 proves uniqueness of a solution to our representation problem (1), Section 4.2 constructs a solution in the deterministic case and, finally, Section 4.3 is devoted to the question of existence in the general stochastic case.

4.1. *Uniqueness.* As a first step to prove uniqueness of a solution to our representation problem (1), let us note the following:

LEMMA 4.1. *If $L$ is a progressively measurable process satisfying* (1), *so is its upper-right-continuous modification*

$$\tilde{L}(t) \triangleq \limsup_{s \searrow t} L(s) = \lim_{\varepsilon \downarrow 0} \sup_{s \in [t, (t+\varepsilon) \wedge \hat{T}]} L(s), \qquad 0 \leq t < \hat{T}.$$

PROOF. Due to Théorème IV.2.33 in Dellacherie and Meyer (1975), the upper-right-continuous process $\tilde{L}$ is again progressively measurable. Moreover, we have for each $\omega \in \Omega$ and all $s \in [0, \hat{T})$ that

$$\sup_{s \leq v \leq t} L(\omega, v) = \sup_{s \leq v \leq t} \tilde{L}(\omega, v)$$

at every point $t \in (s, \hat{T})$ where the increasing function on the left-hand side in this equation does not jump. Since, for fixed $\omega$ and $s$, this happens at most a countable number of times, we obtain

$$\int_s^{\hat{T}} f\Bigl(\omega, t, \sup_{s \leq v \leq t} L(\omega, v)\Bigr)\,dt = \int_s^{\hat{T}} f\Bigl(\omega, t, \sup_{s \leq v \leq t} \tilde{L}(\omega, v)\Bigr)\,dt$$

for every $s \in [0, \hat{T}]$ and all $\omega \in \Omega$. Consequently, we can indeed replace $L$ by $\tilde{L}$ in (1) without changing the optional projection. $\square$

For the sake of completeness, let us note the following.

LEMMA 4.2. *For any $S \in \hat{\mathcal{S}}$ and $T \in \mathcal{S}^{>}(S)$, there is a unique random variable $l_{S,T} \in L^0(\mathcal{F}_S)$ such that*

$$\mathbb{E}\left[\int_S^T f(t, l_{S,T})\,dt \,\middle|\, \mathcal{F}_S\right] = \mathbb{E}[X(S) - X(T)|\mathcal{F}_S].$$

PROOF. Uniqueness follows immediately from strict monotonicity of $f = f(t,l)$ in $l \in \mathbb{R}$. Existence is due to integrability of $f = f(t,l)$ for fixed $l$ and to surjectivity of $l \mapsto f(t,l)$. $\square$

PROOF OF THEOREM 1 (Uniqueness). Fix a stopping time $S \in \hat{\mathcal{S}}$. Consider $T \in \mathcal{S}^{>}(S)$ and use the representation property of $L$ and the integrability assumption (2) to write

$$X(S) = \mathbb{E}\left[\int_S^T f\left(t, \sup_{S \le v \le t} L(v)\right) dt \,\middle|\, \mathcal{F}_S\right] + \mathbb{E}\left[\int_T^{\hat{T}} f\left(t, \sup_{S \le v \le t} L(v)\right) dt \,\middle|\, \mathcal{F}_S\right].$$

As $f(t, \cdot)$ is decreasing, we may estimate the first integrand from above by $f(t, L(S))$ and the second integrand by $f(t, \sup_{T \le v \le t} L(v))$ to obtain

$$X(S) \le \mathbb{E}\left[\int_S^T f\big(t, L(S)\big)\,dt \,\middle|\, \mathcal{F}_S\right] + \mathbb{E}\left[\int_T^{\hat{T}} f\left(t, \sup_{T \le v \le t} L(v)\right) dt \,\middle|\, \mathcal{F}_S\right].$$

From the representation property of $L$ at time $T$, it follows that we may write the second of the above two summands as

$$\mathbb{E}\left[\int_T^{\hat{T}} f\left(t, \sup_{T \le v \le t} L(v)\right) dt \,\middle|\, \mathcal{F}_S\right] = \mathbb{E}[X(T)|\mathcal{F}_S]$$

and, therefore, we get the estimate

$$\mathbb{E}[X(S) - X(T)|\mathcal{F}_S] \le \mathbb{E}\left[\int_S^T f\big(t, L(S)\big)\,dt \,\middle|\, \mathcal{F}_S\right].$$

As $L(S)$ is $\mathcal{F}_S$-measurable [Dellacherie and Meyer (1975), Théorème IV.64b], this shows $L(S) \le l_{S,T}$ almost surely. Since in the above estimate $T \in \mathcal{S}^{>}(S)$ is arbitrary, we deduce

$$L(S) \le \operatorname*{ess\,inf}_{T \in \mathcal{S}^{>}(S)} l_{S,T}.$$

For the converse inequality, consider the sequence of stopping times

$$T^n \triangleq \inf\left\{t \ge S \,\middle|\, \sup_{S \le v \le t} L(v) > K^n\right\} \wedge \hat{T}, \qquad n = 1, 2, \dots,$$

where

$$K^n = \big(L(S) + 1/n\big)\mathbb{1}_{\{L(S)>-\infty\}} - n\mathbb{1}_{\{L(S)=-\infty\}}.$$

Observe that $T^n \in \mathcal{S}^>(S)$ due to the upper-right continuity of $L$. Observe furthermore that this path property also implies $L(T^n) = \sup_{S\le v\le T^n} L(v)$ on $\{T^n < \hat{T}\}$ since on this set $T^n$ is a time of increase for $\sup_{S\le v\le \cdot} L(v)$; confer Remark 3.1. This yields

$$\sup_{S\le v\le t} L(v) = \sup_{T^n\le v\le t} L(v) \qquad \text{for all } t \in [T^n, \hat{T}).$$

Thus, we may write

$$\begin{aligned} X(S) &= \mathbb{E}\left[\int_S^{T^n} f\Big(t, \sup_{S\le v\le t} L(v)\Big)\,dt\,\Big|\mathcal{F}_S\right] + \mathbb{E}\left[\int_{T^n}^{\hat{T}} f\Big(t, \sup_{T^n\le v\le t} L(v)\Big)\,dt\,\Big|\mathcal{F}_S\right] \\ &\ge \mathbb{E}\left[\int_S^{T^n} f(t, K^n)\,dt\,\Big|\mathcal{F}_S\right] + \mathbb{E}[X(T^n)|\mathcal{F}_S], \end{aligned}$$

where the last estimate follows from our definition of $T^n$ and from the representation property of $L$ at time $T^n$. As $K^n$ is $\mathcal{F}_S$-measurable, the above estimate allows us to deduce

$$K^n \ge l_{S,T^n} \ge \operatorname*{ess\,inf}_{T\in\mathcal{S}^>(S)} l_{S,T}.$$

Now note that for $n \uparrow +\infty$, we have $K^n \downarrow L(S)$ and so we obtain

$$L(S) \ge \operatorname*{ess\,inf}_{T\in\mathcal{S}^>(S)} l_{S,T}. \qquad \square$$

4.2. *The deterministic case.* Let us now study the case of certainty where $f$ satisfies Assumption $1'$ and where $X$ can be identified with some deterministic function $x : [0, \hat{T}] \to \mathbb{R}$ such that $x(\hat{T}) = 0$. In this case, Theorem 1 shows that the only candidate for an upper-right-continuous function $l : [0, \hat{T}) \to \mathbb{R}$ with $f(t, \sup_{s\le v\le t} l(v))\mathbb{1}_{[s,\hat{T}]}(t) \in L^1(dt)$ and

$$x(s) = \int_s^{\hat{T}} f\Big(t, \sup_{s\le v\le t} l(v)\Big)\,dt \qquad \text{for all } 0 \le s \le \hat{T} \tag{15}$$

is characterized by

$$l(s) = \inf_{s<t\le\hat{T}} l_{s,t}$$

where $l_{s,t} \in \mathbb{R}$ is the unique constant satisfying

$$x(s) - x(t) = \int_s^t f(u, l_{s,t})\,du.$$

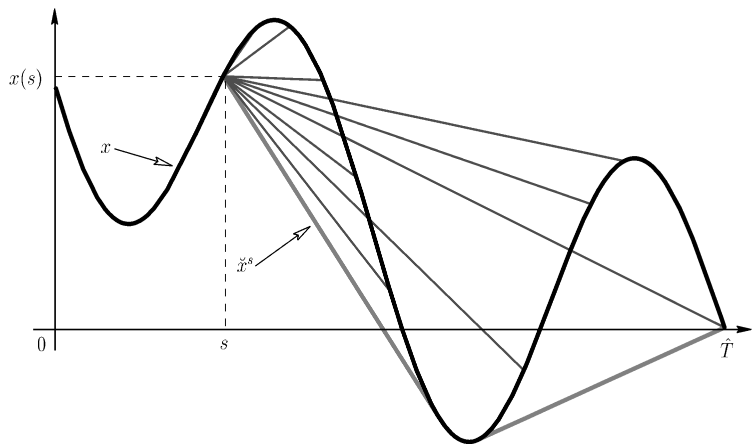


FIG. 1. *A function $x$ (thick black line), its convex envelope $\breve{x}^s$ (thick grey line), and various secants (thin lines) starting in $(s, x(s))$.*

As a motivation for our further steps to solve this problem, let us consider the special case where

$$f(t,l) \equiv -l, \qquad t \in [0, \hat{T}], l \in \mathbb{R}.$$

For this choice of $f$, it is easy to see that $l_{s,t}$ is the difference quotient

$$l_{s,t} = \frac{x(t) - x(s)}{t - s}, \qquad 0 \le s < t \le \hat{T},$$

and, thus, $l(s)$ has to be the smallest slope of a secant in the graph of $x$ which starts in $(s, x(s))$ and which ends in some point $(t, x(t))$ with $t > s$; compare Figure 1. This figure suggests a further representation of $l(s)$, namely as the initial slope $(\partial^+ \breve{x}^s)(s)$ of the convex envelope $\breve{x}^s$ of the restriction $x|_{[s,\hat{T}]}$. Indeed, as we shall see in the subsequent sections, this observation allows us to give a constructive existence proof for a solution to (15) not only in the special case considered in the above example, but also for general deterministic functions $f$ satisfying our Assumption $1'$. The main idea is to pass to a suitably generalized notion of convexity which will be introduced in the following Section 4.2.1. The proof of Theorem 2 will be given after that in Section 4.2.2.

4.2.1. *A time-inhomogeneous notion of convexity.* In this section we shall introduce an inhomogeneous notion of convexity which will prove to be useful for solving the deterministic representation problem (15). This special form of convexity accounts for the time-inhomogeneity introduced to our representation problem by the function $f$. As we shall see, it inherits many properties of usual convexity, the most important being a characterization in terms of derivatives and the existence of an inhomogeneously convex envelope of a given function.

As a framework for this section, we fix a nondegenerate interval $\mathbb{I}$ on the extended real line, that is, a connected set $\mathbb{I} \subset [-\infty, +\infty]$ with $\inf \mathbb{I} < \sup \mathbb{I}$, and we consider a measurable function $g = g(t, l) : \mathbb{I} \times \mathbb{R} \to \mathbb{R}$ which is $dt$-integrable for any $l \in \mathbb{R}$ and continuous and strictly *increasing* from $-\infty$ to $+\infty$ in $l$ for any $t \in \mathbb{I}$.

REMARK 4.3. In the subsequent application to the representation problem (15), the function $g$ will be defined as $g \stackrel{\Delta}{=} -f$ with $f$ as in Assumption 1$'$.

Now, let $x$ be an arbitrary real-valued function on $\mathbb{I}$.

DEFINITION 4.4. *We call $x$ inhomogeneously convex with respect to $g$, or $g$-convex for short, if for all $s, t, u \in \mathbb{I}$ such that $s < t < u$, we have*

$$x(t) \le x(s) + \int_s^t g(v, l_{s,u})\, dv \tag{16}$$

*where $l_{s,u} \in \mathbb{R}$ is the unique constant satisfying*

$$x(u) = x(s) + \int_s^u g(v, l_{s,u})\, dv. \tag{17}$$

*We call $x$ strictly $g$-convex if we always have strict inequality in* (16).

REMARK 4.5. The preceding definition is equivalent to the usual definition of convexity in case the function $g : \mathbb{I} \times \mathbb{R} \to \mathbb{R}$ is time homogeneous in the sense that it does not depend on its first argument.

In complete analogy to usual convexity, there are the following alternative characterizations of $g$-convexity:

PROPOSITION 4.6. *The following properties are equivalent*:

(i) *The function $x$ is* (*strictly*) *$g$-convex.*

(ii) *For all $s, t, u \in \mathbb{I}$ such that $s < t < u$ we have*

$$l_{s,t} \le l_{t,u} \qquad (\textit{resp. } l_{s,t} < l_{t,u}) \tag{18}$$

*where $l_{s,t}$ and $l_{t,u}$ are defined as in* (17).

(iii) *The function $x$ is absolutely continuous on* $\operatorname{int} \mathbb{I}$ *with a density $\dot{x}$ of the form*

$$\dot{x}(t) = g\big(t, l(t)\big), \qquad t \in \operatorname{int} \mathbb{I},$$

*for some* (*strictly*) *increasing function $l : \operatorname{int} \mathbb{I} \to \mathbb{R}$ and, on the boundary, $x$ satisfies*

$$\lim_{s \to t} x(s) \le x(t) \qquad \textit{for } t \in \partial\mathbb{I} \cap \mathbb{I}.$$

*Moreover, for boundary points $t \in \partial\mathbb{I} \cap \mathbb{I}$, $g(\cdot, l(\cdot))$ is Lebesgue-integrable over the interval $[t \wedge t_0, t \vee t_0]$ for any $t_0 \in \operatorname{int} \mathbb{I}$.*

For the proof see Appendix A.1.

CONVENTION 4.7. Since Lebesgue measure has no atoms, we may always assume the increasing function $l : \operatorname{int}\mathbb{I} \to \mathbb{R}$ of Proposition 4.6(iii) to be right continuous. Putting

$$l(t) \stackrel{\Delta}{=} \lim_{s \to t, s \in \operatorname{int}\mathbb{I}} l(s) \in [-\infty, +\infty], \qquad t \in \partial\mathbb{I} \cap \mathbb{I},$$

extends $l$ canonically to all of $\mathbb{I}$. The corresponding density of the $g$-convex function $x$ will be denoted by

$$\partial^+ x(t) = g\big(t, l(t)\big), \qquad t \in \mathbb{I}.$$

Like usual convexity, also $g$-convexity allows for a definition of convex envelopes:

DEFINITION AND PROPOSITION 4.8. *The set $\mathcal{X}$ of $g$-convex functions $\xi : \mathbb{I} \to \mathbb{R}$ which are dominated by $x$ is stable with respect to taking suprema. More precisely, if $\mathcal{X} \neq \varnothing$, there exists a pointwise maximal $g$-convex function $\breve{x} : \mathbb{I} \to \mathbb{R}$ which is dominated by $x$. This function is called the $g$-convex envelope of $x$.*

For the proof see Appendix A.1.

REMARK 4.9. If $\mathbb{I}$ is compact in $[-\infty, +\infty]$ then, for $x$ to possess a $g$-convex envelope $\breve{x}$, it is sufficient that $x$ is bounded from below. If $\mathbb{I}$ is compact in $\mathbb{R}$ this is also necessary.

Let us finally record some properties of $g$-convex envelopes in the following.

PROPOSITION 4.10. *Let $x : \mathbb{I} \to \mathbb{R}$ have $g$-convex envelope $\breve{x} : \mathbb{I} \to \mathbb{R}$ and denote by*

$$x_*(t) \stackrel{\Delta}{=} \liminf_{s \to t} x(s), \qquad t \in \mathbb{I},$$

*its lower-semicontinuous envelope. Then*:

(i) $\breve{x} = x$ *on* $\partial\mathbb{I} \cap \mathbb{I}$, *and* $\breve{x} \leq x_*$ *on* $\operatorname{int}\mathbb{I}$.

(ii) *The unique increasing, right-continuous function* $\breve{l} : \operatorname{int}\mathbb{I} \to \mathbb{R}$ *such that* $g(\cdot, \breve{l}(\cdot))$ *is a density for* $\breve{x}$ *on* $\operatorname{int}\mathbb{I}$ *induces a Borel measure* $d\breve{l}$ *on* $\operatorname{int}\mathbb{I}$ *with support*

$$\operatorname{supp} d\breve{l} \subset \big\{t \in \operatorname{int}\mathbb{I} | \breve{x}(t) = x_*(t)\big\}.$$

(iii) $\breve{x}$ *is absolutely continuous on* $\mathbb{I}$ *iff* $x$ *is lower-semicontinuous in the boundary points contained in* $\partial\mathbb{I} \cap \mathbb{I}$.

(iv) *For $t \in \mathbb{I}$, let $\breve{x}^t$ denote the $g$-convex envelope of the restriction $x|_{\mathbb{I}\cap[t,+\infty]}$. Then we have*

$$\left(\partial^+ \breve{x}^{t_1}\right)(s) \geq \left(\partial^+ \breve{x}^{t_2}\right)(s)$$

*for any $t_1, t_2, s \in \operatorname{int}\mathbb{I}$ such that $t_1 \leq t_2 \leq s$.*

For the proof see Appendix A.1.

4.2.2. *Proof of Theorem* 2. *Uniqueness* of a function $l$ with (4) follows immediately from Theorem 1.

Let us next show that $l$ *with* (5) *indeed satisfies* (4) for any $s \in [0, \hat{T}]$. For ease of notation, we put $g \overset{\Delta}{=} -f$ and we let $\breve{l}^s : [s, \hat{T}) \to [-\infty, +\infty)$ denote the unique right-continuous function such that $g(\cdot, \breve{l}^s(\cdot))$ is a Lebesgue density for the $g$-convex envelope $\breve{x}^s$ of the restriction $x^s \overset{\Delta}{=} x|_{[s,\hat{T}]} (s \in [0, \hat{T}))$. Note that these envelopes do exist by Remark 4.9, because by assumption $x$ is lower-semicontinuous and, hence, bounded from below on the compact interval $[0, \hat{T}] \subset [0, +\infty]$.

In this notation, we have to verify that $l(v) \overset{\Delta}{=} \breve{l}^v(v)$ solves the deterministic representation problem. Apply Proposition 4.10(i) and (iii) to write

$$x(s) = \breve{x}^s(s) - \breve{x}^s(\hat{T}) = -\int_s^{\hat{T}} (\partial^+ \breve{x}^s)(t)\, dt = \int_s^{\hat{T}} f\left(t, \breve{l}^s(t)\right) dt.$$

Obviously, it now suffices to show that, for all $t \in (s, \hat{T})$, we have

$$\breve{l}^s(t) = \sup_{s \leq v \leq t} \breve{l}^v(v). \tag{19}$$

By Proposition 4.10(iv), $\partial^+ \breve{x}^s(v)$ is decreasing in $s \in [0, v]$. Thus,

$$\breve{l}^v(v) \leq \breve{l}^s(v) \leq \breve{l}^s(t)$$

where, for $v \leq t$, the last estimate follows from monotonicity of $\breve{l}^s$. Taking the supremum over $v \in [s, t]$, this proves that "$\geq$" holds true in (19).

To establish the remaining $\leq$ inequality, consider the set

$$\mathcal{V} \overset{\Delta}{=} \left\{v \in [s, t] \mid \breve{x}^s(v) = x(v)\right\}$$

and let $v^* \overset{\Delta}{=} \sup \mathcal{V}$. We claim that

$$\breve{x}^s|_{[v^*,\hat{T}]} = \breve{x}^{v^*}. \tag{20}$$

For this it suffices to show that $\breve{x}^s(v^*) = x(v^*)$. To this end, let $v_n, n = 1, 2, \ldots,$ be a sequence in $\mathcal{V}$ which converges to $v^*$. Using the continuity of $\breve{x}^s$ and the lower-semicontinuity of $x$, we obtain

$$\breve{x}^s(v^*) = \lim_n \breve{x}^s(v_n) = \lim_n x(v_n) \geq \liminf_{v \to v^*} x(v) = x(v^*) \geq \breve{x}^s(v^*).$$

Consequently, equality must hold everywhere in this line and this proves our claim (20).

Now, applying first Proposition 4.10(ii) and then our claim (20), we see that

$$\breve{l}^s(t) = \breve{l}^s(v^*) = \breve{l}^{v^*}(v^*) \leq \sup_{s \leq v \leq t} \breve{l}^v(v),$$

proving "$\leq$" in (19).

We next establish the *upper-semicontinuity* of $l$. From (19) we infer that

$$\breve{l}^s(t) \geq \breve{l}^t(t)$$

for any $t > s$. For $t \downarrow s$, the left-hand side of this inequality converges to $l(s) = \breve{l}^s(s)$, while in the limit its right-hand side is not larger than $\limsup_{t \downarrow s} \breve{l}^t(t) = \limsup_{t \downarrow s} l(t)$. This proves upper-semicontinuity of $l$ from the right.

Now, consider $t < s$ and fix $u \in (s, \hat{T})$. Since $\breve{x}^t$ is $g$-convex with $\breve{x}^t(t) = x(t)$, we have

$$l(t) = \breve{l}^t(t) \leq \lambda\big(t, u, \breve{x}^t(u) - x(t)\big),$$

where $\lambda(t, u, \Delta) \in \mathbb{R}$ is the unique constant $\lambda$ with

$$\int_t^u g(v, \lambda)\, dv = \Delta.$$

As $\lambda(t, s, \Delta)$ is continuous in $(t, s, \Delta)$ and increasing in $\Delta$, the above inequality yields

$$\limsup_{t \uparrow s} l(t) \leq \lambda\left(s, u, \limsup_{t \uparrow s} \{\breve{x}^t(u) - x(t)\}\right).$$

Using $\breve{x}^t \leq \breve{x}^s$ on $[s, \hat{T}]$ and also lower-semicontinuity of $x$, we deduce the estimate

$$\limsup_{t \uparrow s} l(t) \leq \lambda\big(s, u, \breve{x}^s(u) - x(s)\big) = \lambda\big(s, u, \breve{x}^s(u) - \breve{x}^s(s)\big).$$

Due to the $g$-convexity of $\breve{x}^s$, the last expression decreases to $\breve{l}^s(s) = l(s)$ as $u \downarrow s$. This yields upper-semicontinuity of $l$ from the left.

It remains to prove the *converse assertion* that representable $x$ are necessarily lower-semicontinuous. Define

$$i_s(t) \stackrel{\Delta}{=} \mathbb{1}_{(s,\hat{T}]}(t) f\left(t, \sup_{s \leq v \leq t} l(v)\right)$$

such that $x(s) = \int_0^{\hat{T}} i_s(t)\, dt$ for all $s \in [0, \hat{T}]$. Obviously,

$$i_s(t) \geq 0 \wedge f\left(t, \sup_{0 \leq v \leq t} l(v)\right) \in L^1([0, \hat{T}], dt) \tag{21}$$

for every $s \in [0, \hat{T}]$, that is, the family of integrands $(i_s(\cdot), s \in [0, \hat{T}])$ is bounded from below by some Lebesgue-integrable function.

Now, let us show that $x(s) = \int_0^{\hat{T}} i_s(t)\,dt$ is lower-semicontinuous at each point $s^* \in [0, \hat{T}]$. Indeed, on the one hand, we have

$$\lim_{s \downarrow s^*} i_s(t) = \mathbb{1}_{(s^*, \hat{T}]}(t) f\Big(t, \sup_{s^* < v \le t} l(v)\Big) \qquad \text{for every } t \in [0, \hat{T})$$

and, because of estimate (21), we may use Fatou's lemma to obtain

$$\liminf_{s \downarrow s^*} x(s) \ge \int_0^{\hat{T}} \lim_{s \downarrow s^*} i_s(t)\,dt = \int_{s^*}^{\hat{T}} f\Big(t, \sup_{s^* < v \le t} l(v)\Big)\,dt \ge x(s^*).$$

On the other hand, we have

$$\lim_{s \uparrow s^*} i_s(t) = \mathbb{1}_{[s^*, \hat{T}]}(t) f\Big(t, \sup_{s^* \le v \le t} l(v)\Big) \qquad \text{for all } t \in [0, \hat{T}]$$

since $l(\cdot)$ is upper-semicontinuous by assumption. Thus, by Fatou's lemma again,

$$\liminf_{s \uparrow s^*} x(s) \ge \int_0^{\hat{T}} \lim_{s \uparrow s^*} i_s(t)\,dt = \int_{s^*}^{\hat{T}} f\Big(t, \sup_{s^* \le v \le t} l(v)\Big)\,dt = x(s^*).$$

Hence, $\liminf_{s \to s^*} x(s) \ge x(s^*)$ as we wanted to show.

4.3. *Existence in the stochastic case.* Let us now turn to the general stochastic case and assume that the optional process $X$ and the function $f$ satisfy the assumptions of Theorem 3.

A natural approach to prove existence in the stochastic case could be to proceed in a similar way as in the deterministic case. In such an approach, one would specify $L$ as the value process of the nonstandard optimal stopping problems (3) derived in our uniqueness Theorem 1, and try to verify then that this process indeed solves our representation problem (1). However, this would lead to tedious measurability issues as it is not obvious how to choose a *progressively measurable* version of such a value process $L$. Moreover, even granted such a version does exist, the verification that this process solves the representation problem could not be carried out along the same lines as in the deterministic case since, in the present stochastic framework, a suitable notion of convex envelopes does not seem to be available; see, however, our discussion at the end of Section 4.3.

For these reasons, we shall take a different approach, exploiting the connection between our representation problem and the Gittins index presented in Section 3.2. This connection suggests to consider the family of auxiliary optimal stopping problems

$$Y^l(S) = \operatorname*{ess\,inf}_{T \in \mathcal{S}(S)} \mathbb{E}\Big[X(T) + \int_S^T f(t, l)\,dt \Big| \mathcal{F}_S\Big], \qquad S \in \mathcal{S}, l \in \mathbb{R}, \tag{22}$$

and to introduce the associated "Gittins index" $L$ in analogy to (7) as

$$L(\omega, t) \stackrel{\Delta}{=} \sup\{l \in \mathbb{R} \mid Y^l(\omega, t) = X(\omega, t)\} \qquad \text{for } (\omega, t) \in \Omega \times [0, \hat{T}). \tag{23}$$

In contrast to formula (3) obtained in our uniqueness theorem, this representation of $L$ is given in terms of the value functions for *standard* optimal stopping problems. This will allow us to apply the well established theory of Snell envelopes as it is presented in El Karoui (1981) when verifying that $L$ given by (23) does in fact solve our representation problem (1).

To make this precise, we shall first analyze in detail the structure of the auxiliary Gittins problems (22) in the following Section 4.3.1 before we proceed to the proof of Theorem 3 in Section 4.3.2.

4.3.1. *On the family of Gittins problems* (22). Let us start our investigation of the auxiliary Gittins problems (22) and note some consequences of our assumption that $X$ be an optional process of class (D) which is lower-semicontinuous in expectation.

LEMMA 4.11. *Any process $X$ satisfying the assumptions of Theorem* 3 *has the following properties*:

(i) *There are two martingales $M_*$ and $M^*$ such that $M_* \le X \le M^*$, $\mathbb{P}$-a.s.*

(ii) *Almost surely, $X$ has paths which are lower-semicontinuous from the right at every point $t \in [0, \hat{T})$ and which are lower-semicontinuous from the left in $t = \hat{T}$.*

(iii) *$X$ satisfies a conditional version of Fatou's lemma*:

$$\liminf_n \mathbb{E}[X(T^n)|\mathcal{F}_S] \ge \mathbb{E}\Big[\liminf_n X(T^n)\Big|\mathcal{F}_S\Big], \qquad \mathbb{P}\text{-a.s.}$$

*for any $S \in \mathcal{S}$ and for every monotone sequence of stopping times $T^n \in \mathcal{S}(S)$, $n = 1, 2, \ldots$.*

For the proof see Appendix A.2.

In our second lemma we collect those results which rely on techniques from the theory of Gittins indices as developed in El Karoui and Karatzas (1994):

LEMMA 4.12. *Under the assumptions of Theorem* 3, *there is a jointly measurable mapping*

$$Y : \Omega \times [0, \hat{T}] \times \mathbb{R} \to \mathbb{R},$$
$$(\omega, t, l) \mapsto Y^l(\omega, t)$$

*with the following properties*:

(i) *For $l \in \mathbb{R}$ fixed, $Y^l : \Omega \times [0, \hat{T}] \to \mathbb{R}$ is an optional process such that*

$$Y^l(S) = \operatorname*{ess\,inf}_{T \in \mathcal{S}(S)} \mathbb{E}\Big[X(T) + \int_S^T f(t, l)\, dt \Big| \mathcal{F}_S\Big], \qquad \mathbb{P}\text{-a.s.} \tag{24}$$

*for every stopping time $S \in \mathcal{S}$.*

(ii) *For any* $l \in \mathbb{R}$, $S \in \mathcal{S}$, *the stopping time*

$$T_S^l \triangleq \inf\{t \geq S \mid Y^l(t) = X(t)\} \leq \hat{T}$$

*is optimal in* (24), *that is*,

$$Y^l(S) = \mathbb{E}\left[X(T_S^l) + \int_S^{T_S^l} f(t,l)\,dt \,\middle|\, \mathcal{F}_S\right].$$

*Moreover, this stopping time depends on* $l \in \mathbb{R}$ *in a monotone manner: for any* $S \in \mathcal{S}$ *we have*

$$T_S^l(\omega) \leq T_S^{l'}(\omega) \qquad \text{for all } \omega \in \Omega \text{ and any two } l \leq l'.$$

(iii) *For fixed* $(\omega, s) \in \Omega \times [0, \hat{T}]$, *the mapping* $l \mapsto Y^l(\omega, s)$ *is continuously decreasing from*

$$Y^{-\infty}(\omega, s) \triangleq \lim_{l \downarrow -\infty} Y^l(\omega, s) = X(\omega, s).$$

*In particular, there is a canonical extension of* $Y$ *to* $\Omega \times [0, \hat{T}] \times (\mathbb{R} \cup \{-\infty\})$.

(iv) *For every stopping time* $S \in \mathcal{S}$, *the negative random measure* $dY^l(S)$ *associated with the decreasing random mapping* $l \mapsto Y^l(S)$ *can almost surely be disintegrated in the form*

$$\int_{-\infty}^{+\infty} \phi(l)\,dY^l(S) = \mathbb{E}\left[\int_S^{\hat{T}} \left\{\int_{-\infty}^{+\infty} \phi(l) \mathbb{1}_{[S, T_S^l]}(t)\,df(t,l)\right\} dt \,\middle|\, \mathcal{F}_S\right] \tag{25}$$

*for any nonnegative*, $\mathcal{F}_S \otimes \mathcal{B}(\mathbb{R})$*-measurable* $\phi : \Omega \times \mathbb{R} \to \mathbb{R}$.

For the proof see Appendix A.2.

Taking the version of $Y = Y^l(t, \omega)$ given in the preceding lemma, we now can use (23) to define our candidate $L$ for a solution to the representation problem (1).

LEMMA 4.13. *For* $Y$ *as in Lemma* 4.12, *the process* $L$ *defined by* (23) *is optional and takes values in* $[-\infty, +\infty)$ *almost surely. Moreover, for every* $S \in \mathcal{S}$, *each of the following sets is contained in the next*:

$$\begin{aligned} A &\triangleq \left\{(\omega, t, l) \,\middle|\, l > \sup_{S(\omega) \leq v \leq t} L(\omega, v)\right\} \\ &\subset B \triangleq \{(\omega, t, l) \mid T_S^l(\omega) \geq t\} \\ &\subset C \triangleq \left\{(\omega, t, l) \,\middle|\, l \geq \sup_{S(\omega) \leq v < t} L(\omega, v)\right\}, \end{aligned}$$

*and for* $\mathbb{P} \otimes dt$*-a.e.* $(\omega, t) \in \Omega \times [0, \hat{T}]$ *the* $(\omega, t)$*-sections* $A^{(\omega,t)}$, $B^{(\omega,t)}$, $C^{(\omega,t)} \subset \mathbb{R}$ *differ by at most countably many points* $l \in \mathbb{R}$.

For the proof see Appendix A.2.

4.3.2. *Proof of Theorem* 3. With the results of the preceding section at hand, we now can give the proof of our existence theorem in the stochastic case. We proceed in three steps:

*Step* 1. We first prove that $f(t, \sup_{S\le v\le t} L(v))\mathbb{1}_{[S,T_S^l)}(t) \in L^1(\mathbb{P}\otimes dt)$ and

$$X(S) = \mathbb{E}[X(T_S^l)|\mathcal{F}_S] + \mathbb{E}\left[\int_S^{T_S^l} f\left(t, \sup_{S\le v\le t} L(v)\right) dt \middle| \mathcal{F}_S\right]$$

for every $l \in \mathbb{R}$.

Fix $l_0 \in \mathbb{R}$. The definition of $L(S)$ and the monotonicity of $l \mapsto Y^l(S)$ allow us to write

$$X(S) = Y^{L(S)}(S) = Y^{l_0}(S) - \int_{-\infty}^{+\infty} \mathbb{1}_{[L(S)\wedge l_0, l_0)}(l)\, dY^l(S).$$

Due to our disintegration formula (25) for the random measure $dY^l(S)$, the last expression is equal to

$$Y^{l_0}(S) - \mathbb{E}\left[\int_S^{\hat{T}} \left\{\int_{-\infty}^{+\infty} \mathbb{1}_{[L(S)\wedge l_0, l_0]}(l)\mathbb{1}_{\{T_S^l\ge t\}}\, df(t,l)\right\} dt \middle| \mathcal{F}_S\right].$$

Now, let $I$ denote the above conditional expectation. Since for $\mathbb{P}\otimes dt$-a.e. $(\omega, t)$ the sections $B^{(\omega,t)}$ and $C^{(\omega,t)}$ of Lemma 4.13 differ by at most countably many points $l \in \mathbb{R}$, continuity of the measures $df(t,\cdot)$ allows us to replace the set $\{T_S^l \ge t\}$ in the above expression by $\{l \ge \bar{L}(S,t)\}$ where $\bar{L}(S,t) \triangleq \sup_{S\le v\le t} L(v)$. This yields

$$\begin{aligned} I &= \mathbb{E}\left[\int_S^{\hat{T}} \left\{\int_{-\infty}^{+\infty} \mathbb{1}_{[\bar{L}(S,t)\wedge l_0, l_0]}(l)\, df(t,l)\right\} dt \middle| \mathcal{F}_S\right] \\ &= \mathbb{E}\left[\int_S^{\hat{T}} \{f(t,l_0) - f(t, \bar{L}(S,t)\wedge l_0)\}\, dt \middle| \mathcal{F}_S\right]. \end{aligned} \tag{26}$$

We claim that

$$\begin{aligned} &f(t,l_0) - f\big(t, \bar{L}(S,t)\wedge l_0\big) \\ &\quad = \big(f(t,l_0) - f(t, \bar{L}(S,t)\big)\mathbb{1}_{\{T_S^{l_0}\ge t\}}, \qquad dt\text{-a.e. on } [S, \hat{T}). \end{aligned} \tag{27}$$

Indeed, the left-hand side of this equality is equal to

$$\begin{aligned} \big(f(t,l_0) - f\big(t, \bar{L}(S,t)\big)\big)\mathbb{1}_{\{l_0 > \bar{L}(S,t)\}} &\ge \big(f(t,l_0) - f\big(t, \bar{L}(S,t)\big)\big)\mathbb{1}_{\{T_S^{l_0}\ge t\}} \\ &\ge \big(f(t,l_0) - f\big(t, \bar{L}(S,t)\big)\big)\mathbb{1}_{\{l_0\ge \bar{L}(S,t-)\}} \end{aligned}$$

where both estimates are due to the inclusions derived in Lemma 4.13. Since $\bar{L}(S,\cdot)$ is increasing in $t$, we have $\bar{L}(S,t) = \bar{L}(S,t-)$ for Lebesgue-a.e. $t$ and, therefore, the last term in the preceding estimate coincides with the first term $dt$-a.e. This proves our claim (27).

Claim (27) in conjunction with (26) gives us

$$I = \mathbb{E}\left[\int_S^{T_S^{l_0}} \{f(t,l_0) - f(t,\bar{L}(S,t))\}\,dt \,\middle|\, \mathcal{F}_S\right].$$

Since $f(t,l_0)\mathbb{1}_{[S,T_S^{l_0})}(t) \in L^1(\mathbb{P}\otimes dt)$ by Assumption (1), this shows, in particular, that

$$\mathbb{E}\left[\int_S^{T_S^{l_0}} f(t,\bar{L}(S,t))\,dt \,\middle|\, \mathcal{F}_S\right]$$

does exist as a random variable taking values in $\mathbb{R}\cup\{+\infty\}$. Resuming our initial calculation, we see that the above representation of $I$ and optimality of $T_S^{l_0}$ imply

$$\begin{aligned}
X(S) &= Y^{l_0}(S) - I \\
&= \mathbb{E}\left[X(T_S^{l_0}) + \int_S^{T_S^{l_0}} f(t,l_0)\,dt \,\middle|\, \mathcal{F}_S\right] \\
&\quad - \mathbb{E}\left[\int_S^{T_S^{l_0}} \{f(t,l_0) - f(t,\bar{L}(S,t))\}\,dt \,\middle|\, \mathcal{F}_S\right] \\
&= \mathbb{E}[X(T_S^{l_0})|\mathcal{F}_S] + \mathbb{E}\left[\int_S^{T_S^{l_0}} f(t,\bar{L}(S,t))\,dt \,\middle|\, \mathcal{F}_S\right].
\end{aligned}$$

Since $X$ is of class (D), this identity shows that the expectation of the last (a priori) generalized conditional expectation is actually finite. Hence, $f(t,\bar{L}(S,t)) \times \mathbb{1}_{[S,T_S^{l_0})}(t)$ is $\mathbb{P}\otimes dt$-integrable which completes the proof of our first assertion.

*Step* 2. We next show that

$$T_S^{+\infty} \stackrel{\Delta}{=} \lim_{l\uparrow+\infty} T_S^l = \hat{T} \quad \text{and} \quad \lim_{l\uparrow+\infty} \mathbb{E}[X(T_S^l)|\mathcal{F}_S] = 0.$$

Note first that by Lemma 4.12(ii) $l \mapsto T_S^l$ is monotone. Hence, $T_S^{+\infty}$ exists as a monotone limit of stopping times. Moreover, by optimality of $T_S^l$, we have

$$Y^l(S) = \mathbb{E}\left[X(T_S^l) + \int_S^{T_S^l} f(t,l)\,dt \,\middle|\, \mathcal{F}_S\right] \le \mathbb{E}\left[X(\hat{T}) + \int_S^{\hat{T}} f(t,l)\,dt \,\middle|\, \mathcal{F}_S\right]$$

or equivalently, as $X(\hat{T}) = 0$ by assumption,

$$\mathbb{E}[X(T_S^l)|\mathcal{F}_S] \le \mathbb{E}\left[\int_{T_S^l}^{\hat{T}} f(t,l)\,dt \,\middle|\, \mathcal{F}_S\right].$$

Hence, for any $l_0 \in \mathbb{R}$, we have by monotonicity of $f(t,\cdot)$ that

$$
\begin{aligned}
M_*(S) &\le \limsup_{l\uparrow+\infty} \mathbb{E}[X(T_S^l)|\mathcal{F}_S] \le \lim_{l\uparrow+\infty} \mathbb{E}\left[\int_{T_S^l}^{\hat{T}} f(t,l_0)\,dt \,\middle|\, \mathcal{F}_S\right] \\
&= \mathbb{E}\left[\int_{T_S^{+\infty}}^{\hat{T}} f(t,l_0)\,dt \,\middle|\, \mathcal{F}_S\right],
\end{aligned}
\tag{28}
$$

where $M_* \le X$ is the martingale of Lemma 4.11(i). For $l_0 \uparrow +\infty$, the right-hand side in this estimate tends to $-\infty$ on the set $\{\mathbb{P}[T_S^{+\infty} < \hat{T}|\mathcal{F}_S] > 0\}$ while the left-hand side yields an almost surely finite lower bound. Hence, $\mathbb{P}[T_S^{+\infty} < \hat{T}] = 0$.

Let us now show that $\lim_{l\uparrow+\infty} \mathbb{E}[X(T_S^l)|\mathcal{F}_S] = 0$. From (28) and $T_S^{+\infty} = \hat{T}$, we immediately infer that $\limsup_{l\uparrow+\infty} \mathbb{E}[X(T_S^l)|\mathcal{F}_S] \le 0$ almost surely. On the other hand, Lemma 4.11(iii) yields

$$
\liminf_{l\uparrow+\infty} \mathbb{E}[X(T_S^l)|\mathcal{F}_S] \ge \mathbb{E}\left[\liminf_{l\uparrow+\infty} X(T_S^l) \,\middle|\, \mathcal{F}_S\right], \qquad \mathbb{P}\text{-a.s.}
$$

As shown before, $\liminf_{l\uparrow+\infty} X(T_S^l) \ge \liminf_{t\nearrow\hat{T}} X(t)$, and this liminf is 0 by Lemma 4.11(ii).

*Step* 3. The results of Steps 2 allow us to let $l \uparrow +\infty$ in the representation of $X(S)$ derived in Step 1. Indeed, take an arbitrary constant $l_0 = 0$, say, and use the representation obtained in Step 1 to write for $l > l_0 = 0$:

$$
\begin{aligned}
X(S) - \mathbb{E}[X(T_S^l)|\mathcal{F}_S] &= \mathbb{E}\left[\int_S^{T_S^0} f\Big(t, \sup_{S\le v\le t} L(v)\Big)\,dt \,\middle|\, \mathcal{F}_S\right] \\
&\quad + \mathbb{E}\left[\int_{T_S^0}^{T_S^l} \Big\{ f\Big(t, \sup_{S\le v\le t} L(v)\Big) - f(t,0)\Big\}\,dt \,\middle|\, \mathcal{F}_S\right] \\
&\quad + \mathbb{E}\left[\int_{T_S^0}^{T_S^l} f(t,0)\,dt \,\middle|\, \mathcal{F}_S\right].
\end{aligned}
$$

As $l \uparrow +\infty$ the left-hand side in this formula tends to $X(S)$ as shown in Step 2. The integrand in the second summand of the right-hand side is nonpositive by Lemma 4.13. Since $T_S^l \uparrow T_S^{+\infty} = \hat{T}$ by Step 2, monotone convergence implies that the second summand converges to

$$
\mathbb{E}\left[\int_{T_S^0}^{\hat{T}} \Big\{ f\Big(t, \sup_{S\le v\le t} L(v)\Big) - f(t,0)\Big\}\,dt \,\middle|\, \mathcal{F}_S\right];
$$

in particular $\mathbb{E}[\int_{T_S^0}^{\hat{T}} f(t, \sup_{S\le v\le t} L(v))\,dt|\mathcal{F}_S]$ exists as an (a priori) generalized conditional expectation taking values in $\mathbb{R} \cup \{-\infty\}$. Finally, our integrability assumption on $f$ implies that, for $l \uparrow +\infty$, the last summand tends to $\mathbb{E}[\int_{T_S^0}^{\hat{T}} f(t,0)\,dt|\mathcal{F}_S]$. This proves

$$X(S) = \mathbb{E}\left[\int_S^{\hat{T}} f\left(t, \sup_{S \le v \le t} L(v)\right) dt \middle| \mathcal{F}_S\right]$$

from which we immediately infer that in fact the a priori generalized conditional expectation on the right-hand side has finite mean. This finally yields $f(t, \sup_{S\le v\le t} L(v))\mathbb{1}_{[S,\hat{T})}(t) \in L^1(\mathbb{P}\otimes dt)$ and completes our proof.

4.3.3. *Comparison with the deterministic case.* Let us briefly compare the above proof of existence to our proof of existence in the special case where both $X$ and $f$ are deterministic. To this end, let us reconsider the deterministic framework of Section 4.2 and relate the key concept of inhomogeneously convex envelopes with our key device in the stochastic case, the family of processes $Y^l(\cdot)$, $l \in \mathbb{R}$.

In this deterministic setting, it can be shown that $Y^l(s)$ coincides with the time $s$ value of the maximal $(-f)$-convex function $\xi^{s,l}$ on $[s, \hat{T}]$ which is dominated by $x$ on this interval and whose density $\partial^+\xi^{s,l}$ is of the form $\partial^+\xi^{s,l}(t) = -f(t, \breve{l}(t))$ for some right-continuous increasing function $\breve{l} \ge l$; compare Figure 2. The level $L(s)$ can now be reinterpreted as the maximal index $l$ for which this function $\xi^{s,l}$ actually coincides with the $(-f)$-convex envelope of $X|_{[s,\hat{T}]}$. This observation implies also that

$$s \mapsto Y^{\bar{L}(s)}(s) \qquad \text{where } \bar{L}(s) \triangleq \sup_{0\le v\le s} L(v)$$

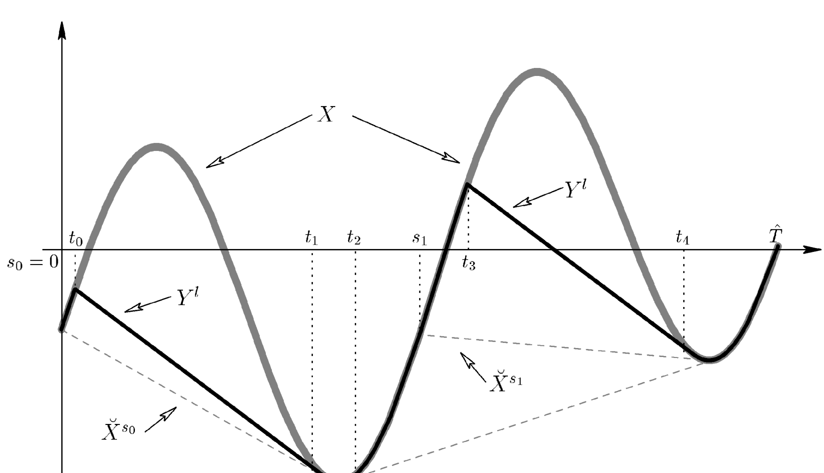


FIG. 2. *A function $X$ (thick grey line), the associated function $Y^l$ for some fixed $l$ (black line) and two convex envelopes $\breve{X}^{s_i}$, $i = 0, 1$, starting in $(s_i, X(s_i))$, respectively (dashed grey lines).*

coincides with the $(-f)$-convex envelope of $X$. For the special function $X$ considered in Figure 2, it turns out that the constant $l$ chosen to compute $Y^l$ coincides with the solution $L$ of the representation problem precisely at the times $t_0$, $t_1$, $t_2$, and $t_3$. Between these points in time, $l$ is either smaller than $L$ and $Y^l$ coincides with $X$, or lager than $L$ and $X$ strictly dominates $Y^l$.

For the stochastic case, the comparison of our two methods to prove existence suggests to consider the semimartingale

$$\check{X}(s) \stackrel{\Delta}{=} \mathbb{E}\left[\int_s^{\hat{T}} f\left(t, \sup_{0\le v\le t} L(v)\right) dt \,\middle|\, \mathcal{F}_s\right], \qquad s \in [0, \hat{T}],$$

as some stochastic kind of inhomogeneously convex envelope for the optional process $X$. In fact, as our analysis of the Skorohod-type obstacle problem in Section 3.4 reveals, $\check{X}$ is the only semimartingale dominated by $X$ with dynamics

$$d\check{X}(t) = f\big(t, A(t)\big)\, dt + dM(t) \quad \text{and} \quad \check{X}(\hat{T}) = 0$$

where $A$ is an adapted, right-continuous, increasing process satisfying the minimality condition

$$\mathbb{E}\int_0^{\hat{T}} |X(s) - \check{X}(s)|\, dA(s) = 0.$$

Hence, one could define a class (D) optional process $X$ to be (inhomogeneously) convex if it coincides with its "convex envelope," that is, with the solution $\check{X}$ of the associated Skorohod problem. It is an open question, however, to which extent such a definition could be justified by additional properties with natural analogues in the deterministic setting.

## APPENDIX

### A.1. Proofs in the deterministic case.

PROOF OF PROPOSITION 4.6. The argument for the characterization of strict convexity being similar, we only prove the characterization of convexity.

(i) $\Rightarrow$ (ii). We shall show $l_{s,t} \le l_{s,u}$ and $l_{s,u} \le l_{t,u}$.

For the first inequality we note that, by definition of $l_{s,t}$ and (i),

$$\int_s^t g(v, l_{s,t})\, dv = x(t) - x(s) \le \int_s^t g(v, l_{s,u})\, dv.$$

Similarly, we obtain the second inequality from

$$\begin{aligned}\int_t^u g(v, l_{s,u})\, dv &= x(u) - \left(x(s) + \int_s^t g(v, l_{s,u})\, dv\right)\\ &\le x(u) - x(t) = \int_t^u g(v, l_{t,u})\, dv.\end{aligned}$$

(ii) $\Rightarrow$ (i). Using the definition of $l_{s,t}$ and $l_{t,u}$, we may write

$$x(u) - x(s) = \int_s^t g(v, l_{s,t})\,dv + \int_t^u g(v, l_{t,u})\,dv.$$

By (ii) and the definition of $l_{s,u}$, this yields

$$\int_s^u g(v, l_{s,u})\,dv \geq \int_s^u g(v, l_{s,t})\,dv.$$

Thus, $l_{s,t} \leq l_{s,u}$ and therefore

$$x(t) - x(s) = \int_s^t g(v, l_{s,t})\,dv \leq \int_s^t g(v, l_{s,u})\,dv$$

as was to be shown.

(iii) $\Rightarrow$ (ii). Because of the boundary conditions, it suffices to show (18) for $s < t < u$ contained in the interior $\operatorname{int}\mathbb{I}$ of our interval. The monotonicity of $l(\cdot)$ implies

$$x(t) - x(s) = \int_s^t g\big(v, l(v)\big)\,dv \leq \int_s^t g\big(v, l(t)\big)\,dv$$

which yields $l_{s,t} \leq l(t)$. Moreover,

$$x(u) - x(t) = \int_t^u g\big(v, l(v)\big)\,dv \geq \int_t^u g\big(v, l(t)\big)\,dv,$$

whence we deduce $l(t) \leq l_{t,u}$.

(ii) $\Rightarrow$ (iii). The same argument as in (ii) $\Rightarrow$ (i) shows that, for $t \in \operatorname{int}\mathbb{I}$ fixed, both $l_{\cdot,t}$ and $l_{t,\cdot}$ are increasing functions on their respective domains. Hence, we may define

$$l^-(t) \stackrel{\Delta}{=} \lim_{s \uparrow t} l_{s,t} \quad \text{and} \quad l^+(t) \stackrel{\Delta}{=} \lim_{s \downarrow t} l_{t,s}.$$

By (18) we have, for $s < t < u$ in $\operatorname{int}\mathbb{I}$,

$$l_{s,t} \leq l^-(t) \leq l^+(t) \leq l_{t,u}.$$

In particular, both $l^-$ and $l^+$ are increasing, real-valued functions on $\operatorname{int}\mathbb{I}$.

We next show that $x$ is absolutely continuous on $\operatorname{int}\mathbb{I}$. To this end, we fix a compact interval $[a, b] \subset \operatorname{int}\mathbb{I}$ and associate with each partition $\tau = \{t_0 = a < t_1 < \cdots < t_n = b\}$ the integrand

$$i^\tau(t) \stackrel{\Delta}{=} \sum_{i=1}^n g\big(t, l_{t_i,t_{i+1}}\big)\mathbb{1}_{[t_i,t_{i+1})}(t), \qquad t \in [a, b].$$

By definition of $l_{\cdot,\cdot}$ and $g$-convexity of $x$, we then have that the function

$$I^\tau(s) \stackrel{\Delta}{=} x(a) + \int_a^s i^\tau(t)\,dt, \qquad s \in [a, b],$$

interpolates $x$ at each point $s \in \tau$ while dominating $x$ otherwise. Moreover, by (ii), refining the partition $\tau$ indefinitely makes $i^\tau$ converge to

$$i(t) \stackrel{\Delta}{=} g\big(t, l^+(t)\big) = g\big(t, l^-(t)\big)$$

in any point $t \in (a, b)$ such that $l^+(t) = l^-(t)$, that is, in all but countably many points. Since, in addition, $g(\cdot, l^+(a)) \le i^\tau \le g(\cdot, l^-(b))$, dominated convergence entails that $I^\tau(s)$ converges to

$$I(s) \stackrel{\Delta}{=} x(a) + \int_a^s g\big(t, l^\pm(t)\big)\, dt$$

for any $s \in [a, b]$ along any sequence of partitions $\tau_1 \subset \tau_2 \subset \cdots$ with mesh $\|\tau_n\|$ tending to 0 as $n \uparrow +\infty$. Since for any given $s \in [a, b]$ we may take a sequence of partitions $\tau_1 \subset \tau_2 \subset \cdots$ which all contain $s$ [whence $x(s) = I^{\tau_n}(s)$ for all $n = 1, 2, \dots$], this yields $x(s) = I^{\tau_n}(s) \to I(s)$ for $n \uparrow +\infty$, that is, $x(s) = I(s)$. In particular, $x$ is absolutely continuous with density

$$\dot{x}(t) = g\big(t, l^-(t)\big) = g\big(t, l^+(t)\big)$$

for almost every $t \in \operatorname{int}\mathbb{I}$. As both $l^-$ and $l^+$ are increasing, either representation of $\dot{x}$ is of the desired form.

To check the boundary conditions $x(t) \ge \lim_{s\to t} x(s)$ for $t \in \partial\mathbb{I} \cap \mathbb{I}$, let us for instance consider the case where $t \stackrel{\Delta}{=} \sup\mathbb{I} \in \partial\mathbb{I} \cap \mathbb{I}$. Note first that $x(t-) = \lim_{s\uparrow t} x(s)$ exists as a number in $\mathbb{R} \cup \{+\infty\}$. Indeed, take an arbitrary $t_0 \in \operatorname{int}\mathbb{I}$, and consider the function $\tilde{x}$ defined by

$$\tilde{x}(s) \stackrel{\Delta}{=} x(t_0) + \int_{t_0}^s \big\{g\big(v, l^+(v)\big) - g\big(v, l^+(t_0)\big)\big\}\, dv, \qquad s \in \operatorname{int}\mathbb{I}.$$

Since $l^+$ is increasing, $\tilde{x}$ is increasing on $(t_0, t)$ and, therefore, has a possibly infinite limit for $s \uparrow t$. By integrability of $g(\cdot, l^+(t_0))$, this property carries over to $x$.

As we know already that property (ii) implies $g$-convexity, we have for any $s \in \operatorname{int}\mathbb{I}$ with $s > t_0$ the estimate

$$x(s) \le x(t) - \int_s^t g\big(v, l_{t_0,t}\big)\, dv.$$

Obviously, the right-hand side converges to $x(t)$ as $s \uparrow t$ while the left-hand side converges to $\lim_{s\uparrow t} x(s) = x(t-)$, establishing the desired boundary condition. Additionally, we obtain that $g(v, l^+(v))\mathbb{1}_{[t_0,t]}(v) \in L^1(dv)$ since otherwise we had $\int_{t_0}^t g(v, l^+(v))\, dv = +\infty$ which would imply $x(t) = +\infty$ in contradiction to our assumption that $x(s) \in \mathbb{R}$ for all $s \in \mathbb{I}$. $\square$

PROOF OF PROPOSITION 4.8. It suffices to show that the pointwise supremum

$$\check{x}(t) \stackrel{\Delta}{=} \sup_{\xi \in \mathcal{X}} \xi(t), \qquad t \in \mathbb{I},$$

of the $g$-convex functions $\xi \le x$ is again $g$-convex. So fix $s < t < u$ in $\mathbb{I}$ and consider $\xi \in \mathcal{X}$. Since $\xi$ is $g$-convex, we have

$$\xi(t) \le \Xi\big(\xi(s), \xi(u)\big)$$

where

$$\Xi(\xi_1, \xi_2) \stackrel{\Delta}{=} \xi_1 + \int_s^t g(v, l)\,dv = \xi_2 - \int_t^u g(v, l)\,dv$$

with $l = l(\xi_1, \xi_2) \in \mathbb{R}$ such that

$$\xi_1 + \int_s^u g(v, l)\,dv = \xi_2.$$

It is easy to see that the function $\Xi$ is increasing in both arguments. Thus,

$$\xi(t) \le \Xi\big(\breve{x}(s), \breve{x}(u)\big).$$

As this holds true for any $\xi \in \mathcal{X}$, we deduce

$$\breve{x}(t) \le \Xi\big(\breve{x}(s), \breve{x}(u)\big),$$

which means that indeed $\breve{x}$ is $g$-convex. □

PROOF OF PROPOSITION 4.10. (i) Let $\xi$ be an arbitrary $g$-convex function dominated by $x$. Define $\tilde{\xi} \stackrel{\Delta}{=} \xi$ on $\operatorname{int}\mathbb{I}$ and put $\tilde{\xi} \stackrel{\Delta}{=} x$ on $\partial\mathbb{I} \cap \mathbb{I}$. Then $\tilde{\xi}$ is another $g$-convex function dominated by $x$. Since $\breve{x}$ is the largest of these functions, this yields in particular $x = \breve{x}$ on $\partial\mathbb{I} \cap \mathbb{I}$. The property $\breve{x} \le x_*$ on $\operatorname{int}\mathbb{I}$ holds since, on this set, $\breve{x}$ is continuous and dominated by $x$.

(ii) Consider $t \in \mathbb{I}$ with $\breve{x}(t) < x_*(t)$. By (i), $t \in \operatorname{int}\mathbb{I}$ and we have to show that $t \notin \operatorname{supp} d\breve{l}$. To this end, we note first that, by assumption on $t$, there are real numbers $c, \delta > 0$ such that

$$\breve{x}(s) + c \le x(s) \qquad \text{for all } s \in [t-\delta, t+\delta] \subset \mathbb{I}.$$

For $0 < h \le \delta$, consider the function $x^h$ defined by $x^h \stackrel{\Delta}{=} \breve{x}$ on $\mathbb{I} \setminus (t-h, t+h)$ and

$$x^h(s) \stackrel{\Delta}{=} \breve{x}(t-h) + \int_{t-h}^s g(v, l^h)\,dv \qquad \text{for } s \in (t-h, t+h)$$

where $l^h \in \mathbb{R}$ is the unique constant satisfying

$$\breve{x}(t-h) + \int_{t-h}^{t+h} g(v, l^h)\,dv = \breve{x}(t+h).$$

As $\breve{x}$ is $g$-convex, we have $\breve{x} \le x^h$ on $[t-h, t+h]$ and, hence, $\breve{x} \le x^h$ on all of $\mathbb{I}$.

Since $\sup_{[t-h,t+h]} x^h$ depends continuously on $h$ through $\breve{x}(t \pm h)$ and because $\breve{x} + c \le x$ on $[t-\delta, t+\delta]$, we may choose $h > 0$ small enough to ensure $x^h \le x$ on this interval and, hence, even on all of $\mathbb{I}$. Then, by construction, $x^h$ is a $g$-convex function dominated by $x$ and, thus, $x^h$ is also dominated by $\breve{x}$.

Altogether, we find that in fact $x^h$ has to coincide with $\breve{x}$. This implies $\breve{l} \equiv l^h$ on $(t-h, t+h)$ and, in particular, $t \notin \operatorname{supp} d\breve{l}$.

(iii) We know already that $\check{x}$ is absolutely continuous on $\operatorname{int}\mathbb{I}$. Thus, in order to establish this property on all of $\mathbb{I}$, it suffices to show that $\check{x}$ is continuous on the boundary points $\inf\mathbb{I}$, $\sup\mathbb{I}$ if these are contained in $\mathbb{I}$. The argument for the inf case being similar, we restrict ourselves to show continuity of $\check{x}$ in case $b \stackrel{\Delta}{=} \sup\mathbb{I} \in \mathbb{I}$.

By Proposition 4.6, $\lim_{t\to b}\check{x}(t)$ exists and is $\leq \check{x}(b) = x(b)$. For the converse inequality, we distinguish two cases.

If $b = \sup\operatorname{supp} d\check{l}$ with $\check{l}$ as in (ii), there is a sequence of points $t_n \in \operatorname{supp} d\check{l}$ which increases to $b$. By (ii), we thus have

$$\lim_{t\to b}\check{x}(t) = \lim_n \check{x}(t_n) = \lim_n x_*(t_n) \geq \liminf_{t\to b} x_*(t) = x_*(b) = x(b)$$

which establishes the converse inequality in this case.

If $b > \tau \stackrel{\Delta}{=} \sup\operatorname{supp} d\check{l}$, then

$$\check{x}(t) = \check{x}(s) + \int_s^t g\big(v, \check{l}(\tau)\big)\,dv \qquad \text{for all } \tau \leq s \leq t < b.$$

Thus, in case $\lim_{t\to b}\check{x}(t) < x(b) = x_*(b)$, there is a constant $c > 0$ such that for $s < b$ large enough we have $\check{x}(t) + c \leq x(t)$ for all $t \in [s, b)$. This, however, contradicts the maximality of $\check{x}$ as a $g$-convex function dominated by $x$.

(iv) Consider $s, t_1, t_2 \in \operatorname{int}\mathbb{I}$ with $t_1 < t_2 \leq s$ and put

$$u \stackrel{\Delta}{=} \inf\big\{t \geq s \mid \check{x}^{t_1}(t) = \check{x}^{t_2}(t)\big\}.$$

As $\check{x}^{t_1} \leq \check{x}^{t_2}$ on $[s, +\infty] \cap \mathbb{I}$, we then have

$$\check{x}^{t_1}|_{[u,+\infty]\cap\mathbb{I}} = \check{x}^{t_2}|_{[u,+\infty]\cap\mathbb{I}}. \tag{29}$$

If $u = s$, this immediately yields that our assertion

$$\partial^+\check{x}^{t_1}(s) \geq \partial^+\check{x}^{t_2}(s)$$

holds true with equality.

In case $u > s$ and $u \in \mathbb{I}$, let $\check{l}^1_{s,u}, \check{l}^2_{s,u}$ denote the constants associated via (17) with $\check{x}^{t_1}$ and $\check{x}^{t_2}$, respectively; let furthermore $\check{l}^1, \check{l}^2$ be the right-continuous increasing functions such that $\partial^+\check{x}^{t_1} = g(\cdot, \check{l}^1(\cdot))$, $\partial^+\check{x}^{t_2} = g(\cdot, \check{l}^2(\cdot))$. Since $\check{x}^{t_1}(s) < \check{x}^{t_2}(s)$, it follows that $\check{l}^1_{s,u} > \check{l}^2_{s,u}$. Using (iii), our identity (29) and monotonicity of $\check{l}^2$, we thus obtain the series of (in)equalities

$$\partial^+\check{x}^{t_1}(s) = g(s, \check{l}^1_{s,u}) > g(s, \check{l}^2_{s,u}) \geq g(s, \check{l}^2_{s,u}) \geq g\big(s, \check{l}^2(s)\big) = \partial^+\check{x}^{t_2}(s)$$

as claimed.

Finally, if $u > s$ is not contained in $\mathbb{I}$, then $\check{x}^{t_1} < \check{x}^{t_2} \leq x_*$ on $[s, u)$. By (iii), this implies

$$\check{x}^{t_2}(t) = \check{x}^{t_2}(s) + \int_s^t g\big(v, \check{l}^1(s)\big)\,dv.$$

Hence, by integrability of $g(\cdot, \text{const.})$, we may extend both $\check{x}^{t_2}$ and $x$ canonically to $\mathbb{I} \cup \{u\}$ and apply the reasoning of the preceding case to conclude the assertion. $\square$

**A.2. Proofs in the stochastic case.**

A.2.1. *Proof of Lemma* 4.11. (i) By Proposition 2.29 in El Karoui (1981), the optional versions of the Snell envelopes

$$J_*(S) = \operatorname*{ess\,sup}_{T \in \mathcal{S}(S)} \mathbb{E}[X^-(T)|\mathcal{F}_S] \quad \text{and} \quad J^*(S) = \operatorname*{ess\,sup}_{T \in \mathcal{S}(S)} \mathbb{E}[X^+(T)|\mathcal{F}_S], \qquad S \in \mathcal{S},$$

are of class (D) since so is $X = X^+ - X^-$. By the same proposition, $J_*$ (resp. $J^*$) can be written as the difference of a martingale $-M_*$ (resp. $M^*$) and a nonnegative increasing process. It follows that

$$M_* \le -J_* \le -X^- \le X \le X^+ \le J^* \le M^*$$

which proves assertion (i).

(ii) Since $X$ is optional and of class (D), pathwise lower-semicontinuity from the right follows from Dellacherie and Lenglart (1982). In order to prove $\liminf_{t \uparrow \hat{T}} X(t) \ge X(\hat{T}) = 0$ almost surely, suppose to the contrary that for some $\varepsilon > 0$ we have $\mathbb{P}[\liminf_{t \uparrow \hat{T}} X(t) < -2\varepsilon] > 0$. Put $T^0 \triangleq 0$ and define

$$T^n \triangleq \inf\{t \ge T^{n-1} \vee S^n \mid X(t) \le -\varepsilon\} \wedge \hat{T}$$

where $S^n, n = 1, 2, \ldots,$ may be any sequence of stopping times announcing $\hat{T}$. Then obviously $T^n \nearrow \hat{T}$ and, since the paths of $X$ are lower-semicontinuous from the right, it holds that $X(T^n) \le -\varepsilon$ on $\{T^n < \hat{T}\}$ while $X(T^n) = 0$ on $\{T^n = \hat{T}\}$. Hence, we have

$$\mathbb{E}X(T^n) = \mathbb{E}\big[X(T^n)\mathbb{1}_{\{T^n < \hat{T}\}}\big] \le -\varepsilon\mathbb{P}[T^n < \hat{T}].$$

Since the process $X$ is lower-semicontinuous in expectation we may let $n \uparrow +\infty$ in the above relation to deduce

$$\mathbb{E}X(\hat{T}) \le \liminf_n \mathbb{E}X(T^n) \le -\varepsilon\mathbb{P}[T^n < \hat{T} \text{ for all } n = 1, 2, \ldots].$$

However, as $\mathbb{E}X(\hat{T}) = 0$, this is a contradiction to our initial assumption that the event

$$\Big\{\liminf_{t \uparrow \hat{T}} X(t) < -2\varepsilon\Big\} \subset \{T^n < \hat{T} \text{ for all } n = 1, 2, \ldots\}$$

has strictly positive probability.

(iii) Apply Fatou's lemma to $X(T^n) - M_*(T^n) \ge 0, n = 1, 2, \ldots,$ to deduce

$$\liminf_n \mathbb{E}[X(T^n) - M_*(T^n)|\mathcal{F}_S] \ge \mathbb{E}\Big[\liminf_n \{X(T^n) - M_*(T^n)\}\Big|\mathcal{F}_S\Big].$$

Since $M_*$ is a martingale we may rewrite the left-hand side in this expression as $\liminf_n \mathbb{E}[X(T^n)|\mathcal{F}_S] - M_*(S)$. Furthermore, the martingale property of $M_*$ in conjunction with the monotonicity of $T^n, n = 1, 2, \ldots,$ implies that $\lim_n M_*(T^n)$ exists almost surely and in $L^1(\mathbb{P})$ so that the right-hand side equals $\mathbb{E}[\liminf_n X(T^n) - \lim_n M_*(T^n)|\mathcal{F}_S] = \mathbb{E}[\liminf_n X(T^n)|\mathcal{F}_S] - M_*(S)$.

A.2.2. *Proof of Lemma* 4.12. The proof of Lemma 4.12 is rather lengthy and technical. We therefore split it into several parts and start with some preliminaries.

*Preliminaries.* Due to our assumptions on $f$ and $X$, we may apply Théorème 2.28 in El Karoui (1981) to obtain existence of optional processes $\tilde{Y}^l$ ($l \in \mathbb{R}$) such that

$$\tilde{Y}^l(S) = \operatorname*{ess\,inf}_{T \in \mathcal{S}(s)} \mathbb{E}\left[X(T) + \int_S^T f(t,l)\,dt \middle| \mathcal{F}_S\right] \leq X(S)$$

for every stopping time $S \in \mathcal{S}$ and every $l \in \mathbb{R}$. Moreover, Théorème 2.41 in El Karoui (1981) implies that, for $S \in \mathcal{S}$ fixed,

$$\tilde{T}_S^l \triangleq \inf\{t \geq S \mid \tilde{Y}^l(t) = X(t)\} \leq \hat{T}$$

is optimal in the sense that

$$\tilde{Y}^l(S) = \mathbb{E}\left[X(\tilde{T}_S^l) + \int_S^{\tilde{T}_S^l} f(t,l)\,dt \middle| \mathcal{F}_S\right].$$

For $l, l' \in \mathbb{R}$ with $l \leq l'$, the monotonicity of $f(t,\cdot)$, $0 \leq t \leq \hat{T}$, yields

$$\mathbb{E}\left[X(T) + \int_S^T f(t,l')\,dt \middle| \mathcal{F}_S\right] \leq \mathbb{E}\left[X(T) + \int_S^T f(t,l)\,dt \middle| \mathcal{F}_S\right]$$

for all $T \in \mathcal{S}(S)$. As $\tilde{Y}^{l'}(S)$ [resp., $\tilde{Y}^l(S)$] is the essential infimum of the left-hand (resp., the right-hand) side of this inequality where $T$ ranges over $\mathcal{S}(S)$, this implies

$$\tilde{Y}^{l'}(S) \leq \tilde{Y}^l(S), \qquad \mathbb{P}\text{-a.s.} \tag{30}$$

In addition, we have

$$\begin{aligned}
\tilde{Y}^l(S) &\leq \mathbb{E}\left[X(\tilde{T}_S^{l'}) + \int_S^{\tilde{T}_S^{l'}} f(t,l)\,dt \middle| \mathcal{F}_S\right] \\
&= \mathbb{E}\left[X(\tilde{T}_S^{l'}) + \int_S^{\tilde{T}_S^{l'}} f(t,l')\,dt \middle| \mathcal{F}_S\right] + \mathbb{E}\left[\int_S^{\tilde{T}_S^{l'}} \{f(t,l) - f(t,l')\}\,dt \middle| \mathcal{F}_S\right] \\
&= \tilde{Y}^{l'}(S) + \mathbb{E}\left[\int_S^{\tilde{T}_S^{l'}} \{f(t,l) - f(t,l')\}\,dt \middle| \mathcal{F}_S\right]
\end{aligned}$$

where the last equality follows from optimality of $\tilde{T}_S^{l'}$. For $l \leq l'$, we have $f(t,l) - f(t,l') \geq 0$ for any $t \in [0,\hat{T}]$, and thus the preceding estimate yields

$$\tilde{Y}^l(S) \leq \tilde{Y}^{l'}(S) + \int_0^{\hat{T}} |f(t,l') - f(t,l)|\,dt, \qquad \mathbb{P}\text{-a.s.} \tag{31}$$

Since both estimates (30) and (31) hold true for every stopping time $S \in \mathcal{S}$, optionality of both $\tilde{Y}^l$ and $\tilde{Y}^{l'}$ entails the pathwise estimate

$$\tilde{Y}^{l'}(s) \le \tilde{Y}^l(s) \le \tilde{Y}^{l'}(s) + \int_0^{\hat{T}} |f(t,l') - f(t,l)|\,dt \qquad \text{for all } s \in [0,\hat{T}], \mathbb{P}\text{-a.s.}$$

by Meyer's optional section theorem. In fact, we may even choose $\tilde{Y}^l$ for $l \in \mathbb{Q}$ such that the above relation holds true simultaneously at each point $\omega \in \Omega$ for all rational $l \le l'$. Similarly, we may assume that $\tilde{Y}^l(\omega,t) \le X(\omega,t)$ for all $l \in \mathbb{Q}$ and any $(\omega,t) \in \Omega \times [0,\hat{T}]$.

With this choice of the auxiliary processes $\tilde{Y}^l, l \in \mathbb{Q}$, we now come to the following:

CONSTRUCTION OF $Y$ AND PROOF OF LEMMA 4.12(i). For each $l \in \mathbb{R}$, define the process

$$Y^l(s) \triangleq \lim_{\mathbb{Q}\ni r\uparrow l} \tilde{Y}^r(s) = \inf_{l>r\in\mathbb{Q}} \tilde{Y}^r(s), \qquad s \in [0,\hat{T}].$$

We claim that $Y^l$ *is indistinguishable from* $\tilde{Y}^l$ *for every* $l \in \mathbb{R}$. Indeed, $Y^l$ is obviously optional. As $\tilde{Y}^r \le \tilde{Y}^l$ for all rational $r < l$, we also have $Y^l \le \tilde{Y}^l$. For the remaining converse inequality, fix $S \in \mathcal{S}$ and note that, for every $T \in \mathcal{S}(S)$,

$$\begin{aligned} Y^l(S) &= \lim_{\mathbb{Q}\ni r\uparrow l} \tilde{Y}^r(S) \\ &\le \liminf_{\mathbb{Q}\ni r\uparrow l} \mathbb{E}\left[X(T) + \int_S^T f(t,r)\,dt \,\middle|\, \mathcal{F}_S\right] = \mathbb{E}\left[X(T) + \int_S^T f(t,l)\,dt \,\middle|\, \mathcal{F}_S\right]. \end{aligned}$$

Since this estimate holds true for all $T \in \mathcal{S}(S)$, we may pass to the essential infimum on its right-hand side to obtain $Y^l(S) \le \tilde{Y}^l(S)$ almost surely. By optionality, this entails $Y^l(t) \le \tilde{Y}^l(t)$ for all $t \in [0,\hat{T}]$, $\mathbb{P}$-a.s., which is the asserted converse inequality. $\square$

PROOF OF LEMMA 4.12(ii) AND (iii). $Y^l$ and $\tilde{Y}^l$ being indistinguishable by construction, *optimality of* $T_S^l$ follows from optimality of $\tilde{T}_S^l$.

To prove the first part of assertion (iii), recall that we have chosen $\tilde{Y}^l, l \in \mathbb{Q}$, such that

$$\tilde{Y}^{l'}(\omega,s) \le \tilde{Y}^l(\omega,s) \le \tilde{Y}^{l'}(\omega,s) + \int_0^{\hat{T}} |f(\omega,t,l') - f(\omega,t,l)|\,dt$$

for all $\omega \in \Omega$, $s \in [0,\hat{T}]$ and all rational $l \le l'$. Taking rational limits, we infer from this that

$$Y^{l'}(\omega,s) \le Y^l(\omega,s) \le Y^{l'}(\omega,s) + \int_0^{\hat{T}} |f(\omega,t,l') - f(\omega,t,l)|\,dt$$

for all $\omega \in \Omega$, $s \in [0, \hat{T}]$ and all *real* $l \leq l'$. This inequality proves the claimed *continuity and monotonicity of* $l \mapsto Y^l(\omega, s)$.

We next show that, for $S \in \mathcal{S}$ fixed, we have $T_S^l(\omega) \leq T_S^{l'}(\omega)$ *simultaneously for all* $l \leq l'$ *and all* $\omega \in \Omega$. Indeed, by construction, we have $Y^{l'}(\omega, s) \leq Y^l(\omega, s) \leq X(\omega, s)$ for every $l' \leq l$, $s \in [0, \hat{T}]$ and all $\omega \in \Omega$. For fixed $\omega$, this yields

$$\{t \geq S(\omega) \mid Y^{l'}(\omega, t) = X(\omega, t)\} \subset \{t \geq S(\omega) \mid Y^l(\omega, t) = X(\omega, t)\}$$

whence $T_S^{l'}(\omega) \geq T_S^l(\omega)$ by definition of these stopping times.

To complete the proof of (iii), we next determine *the limit* $Y^{-\infty}$. By optimality of $T_S^l$, we have

$$X(S) \geq Y^l(S) = \mathbb{E}\left[X(T_S^l) + \int_S^{T_S^l} f(t, l)\, dt \,\middle|\, \mathcal{F}_S\right]$$

for any $l \in \mathbb{R}$. Letting $l \downarrow -\infty$, this entails

$$X(S) \geq Y^{-\infty}(S) \geq \liminf_{l \downarrow -\infty} \mathbb{E}[X(T_S^l) | \mathcal{F}_S] + \liminf_{l \downarrow -\infty} \mathbb{E}\left[\int_S^{T_S^l} f(t, l)\, dt \,\middle|\, \mathcal{F}_S\right]. \tag{32}$$

From the monotonicity of $l \mapsto T_S^l$ we deduce that $T_S^{-\infty} \triangleq \lim_{l \downarrow -\infty} T_S^l$ exists. Moreover, from Lemma 4.11(iii) we may infer the estimate

$$\liminf_{l \downarrow -\infty} \mathbb{E}[X(T_S^l) | \mathcal{F}_S] \geq \mathbb{E}\left[\liminf_{l \downarrow -\infty} X(T_S^l) \,\middle|\, \mathcal{F}_S\right] \geq \mathbb{E}[X(T_S^{-\infty}) | \mathcal{F}_S] \tag{33}$$

for the first summand on the right-hand side of (32). Here, the second inequality follows by pathwise lower-semicontinuity from the right of $X$ [Lemma 4.11(ii)].

The second summand can be estimated from below by

$$\begin{aligned} &\liminf_{l \downarrow -\infty} \mathbb{E}\left[\int_S^{T_S^l} f(t, l)\, dt \,\middle|\, \mathcal{F}_S\right] \\ &\quad \geq \operatorname*{ess\,sup}_{l_0 \in \mathbb{R}} \liminf_{l \downarrow -\infty} \mathbb{E}\left[\int_S^{T_S^l} f(t, l_0)\, dt \,\middle|\, \mathcal{F}_S\right] \\ &\quad = \mathbb{E}\left[\int_S^{T_S^{-\infty}} f(t, -\infty)\, dt \,\middle|\, \mathcal{F}_S\right] \\ &\quad = +\infty \mathbb{1}_{\{T_S^{-\infty} > S\}}. \end{aligned} \tag{34}$$

Hence, it follows from (32) that $T_S^{-\infty} = S$ almost surely. Combining this with our estimates (32)–(34) yields $Y^{-\infty}(S) = X(S)$ almost surely as claimed. $\square$

It finally remains to prove our version of the Envelope theorem.

PROOF OF LEMMA 4.12(iv). Fix $S \in \mathcal{S}$ and $l_* < l^*$ in $\mathbb{R}$. We have to show that $Y^{l^*}(S) - Y^{l_*}(S)$ is a version of the conditional expectation

$$\mathbb{E}\left[\int_S^{\hat{T}} \left\{\int_{l_*}^{l^*} \mathbb{1}_{[S,T_S^l]}(t)\, df(t,l)\right\} dt \,\middle|\, \mathcal{F}_S\right].$$

To this end, fix a set $A \in \mathcal{F}_S$ and consider a partition $\tau = \{l_* = l_0 < l_1 < \cdots < l_{n+1} = l^*\}$ of the interval $[l_*, l^*]$. Write

$$\mathbb{E}[(Y_S^{l^*} - Y_S^{l_*})\mathbb{1}_A] = \sum_{i=0}^{n} \mathbb{E}[(Y_S^{l_{i+1}} - Y_S^{l_i})\mathbb{1}_A]$$

and use optimality of $T_S^{l_{i+1}}$ and $T_S^{l_i}$, respectively, to estimate

$$\begin{aligned}\mathbb{E}[(Y_S^{l^*} - Y_S^{l_*})\mathbb{1}_A] &\geq \sum_{i=0}^{n} \mathbb{E}\left[\left(X(T_S^{l_{i+1}}) + \int_S^{T_S^{l_{i+1}}} f(t,l_{i+1})\, dt\right)\mathbb{1}_A\right] \\ &\qquad - \mathbb{E}\left[\left(X(T_S^{l_{i+1}}) + \int_S^{T_S^{l_{i+1}}} f(t,l_i)\, dt\right)\mathbb{1}_A\right] \\ &= \sum_{i=0}^{n} \mathbb{E}\left[\int_S^{T_S^{l_{i+1}}} \{f(t,l_{i+1}) - f(t,l_i)\}\, dt\, \mathbb{1}_A\right] \triangleq I^\tau\end{aligned} \tag{35}$$

and similarly

$$\mathbb{E}[(Y_S^{l^*} - Y_S^{l_*})\mathbb{1}_A] \leq \sum_{i=0}^{n} \mathbb{E}\left[\int_S^{T_S^{l_i}} \{f(t,l_{i+1}) - f(t,l_i)\}\, dt\, \mathbb{1}_A\right] \triangleq II^\tau. \tag{36}$$

We may rewrite $I^\tau$ in terms of the measures $df(t,\cdot)$, $t \in [0,\hat{T}]$, as

$$\begin{aligned}I^\tau &= \sum_{i=0}^{n} \mathbb{E}\left[\int_S^{T_S^{l_{i+1}}} \left\{\int_{-\infty}^{+\infty} \mathbb{1}_{[l_i,l_{i+1})}(l)\, df(t,l)\right\} dt\, \mathbb{1}_A\right] \\ &= \mathbb{E}\left[\int_S^{\hat{T}} \left\{\int_{-\infty}^{+\infty} \sum_{i=0}^{n} \mathbb{1}_{[S,T_S^{l_{i+1}})}(t)\mathbb{1}_{[l_i,l_{i+1})}(l)\, df(t,l)\right\} dt\, \mathbb{1}_A\right].\end{aligned}$$

For mesh $\|\tau\|$ tending to zero, the above sum of indicator products converges to $\mathbb{1}_{[l_*,l^*)}(l)\mathbb{1}_{[S,T_S^{l+}]}(t)$ and is dominated uniformly in $\tau$ by $\mathbb{1}_{[l_*,l^*)}(l)\mathbb{1}_{[S,\hat{T}]}(t)$. Since the latter product is in $L^1(\mathbb{P} \otimes dt \otimes df(t,l))$ due to our integrability assumption on $f$, we may conclude by dominated convergence that

$$\lim_{\|\tau\|\to 0} I^\tau = I \triangleq \mathbb{E}\left[\int_S^{\hat{T}} \left\{\int_{-\infty}^{+\infty} \mathbb{1}_{[l_*,l^*)}(l)\mathbb{1}_{[S,T_S^{l+}]}(t)\, df(t,l)\right\} dt\, \mathbb{1}_A\right].$$

An analogous argument shows

$$\lim_{\|\tau\|\to 0} II^\tau = II \triangleq \mathbb{E}\left[\int_S^{\hat{T}} \left\{\int_{-\infty}^{+\infty} \mathbb{1}_{[l_*,l^*)}(l)\mathbb{1}_{[S,T_S^{l-}]}(t)\, df(t,l)\right\} dt\, \mathbb{1}_A\right].$$

For every $\omega \in \Omega$, the set $\{l \in \mathbb{R} \mid T_S^{l-}(\omega) < T_S^{l+}(\omega)\}$ is countable due to the monotonicity of $T_S^l(\omega)$ in $l$. In conjunction with our estimates (35) and (36), this yields the identities

$$I = II = \mathbb{E}\big[\big(Y_S^{l^*} - Y_S^{l_*}\big)\mathbb{1}_A\big]. \tag{37}$$

Moreover, monotonicity of $T_S^{l-}$, $T_S^l$ and $T_S^{l+}$ in conjunction with $T_S^{l-} \leq T_S^l \leq T_S^{l+}$ and $df(t,\cdot) \leq 0$ implies

$$I \geq \mathbb{E}\bigg[\int_S^{\hat{T}} \bigg\{\int_{-\infty}^{+\infty} \mathbb{1}_{[l_*,l^*)}(l)\mathbb{1}_{[S,T_S^l]}(t)\, df(t,l)\bigg\}\, dt\, \mathbb{1}_A\bigg] \geq II.$$

Together with (37), the preceding inequality finally implies

$$\mathbb{E}\big[\big(Y_S^{l^*} - Y_S^{l_*}\big)\mathbb{1}_A\big] = \mathbb{E}\bigg[\int_S^{\hat{T}} \bigg\{\int_{-\infty}^{+\infty} \mathbb{1}_{[l_*,l^*)}(l)\mathbb{1}_{[S,T_S^l]}(t)\, df(t,l)\bigg\}\, dt\, \mathbb{1}_A\bigg].$$

As $A \in \mathcal{F}_S$ is arbitrary, this completes the proof of assertion (iv). □

A.2.3. *Proof of Lemma* 4.13. The process $L$ is optional since, for every $l \in \mathbb{R}$, we have

$$\big\{(\omega,t) \in \Omega \times [0,\hat{T}] \mid L(\omega,t) > l\big\} = \bigcup_{l<r\in\mathbb{Q}} \{Y^r = X\},$$

where the latter set is optional by optionality of $Y^r$ and $X$. To see that $L$ takes values in $[-\infty,+\infty)$, consider $S \in \hat{\mathcal{S}}$ and note that on $\{L(S) = +\infty\}$ we have $X(S) = Y^l(S)$ for all $l \in \mathbb{R}$ almost surely. This entails, in particular, that

$$X(S) \leq \mathbb{E}[X(\hat{T})|\mathcal{F}_S] + \mathbb{E}\bigg[\int_S^{\hat{T}} f(t,l)\, dt\bigg|\mathcal{F}_S\bigg] \qquad \text{on } \{L(S) = +\infty\} \text{ for all } l \in \mathbb{R}$$

almost surely. Letting $l \uparrow +\infty$, this implies

$$\{L(S) = +\infty\} \subset \{X(S) = -\infty\}$$

up to a $\mathbb{P}$-null set. The right event has probability zero by assumption on $X$ and, thus, also $\mathbb{P}[L(S) = +\infty] = 0$.

The claimed inclusions $A \subset B \subset C$ are easily derived from the definitions of $L$ and $T_S^l$. Moreover, for $(\omega,t) \in \Omega \times [0,\hat{T}]$ such that the running supremum $\sup_{S(\omega)\leq v\leq \cdot} L(v)$ does not jump at time $t$, the only point contained in the difference of the $(\omega,t)$-sections $C^{(\omega,t)} \backslash A^{(\omega,t)}$ is $l = \sup_{S(\omega)\leq v<t} L(v)$.

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INSTITUT FÜR MATHEMATIK
HUMBOLDT-UNIVERSITÄT ZU BERLIN
UNTER DEN LINDEN 6
D-10099 BERLIN
GERMANY
E-MAIL: pbank@mathematik.hu-berlin.de
URL: www.math.hu-berlin.de/~pbank

CENTRE DE MATHÉMATIQUES APPLIQUÉES
ÉCOLE POLYTECHNIQUE
F-91128 PALAISEAU
FRANCE
E-MAIL: elkaroui@cmapx.polytechnique.fr
URL: www.cmap.polytechnique.fr/~elkaroui