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LIMIT THEOREMS FOR COUPLED CONTINUOUS TIME RANDOM WALKS[1]

BY PETER BECKER-KERN, MARK M. MEERSCHAERT[2]
AND HANS-PETER SCHEFFLER

University of Dortmund, University of Nevada and University of Dortmund

Scaling limits of continuous time random walks are used in physics to model anomalous diffusion, in which a cloud of particles spreads at a different rate than the classical Brownian motion. Governing equations for these limit processes generalize the classical diffusion equation. In this article, we characterize scaling limits in the case where the particle jump sizes and the waiting time between jumps are dependent. This leads to an efficient method of computing the limit, and a surprising connection to fractional derivatives.

1. Introduction. Continuous time random walks (CTRW) were introduced in [21] to study random walks on a lattice. They are now used in physics to model a wide variety of phenomena connected with anomalous diffusion [9, 26, 30]. A CTRW is a random walk subordinated to a renewal process. The random walk increments represent the magnitude of particle jumps, and the renewal epochs represent the times of the particle jumps. CTRW are also called *renewal reward processes* (see, e.g., [33], where applications are given to queuing theory). The usual assumption is that the CTRW is uncoupled, meaning that the random walk is independent of the subordinating renewal process. In this case, if the time between renewals has finite mean, then the renewal process is asymptotically equivalent to a constant multiple of the time variable, and the CTRW behaves like the original random walk for large time [3, 10]. In many physical applications, the waiting time between renewals has infinite mean [28]. In [19] we showed that the scaling limit of an uncoupled CTRW with infinite mean waiting time is of the form $A(E(t))$, where $A(t)$ is the scaling limit of the underlying random walk and $E(t)$ is the hitting time process for a stable subordinator independent of $A(t)$. In some applications it becomes important to consider coupled CTRW, where the waiting time between jumps and the jump sizes are not assumed independent [9, 28]. In this article, we extend the results in [19] by computing the scaling limits of coupled CTRW models. This case is mathematically more delicate, and leads to an interesting connection with fractional derivatives. Since the space and time

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processes usually scale differently, CTRW limit theorems are a natural application for the theory of operator stable laws and their generalized domains of attraction. It also turns out that the log-characteristic function, or symbol, of certain operator-stable laws can be used to simplify the computation of CTRW scaling limits.

2. Continuous time random walks. Let $J_1, J_2, \dots$ be nonnegative independent and identically distributed (i.i.d.) random variables that model the waiting times between jumps of a particle. We set $T(0) = 0$ and $T(n) = \sum_{j=1}^{n} J_j$, the time of the nth jump. The particle jumps are given by i.i.d. random vectors $Y_1, Y_2, \dots$ on $\mathbb{R}^d$. Let $S(0) = 0$ and $S(n) = \sum_{i=1}^{n} Y_i$, the position of the particle after the nth jump. For $t \geq 0$, let

$$N_t = \max\{n \geq 0 : T(n) \leq t\} \tag{2.1}$$

be the number of jumps up to time t and define the stochastic process $\{X(t)\}_{t\geq 0}$ by

$$X(t) = S(N_t) = \sum_{i=1}^{N_t} Y_i. \tag{2.2}$$

Then $X(t)$ is the position of the particle at time t. We call $\{X(t)\}_{t\geq 0}$ a *continuous time random walk*.

Assume that for some invertible linear operators A_n on $\mathbb{R}^d$ and $b_n > 0$ we have

$$\big(A_n S(n), b_n T(n)\big) \Rightarrow (A, D) \qquad \text{as } n \to \infty, \tag{2.3}$$

where D is nondegenerate and A has a full distribution, meaning that it is not supported on any proper hyperplane of $\mathbb{R}^d$. Here $\Rightarrow$ denotes convergence in distribution. Then, by projecting on $\mathbb{R}^d$ and $\mathbb{R}$, respectively, it follows that $D > 0$ almost surely is some stable law with index $0 < \beta < 1$ and A is operator stable on $\mathbb{R}^d$ with some exponent written here as $(1/\beta)E$; see [18], Chapter 8.3.2, for details. Note that it follows from Theorem 7.2.1 of [18] that $\operatorname{Re}\lambda \geq \beta/2$ for any eigenvalue λ of E. For a probability measure ρ on $[0, \infty)$, let $\mathcal{L}(\rho)(s) = \int_0^\infty e^{-st}\, d\rho(t)$, $s \geq 0$, denote its Laplace transform and let P_X denote the distribution of a random variable X. Then, in view of [25], Example 24.12, for a suitable choice of norming constants b_n in (2.3) we have

$$\mathcal{L}(P_D)(s) = \exp(-s^\beta). \tag{2.4}$$

In the following discussion, we can, and hence will, assume (without loss of generality) that the limit D in (2.3) has the form (2.4).

For $t \geq 0$, let $S(t) = \sum_{i=1}^{[t]} Y_i$ and $T(t) = \sum_{i=1}^{[t]} J_i$, where $[t]$ denotes the integer part of t. Recall from Theorem 8.2.17 in [18], that, without loss of generality, there exists a norming function b, regularly varying with exponent $-1/\beta$. This means that $b(\lambda t)/b(t) \to \lambda^{-1/\beta}$ as $t \to \infty$, for any $\lambda > 0$. Moreover, by Theorem 8.1.5 of [18], there exists a function $B \in \mathrm{RV}(-(1/\beta)E)$ that is $B(c)$ is an invertible linear operator on $\mathbb{R}^d$ for any $c > 0$ and $B(\lambda c)B(c)^{-1} \to \lambda^{-(1/\beta)E}$ as $c \to \infty$ for any $\lambda > 0$. Then (2.3) holds with $A_n = B(n)$ and $b_n = b(n)$. The $\mathrm{GL}(\mathbb{R}^{d+1})$-valued function $\operatorname{diag}(B(c), b(c))$ is regularly varying with exponent $\operatorname{diag}(-(1/\beta)E, -1/\beta)$ and we can write (2.3) in the form

$$\operatorname{diag}\big(B(n), b(n)\big)\big(S(n), T(n)\big) \Rightarrow \big(A, D\big) \qquad \text{as } n \to \infty. \tag{2.5}$$

Since the components A and D of the operator-stable vector (A, D) are in general dependent, we first investigate the structure of the distribution of (A, D) in terms of the Lévy measure of its infinitely divisible distribution. This result is later used to compute various interesting examples of our CTRW limits. We prove a little more general result which is also of independent interest. Since $D > 0$ almost surely, it is more natural to use the so-called Fourier–Laplace transform (FLT) of the distribution of (A, D) instead of its Fourier transform.

For suitable functions g on $\mathbb{R}^d \times \mathbb{R}_+$ we define the *Fourier–Laplace transform*

$$\mathcal{FL}(g)(k, s) = \int_{\mathbb{R}^d} \int_0^\infty e^{i\langle x,k\rangle} e^{-st} g(x, t)\, dt\, dx, \tag{2.6}$$

where $(k, s) \in \mathbb{R}^d \times (0, \infty)$. Similarly, if μ is a bounded Borel measure on $\mathbb{R}^d \times \mathbb{R}_+$,

$$\mathcal{FL}(\mu)(k, s) = \int_{\mathbb{R}^d} \int_0^\infty e^{i\langle x,k\rangle} e^{-st} \mu(dt, dx)$$

is the FLT of μ. It follows from a general theory of FLTs on semigroups that $\mathcal{FL}$ has similar properties as the usual Fourier transform of probability measures (see, e.g., [22], Theorem 1, and [4]). Note that if g is Lebesgue-integrable on $\mathbb{R}^d \times \mathbb{R}_+$, then $\mathcal{FL}(g)$ exists, but $\mathcal{FL}$ is defined on a larger class of measurable functions by the integral formula (2.6).

Infinitely divisible distributions are characterized by the Lévy–Khinchin formula of its log-characteristic function. This concept carries over to the FLT setting. In fact we have:

LEMMA 2.1. *There exists a unique continuous function* $\psi : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{C}$ *such that* $\psi(0, 0) = 0$, $\operatorname{Re}\psi \geq 0$ *and*

$$\mathcal{FL}\big(P_{(A,D)}\big)(k, s) = \exp\big(-\psi(k, s)\big) \tag{2.7}$$

for all $(k, s) \in \mathbb{R}^d \times \mathbb{R}_+$. *We call* ψ *the log-FLT of* (A, D). *Moreover, there exist uniquely determined* $(a, b) \in \mathbb{R}^d \times \mathbb{R}_+$, *a positive quadratic form* Q *on* $\mathbb{R}^d$ *and a measure* ϕ *on* $\mathbb{R}^d \times \mathbb{R}_+ \setminus \{(0, 0)\}$ *such that*

$$\begin{aligned}\psi(k, s) &= i\langle a, k\rangle + bs + Q(k)\\ &\quad + \int_{\mathbb{R}^d \times \mathbb{R}_+ \setminus \{(0,0)\}} \left(1 - e^{i\langle k,x\rangle} e^{-st} + \frac{i\langle k, x\rangle}{1 + \|x\|^2}\right) \phi(dx, dt).\end{aligned}$$

The measure ϕ is finite outside every neighborhood of the origin and

$$\int_{0<\|x\|^2+t\le 1} (\|x\|^2 + t)\phi(dx, dt) < \infty.$$

Measure ϕ is called the Lévy measure of (A, D).

PROOF. The existence of ψ with $\psi(0,0) = 0$ and $\operatorname{Re}\psi \ge 0$ follows from Theorem 6 in [22] or Theorem 4.3.19 in [4]. Note that then $\mathcal{FL}(P_{(A,D)})(k,s) \neq 0$ for all $(k,s) \in \mathbb{R}^d \times \mathbb{R}_+$. Since $(k,s) \mapsto \mathcal{FL}(P_{(A,D)})(k,s)$ is continuous, a simple variant of Theorem 3.4.1 of [18] implies that ψ is continuous and uniquely determined. That ψ has the desired form follows from (1) on page 345 of [22]. □

In the following text, Lévy measure always means the Lévy measure of the log-FLT. Now the limit (A, D) can be characterized as follows:

THEOREM 2.2. *Assume that (Y_i, J_i) are i.i.d. $\mathbb{R}^d \times [0,\infty)$-valued random vectors. If there exist $B \in \mathrm{RV}(-(1/\beta)E)$, $b \in \mathrm{RV}(-1/\beta)$ for some $0 < \beta < 1$ and some exponent E, with $\operatorname{Re}\lambda > \beta/2$ for all eigenvalues λ of E, such that for some nonrandom $d_n \in \mathbb{R}^d$,*

$$(B(n)S(n) - d_n, b(n)T(n)) \Rightarrow (A, D) \qquad \text{as } n \to \infty \tag{2.8}$$

for some full limit A on $\mathbb{R}^d$ and some nondegenerate D, then we have: There exists a constant $K > 0$, a probability measure ω on $\mathbb{R}^d$ and a Lévy measure ϕ_1 on $\mathbb{R}^d$ with $t^{(1/\beta)E}\phi_1 = t \cdot \phi_1$ for all $t > 0$ such that the Lévy measure ϕ of (A, D) has the form

$$\phi(dy, dt) = \varepsilon_0(dt)\phi_1(dy) + \mathbb{1}_{\mathbb{R}^d\times(0,\infty)}(y,t)\phi_2(dy, dt), \tag{2.9}$$

where

$$\phi_2(dy, dt) = (t^E\omega)(dy)K\beta t^{-\beta-1}\,dt. \tag{2.10}$$

Conversely, if $0 < \beta < 1$, $K > 0$, ω is a probability measure on $\mathbb{R}^d$ and $\operatorname{Re}\lambda > \beta/2$ for all eigenvalues λ of E, we take J_i i.i.d. as D, where D is a β-stable subordinator with $\mathbb{E}[e^{-sD}] = \exp\{-K\Gamma(1-\beta)s^\beta\}$ and we define $\bar{Y}_i$ by $P\{\bar{Y}_i \in S | J_i = t\} = (t^E\omega)(S)$ for Borel sets $S \subset \mathbb{R}^d$. Moreover, choose $\tilde{Y}_i$ i.i.d. $\mathbb{R}^d$-valued and independent of $(\bar{Y}_i, J_i)$ with distribution $\tilde{\mu}$, where $\tilde{\mu}$ is infinitely divisible with Lévy representation $[0, 0, \phi_1]$ for some Lévy measure ϕ_1 on $\mathbb{R}^d$ with $t^{(1/\beta)E}\phi_1 = t \cdot \phi_1$ for all $t > 0$. Then, if we set $Y_i = \tilde{Y}_i + \bar{Y}_i$, the i.i.d. sequence (Y_i, J_i) satisfies (2.8), *where the Lévy measure ϕ of (A, D) has the form* (2.9) *and* (2.10).

PROOF. It follows from Theorem 7.2.1 together with Theorem 8.3.24 of [18] that the limit (A, D) in (2.8) is operator-stable with exponent $F = \operatorname{diag}((1/\beta)E, 1/\beta)$ and hence, by a variant of Lemma 7.1.6 of [18], we have

$$c \cdot \phi = c^F \phi \qquad \text{for all } c > 0, \tag{2.11}$$

where ϕ denotes the Lévy measure of (A, D). Note that since $D > 0$ almost surely, ϕ is supported on $\mathbb{R}^d \times [0, \infty)$.

Now define for Borel sets $B_1 \subset \mathbb{R}^d \setminus \{0\}$ and $r > 0$,

$$\phi_1(B_1) = \phi(B_1 \times \{0\}) \quad \text{and} \quad \phi_2\big(B_1 \times (r, \infty)\big) = \phi\big(B_1 \times (r, \infty)\big).$$

Then (2.9) holds. Moreover, it is easy to see that ϕ_1 is a Lévy measure on $\mathbb{R}^d$ and ϕ_2 is a Lévy measure on $\mathbb{R}^d \times (0, \infty)$. For $c > 0$ we get from (2.11) that

$$\begin{aligned} c \cdot \phi_1(B_1) &= c \cdot \phi(B_1 \times \{0\}) = (c^F \phi)(B_1 \times \{0\}) = \phi\big(c^{-F}(B_1 \times \{0\})\big) \\ &= \phi\big(c^{-(1/\beta)E} B_1 \times \{0\}\big) = \phi_1\big(c^{-(1/\beta)E} B_1\big) = c^{(1/\beta)E} \phi_1(B_1), \end{aligned}$$

so $t \cdot \phi_1 = t^{(1/\beta)E} \phi_1$ for all $t > 0$.

It remains to show now that ϕ_2 can be written in the form (2.10). To do so, for a Borel set $B \subset \mathbb{R}^d$ and $r > 0$, let

$$S(B, r) = \{(t^E y, t) : t > r, y \in B\}$$

and note that all sets of this form are a $\bigcap$-stable generator of the Borel sets of $\mathbb{R}^d \times (0, \infty)$. This follows from the fact that the mapping $\psi(y, t) = (t^E y, t)$ is a homeomorphism from $\mathbb{R}^d \times (0, \infty)$ to $\mathbb{R}^d \times (0, \infty)$. Compute that $S(B, r) = r^{\beta F} S(B, 1)$ and hence by (2.11),

$$\phi\big(S(B, r)\big) = \phi\big(r^{\beta F} S(B, 1)\big) = (r^{-\beta F} \phi)\big(S(B, 1)\big) = r^{-\beta} \cdot \phi\big(S(B, 1)\big). \tag{2.12}$$

On the other hand, we have for any probability measure ω on $\mathbb{R}^d$ and any $K > 0$,

$$\begin{aligned} \int_{S(B,r)} (t^E \omega)(dy) K \beta t^{-\beta-1}\, dt &= \int_r^\infty \int_{t^E B} (t^E \omega)(dy) K \beta t^{-\beta-1}\, dt \\ &= \int_r^\infty \omega(B) K \beta t^{-\beta-1}\, dt \\ &= \omega(B) K r^{-\beta}. \end{aligned} \tag{2.13}$$

Now we define $\omega(B) = (1/K)\phi(S(B, 1))$, where $K = \phi(S(\mathbb{R}^d, 1)) > 0$ since ϕ is not the zero measure on $\mathbb{R}^d \times \mathbb{R}_+$. Then, by (2.12) and (2.13) we obtain

$$\phi\big(S(B, r)\big) = \int_{S(B,r)} (t^E \omega)(dy) K \beta t^{-\beta-1}\, dt$$

showing that (2.10) holds.

For the proof of the converse, choose $A_1, \tilde{Y}_i$ i.i.d. and $\mathbb{R}^d$-valued with distribution $\tilde{\mu}$, where $\tilde{\mu}$ is infinitely divisible with Lévy representation $[0, 0, \phi_1]$.

Since $t^{(1/\beta)E}\phi_1 = t \cdot \phi_1$ for all $t > 0$, it follows from a variant of Lemma 7.1.6 of [18] that $t^{(1/\beta)E}\tilde{\mu} = \tilde{\mu}^t$ for all $t > 0$. Hence

$$n^{-(1/\beta)E} \sum_{i=1}^{n} \tilde{Y}_i \stackrel{d}{=} A_1 \qquad \text{for all } n \geq 1. \tag{2.14}$$

Independent of $(\tilde{Y}_i)$ choose J_i i.i.d. as D, where D is β-stable with $\mathbb{E}(e^{-sD}) = \exp\{-K\Gamma(1-\beta)s^{\beta}\}$ and denote its distribution by ρ. By stability we have $n^{-1/\beta}\sum_{i=1}^{n} J_i \Rightarrow D$ as $n \to \infty$ and hence, by Theorem 3.3.8 of [18], we have

$$n \cdot (n^{-1/\beta}\rho) \to \bar{\phi} \qquad \text{as } n \to \infty,$$

where $\bar{\phi}$ is the Lévy measure of ρ. A direct calculation shows that $\bar{\phi}(r, \infty) = Kr^{-\beta}$ and $\bar{\phi}(-\infty, -r) = 0$ for all $r > 0$ (cf. [18], page 266). Independent of $(\tilde{Y}_i)$ we now choose $\bar{Y}_i$ i.i.d. such that $P\{\bar{Y}_i \in S | J_i = t\} = (t^E\omega)(S)$ for all $t > 0$ and any Borel set $S \subset \mathbb{R}^d$. We will now show that for some nonrandom $(d_n, e_n) \in \mathbb{R}^d \times \mathbb{R}$, we have

$$n^{-F} \sum_{i=1}^{n} (\bar{Y}_i, J_i) - (d_n, e_n) \Rightarrow (A_2, D), \tag{2.15}$$

where the distribution of (A_2, D) has the Lévy representation $[a, 0, \phi_2]$ for some $a \in \mathbb{R}^{d+1}$ and ϕ_2 as in (2.10).

To show (2.15), let μ denote the distribution of $(\bar{Y}_1, J_1)$. Then for continuity sets $B_1 \subset \mathbb{R}^d \setminus \{0\}$ and $B_2 \subset \mathbb{R}_+ \setminus \{0\}$, both bounded away from zero, we get

$$\begin{aligned}
n \cdot (n^{-F}\mu)(B_1 \times B_2) &= n \cdot \mu\big(\big(n^{(1/\beta)E}B_1\big) \times (n^{1/\beta}B_2)\big) \\
&= nP\{\bar{Y}_1 \in n^{(1/\beta)E}B_1, J_1 \in n^{1/\beta}B_2\} \\
&= n\int P\{\bar{Y}_1 \in n^{(1/\beta)E}B_1 | J_1 = t\}\mathbb{1}_{B_2}(n^{-1/\beta}t)\rho(dt) \\
&= n\int (t^E\omega)\big(n^{(1/\beta)E}B_1\big)\mathbb{1}_{B_2}(n^{-1/\beta}t)\rho(dt) \\
&= n\int \big((n^{1/\beta}u)^E\omega\big)\big(n^{(1/\beta)E}B_1\big)\mathbb{1}_{B_2}(u)\rho(n^{1/\beta}\,du) \\
&= \int (u^E\omega)(B_1)\mathbb{1}_{B_2}(u)n \cdot (n^{-1/\beta}\rho)(du) \\
&\to \int (u^E\omega)(B_1)\mathbb{1}_{B_2}(u)\bar{\phi}(du) \\
&= \int (u^E\omega)(B_1)\mathbb{1}_{B_2}(u)K\beta u^{-\beta-1}\,du \\
&= \phi_2(B_1 \times B_2)
\end{aligned}$$

as $n \to \infty$, since $u \mapsto (u^E \omega)(B_1)$ is a bounded continuous function and the measure $\mathbb{1}_{B_2}(u) K\beta u^{-\beta-1}\,du$ is finite. Hence

$$n \cdot (n^{-F}\mu) \to \phi_2 \qquad \text{as } n \to \infty. \tag{2.16}$$

We show now that ϕ_2 is indeed a Lévy measure on $\mathbb{R}^{d+1}$. For Borel sets B_1 and B_2 as above and $s > 0$, a simple change of variable yields

$$s^F \phi_2(B_1 \times B_2) = s \cdot \phi_2(B_1 \times B_2).$$

Now Lemma 6.3.11 of [18] (take $\zeta = 2$ and note that the $\mathcal{M}$-full assumption is superfluous) implies that ϕ_2 is a Lévy measure.

Let L denote the smallest subspace of $\mathbb{R}^{d+1}$ supporting μ. By a change of variable, it follows easily that for any $s > 0$,

$$(s^F \mu)(dy, dt) = (t^E \omega)(dy)\rho^s(dt),$$

where ρ^s is the s-fold convolution power of ρ. Since $\operatorname{supp}(\rho^s) = [0, \infty)$ for all $s > 0$, it follows that $\operatorname{supp}(s^F \mu) = \operatorname{supp}(\mu)$ for all $s > 0$. Then Lemma 1.1.9 of [18] implies $s^F L = L$ for all $s > 0$. It follows from (2.10) that ϕ_2 is also supported on L, and not on any proper subspace of L. Let μ', ϕ_2' denote the restriction of μ, ϕ_2 to L and let A_n denote the restriction of n^F to L. Since $s^F \phi_2 = s \cdot \phi_2$ for any $s > 0$, we also have $A_n \phi_2' = n \cdot \phi_2'$ for each n. Then the infinitely divisible law ν with Lévy representation $[0, 0, \phi_2']$ satisfies $\nu^n = A_n \nu$, and hence ν is a full operator stable law on L with Lévy measure ϕ_2'. We also have $n(A_n \mu') \to \phi_2'$ and then Corollary 8.2.12 of [18] shows that $A_n(\mu')^n * \varepsilon_{S(n)} \Rightarrow \nu$ for some centering constants $S(n) \in L$. Then (2.15) holds with the limit (A_2, D) supported on L.

By independence, (2.14) together with (2.15) imply

$$\left(n^{-F} \sum_{i=1}^{n} (\bar{Y}_i, J_i) - (d_n, e_n), n^{-F} \sum_{i=1}^{n} (\tilde{Y}_i, 0) \right) \Rightarrow \big((A_2, D), (A_1, 0)\big). \tag{2.17}$$

Note that the distribution of $(A_1, 0)$ has Lévy measure $\varepsilon_0(dt)\phi_1(dy)$. By continuous mapping, (2.17) implies

$$n^{-F} \sum_{i=1}^{n} (\bar{Y}_i + \tilde{Y}_i, J_i) - (d_n, e_n) \Rightarrow (A_1 + A_2, D)$$

as $n \to \infty$. Since (A_2, D) has Lévy measure ϕ_2 of the form (2.10), $(A_1, 0)$ has Lévy measure $\varepsilon_0(dt)\phi_1(dy)$, and (A_2, D) and $(A_1, 0)$ are independent, the Lévy measure of $(A_1 + A_2, D)$ has the form (2.9). Note that since D is β-stable with $0 < \beta < 1$, Theorem 8.2.16 of [18] shows that we can take $e_n = 0$ for all n. This concludes the proof. □

Theorem 2.2 implies an interesting characterization of the independence of A and D in (2.8) in terms of the measure ω in (2.10).

COROLLARY 2.3. *In the situation of Theorem* 2.2, *the random variables* A *and* D *in* (2.8) *are independent if and only if* $\omega = \varepsilon_0$ *in* (2.10).

PROOF. Assume first that $\omega = \varepsilon_0$. Then $t^E \omega = \varepsilon_0$ for all $t > 0$ and hence by (2.9) we have, for the Lévy measure ϕ of (A, D), that

$$\phi(dy, dt) = \varepsilon_0(dt)\phi_1(dy) + \varepsilon_0(dy)K\beta t^{-\beta-1}\, dt.$$

Then, in view of the Lévy representation, it follows that $\mathbb{E}[\exp(i\langle A, k\rangle - sD)] = \mathbb{E}[\exp(i\langle A, k\rangle)] \cdot \mathbb{E}[\exp(-sD)]$ and hence A and D are independent.

Conversely, if A and D are independent, in view of the Lévy representation, the Lévy measure ϕ of (A, D) has the form

$$\phi(dy, dt) = \varepsilon_0(dt)\phi_1(dy) + \varepsilon_0(dy)\bar{\phi}(dt),$$

where $\bar{\phi}(dt) = K\beta t^{-\beta-1}$ for some $K > 0$. On the other hand, it also has the form (2.9). Hence, by uniqueness of the Lévy measure and (2.10), we obtain $t^E \omega = \varepsilon_0$ for all $t > 0$ and hence $\omega = \varepsilon_0$. This concludes the proof. □

Since we are interested in convergence of stochastic processes in Skorokhod spaces we need some further notation. If S is a complete separable metric space, let $D([0, \infty), S)$ denote the space of all right-continuous S-valued functions on $[0, \infty)$ with limits from the left and endow $D([0, \infty), S)$ with the J_1 topology introduced in [29]. Note that by definition all the sample paths of the process $\{(S(t), T(t))\}_{t\geq 0}$ belong to $D([0, \infty), \mathbb{R}^d \times [0, \infty))$. Now let $\{(A(t), D(t))\}_{t\geq 0}$ denote the operator Lévy motion generated by (A, D). That is, the process $\{(A(t), D(t))\}_{t\geq 0}$ has stationary independent increments with $(A(t), D(t)) \stackrel{d}{=} t^{\mathrm{diag}((1/\beta)E, 1/\beta)}(A, D)$, where $\stackrel{d}{=}$ means equality in distribution. Note that in view of [25], page 197, we can assume without loss of generality that all the sample paths of that process also belong to $D([0, \infty), \mathbb{R}^d \times [0, \infty))$. It then follows from Example 11.2.18 in [18] together with Theorem 4.1 in [19] that

$$\{\mathrm{diag}\,(B(c), b(c))(S(ct), T(ct))\}_{t\geq 0} \Rightarrow \{(A(t), D(t))\}_{t\geq 0} \tag{2.18}$$

in $J_1 - D([0, \infty), \mathbb{R}^d \times \mathbb{R}_+)$ as $c \to \infty$. There is another topology on $D([0, \infty), S)$ called the M_1 topology, which is more suitable for our purposes. It is weaker than the J_1 topology, and hence (2.18) also holds in the M_1 topology; see [33] and [32] for details. For an element $x \in D([0, \infty), S)$, let

$$\mathrm{Disc}(x) = \{t \geq 0 : x(t-) \neq x(t)\}$$

denote the set of discontinuity points of x.

3. The limit theorem. In this section we prove the main result of this article. We show that under a certain condition, a scaling limit of the CTRW process $\{X(t)\}_{t\geq 0}$ converges weakly on $D([0,\infty),\mathbb{R}^d)$ in the M_1 topology, and we investigate the limit. Moreover, if this additional condition is dropped, we still get convergence in distribution for each fixed point in time.

Define the hitting time process of the stable subordinator $\{D(t)\}_{t\geq 0}$ by

$$E(t) = \inf\{x \geq 0 : D(x) > t\}. \tag{3.1}$$

Note that $\{E(t)\}_{t\geq 0}$ has almost surely continuous nondecreasing sample paths. Moreover, it is easy to see that $\{E(t)\}_{t\geq 0}$ is strictly increasing at some $t_0 > 0$ if and only if $\{D(t)\}_{t\geq 0}$ is continuous at $E(t_0)$. Furthermore, for $x, t \geq 0$ we have

$$\{E(t) \leq x\} = \{D(x) \geq t\}. \tag{3.2}$$

See [19] and [5] for more information on the hitting time process $\{E(t)\}_{t\geq 0}$.

Recall from Section 2 that the norming function b in (2.5) is regularly varying with index $-1/\beta$. Hence $1/b$ is regularly varying with index $1/\beta > 0$ so, by Property 1.5.5 of [27], there exists a regularly varying function $\tilde{b}$ with index β such that $1/b(\tilde{b}(c)) \sim c$ as $c \to \infty$. Here we use the notation $f \sim g$ for positive functions f, g if and only if $f(c)/g(c) \to 1$ as $c \to \infty$. Equivalently we have

$$b\big(\tilde{b}(c)\big) \sim \frac{1}{c} \qquad \text{as } c \to \infty. \tag{3.3}$$

Then, since the norming function B in (2.3) is $\mathrm{RV}(-(1/\beta)E)$, the function $\tilde{B}(c) = B(\tilde{b}(c))$ is $\mathrm{RV}(-E)$.

THEOREM 3.1. *Assume that* (Y_i, J_i) *are i.i.d.* $\mathbb{R}^d \times [0,\infty)$*-valued random vectors and that* (2.3) *holds. If*

$$\operatorname{Disc}\big(\{A(t)\}_{t\geq 0}\big) \cap \operatorname{Disc}\big(\{D(t)\}_{t\geq 0}\big) = \varnothing \qquad \textit{almost surely}, \tag{3.4}$$

then

$$\{\tilde{B}(c)X(ct)\}_{t\geq 0} \Rightarrow \{M(t)\}_{t\geq 0} \qquad \textit{in } M_1 - D\big([0,\infty),\mathbb{R}^d\big) \textit{ as } c \to \infty, \tag{3.5}$$

where $\{M(t)\}_{t\geq 0} = \{A(E(t))\}_{t\geq 0}$ *is a subordinated process with* $\{A(t)\}_{t\geq 0}$ *being the first coordinate of the limit in* (2.18) *and* $\{E(t)\}_{t\geq 0}$ *defined in* (3.1) *using the second coordinate of the limit in* (2.18).

PROOF. As indicated above, (2.3) implies (2.18). We use the continuous mapping approach together with the continuity of certain functionals on $D([0,\infty),\mathbb{R}^d)$ as in [33, 32]. In fact, define the mappings $\Psi_c : D(\mathbb{R}_+,\mathbb{R}^d) \times D(\mathbb{R}_+,\mathbb{R}_+) \to D(\mathbb{R}_+,\mathbb{R}^d) \times D(\mathbb{R}_+,\mathbb{R}_+)$, $\Psi_c(x,y) = (x, (cb(\tilde{b}(c)))^{-1}y)$. Then in view of (3.3) we have $\Psi_c(x,y) \to (x,y)$ in the J_1 topology, where as usual the topology on product spaces is the product topology. Hence, by (2.18), using Theorem 3.4.4 of [33], we obtain

$$(3.6)\quad \left(\{B(\tilde{b}(c))S(\tilde{b}(c)t)\}_{t\ge 0}, \left\{\frac{1}{c}T(\tilde{b}(c)t)\right\}_{t\ge 0}\right) \Rightarrow (\{A(t)\}_{t\ge 0}, \{D(t)\}_{t\ge 0})$$

as $c\to\infty$ in the J_1 topology. For a nondecreasing $y\in D(\mathbb{R}_+,\mathbb{R}_+)$, define the inverse $y^{-1}(t)=\inf\{s\ge 0: y(s)>t\}$ and note that by (3.1) we have $E(t)=D^{-1}(t)$. Furthermore, $\inf\{s\ge 0:(1/c)T(\tilde{b}(c)s)>t\}=(1/\tilde{b}(c))(N_{ct}+1)$. Define the mapping $\Phi: D(\mathbb{R}_+,\mathbb{R}^d)\times D(\mathbb{R}_+,\mathbb{R}_+)\to D(\mathbb{R}_+,\mathbb{R}^d)\times D(\mathbb{R}_+,\mathbb{R}_+)$, $\Phi(x,y)=(x,y^{-1})$. Then Corollary 13.6.3 together with Theorem 3.4.3 of [33], since (3.6) also holds in the weaker M_1 topology, imply

$$(3.7)\quad \left(\{B(\tilde{b}(c))S(\tilde{b}(c)t)\}_{t\ge 0}, \left\{\frac{1}{\tilde{b}(c)}(N_{ct}+1)\right\}_{t\ge 0}\right) \Rightarrow (\{A(t)\}_{t\ge 0}, \{E(t)\}_{t\ge 0})$$

as $c\to\infty$ in the M_1 topology. Next define the mappings $\Xi_c: D(\mathbb{R}_+,\mathbb{R}^d)\times D(\mathbb{R}_+,\mathbb{R}_+)\to D(\mathbb{R}_+,\mathbb{R}^d)\times D(\mathbb{R}_+,\mathbb{R}_+)$, $\Xi_c(x,y)=(x,y-1/\tilde{b}(c))$. Then, since $\tilde{b}(c)\to\infty$, $\Xi_c(x_c,y_c)\to(x,y)$ as $c\to\infty$ in the M_1 topology, whenever $x_c\to x$, $y_c\to y$ in M_1 and y is continuous. This follows from the fact that for continuous y, the convergence $y_c\to y$ in M_1 is equivalent to the uniform convergence on compact sets; see [33], Chapter 3.3, for details. Another application of Theorem 3.4.4 of [33] to (3.7) yields

$$(3.8)\quad \left(\{B(\tilde{b}(c))S(\tilde{b}(c)t)\}_{t\ge 0}, \left\{\frac{1}{\tilde{b}(c)}N_{ct}\right\}_{t\ge 0}\right) \Rightarrow (\{A(t)_{t\ge 0}\}, \{E(t)\}_{t\ge 0})$$

as $c\to\infty$ in the M_1 topology. We now want to apply Theorem 13.2.4 together with Theorem 3.4.4 of [33]. Note that condition (ii) of Theorem 13.2.4 is always true since $E(t)$ is continuous. Moreover, condition (i) of Theorem 13.2.4 follows from condition (3.4) using the fact that $E(t)$ is strictly increasing in t if and only if $D(\cdot)$ is continuous in $E(t)$. An application of Theorem 13.2.4 and Theorem 3.4.4 of [33] shows that (3.8) implies

$$\{\tilde{B}(c)X(ct)\}_{t\ge 0}=\{B(\tilde{b}(c))S(N_{ct})\}_{t\ge 0}\Rightarrow\{A(E(t))\}_{t\ge 0}$$

as $c\to\infty$ in the M_1 topology. This concludes the proof. □

REMARK 3.2. (a) If the processes $\{A(t)\}_{t\ge 0}$ and $\{D(t)\}_{t\ge 0}$ are independent, a conditioning argument together with the well-known fact that any Lévy process has almost surely no fixed point of discontinuity implies that condition (3.4) holds.

(b) If $\{A(t)\}_{t\ge 0}$ is a multivariate Brownian motion, then $\mathrm{Disc}(\{A(t)\}_{t\ge 0})=\varnothing$ almost surely and hence (3.4) is trivially fulfilled.

(c) If we have $Y_n = J_n$ pointwise for all $n \geq 1$ and $b(n)T(n) \Rightarrow D$, then $\{b(c)T(ct)\}_{t\geq 0} \Rightarrow \{D(t)\}_{t\geq 0}$ in $J_1 - D([0,\infty),\mathbb{R})$. Since $S(t) = T(t)$ pointwise, if we set $B(c) = b(c)$, using the continuous mapping $x \mapsto (x,x)$ from $D([0,\infty),\mathbb{R})$ to $D([0,\infty),\mathbb{R}) \times D([0,\infty),\mathbb{R})$, we get

$$\{(B(c)S(ct), b(c)T(ct))\}_{t\geq 0} \Rightarrow \{(A(t), D(t))\}_{t\geq 0}$$

in the J_1 topology, where now $A(t) = D(t)$ pointwise. Then condition (3.4) does not hold. This special case was also considered in [10] for one fixed point $t > 0$ in time.

We have not been able to prove process convergence in cases like Remark 3.2(c), where (3.4) does not hold. However, for many physics applications it is sufficient to show that $\tilde{B}(c)X(ct) \Rightarrow M(t) = A(E(t))$ as $c \to \infty$ for each fixed $t > 0$. We prove this now, under the more general assumptions laid out in the beginning of Section 2. We begin with a lemma that shows how to compute the limit distribution using fractional derivatives. For suitable functions $u:\mathbb{R}_+ \to \mathbb{R}$, we define the fractional derivative $\partial^\beta u(t)/\partial t^\beta$ of order $0 < \beta < 1$ as the inverse Laplace transform of $s^\beta \mathcal{L}u(s)$, where $\mathcal{L}u(s) = \int_0^\infty e^{-st}u(t)\,dt$ is the usual Laplace transform of u. This generalizes the well-known formula for derivatives and their Laplace transforms to noninteger orders; see [24] for details. Recall from Section 2 that if (2.3) holds, then (2.4) also holds for a suitable sequence of norming constants (b_n). Hence that assumption on D entails no loss of generality.

LEMMA 3.3. *Assume that (Y_i, J_i) are i.i.d. $\mathbb{R}^d \times [0,\infty)$-valued random vectors. Assume that* (2.3) *and* (2.4) *hold, and fix any $t > 0$. Then for all Borel sets $M \subset \mathbb{R}^d$ whose boundary has zero Lebesgue measure, we have*

$$\lim_{c\to\infty} P\{\tilde{B}(c)X(ct) \in M\} = \int_0^\infty \frac{\partial^\beta}{\partial t^\beta} H_s(t)\,ds, \tag{3.9}$$

where

$$H_s(t) = P\{A(s) \in M, D(s) \leq t\}. \tag{3.10}$$

PROOF. For a given set $M \subset \mathbb{R}^d$, $c > 0$ and $t, s \geq 0$ let

$$F_s^{(c)}(t) = P\{\tilde{B}(c)S(s) \in M, T(s) \leq ct\}.$$

Since by [8], Theorem 4.10.2, the distribution of (A, D) has a bounded C^∞ density, every set M whose boundary has Lebesgue measure zero is a continuity set of the distribution of A. Moreover, the mapping $(s,t) \mapsto H_s(t)$ is continuous.

Now let $s(c) \to s \geq 0$ as $c \to \infty$. Then it follows from (2.5), using Proposition 3.3.7 of [18], that

$$\begin{aligned} &P\{B(\tilde{b}(c))S(\tilde{b}(c)s(c)) \in M, b(\tilde{b}(c))T(\tilde{b}(c)s(c)) \leq t\} \\ &\quad \to P\{A(s) \in M, D(s) \leq t\} = H_s(t) \end{aligned} \tag{3.11}$$

for $c \to \infty$. Note that the proof of Proposition 3.3.7 in [18] remains true in the case $s = 0$. Let $\delta(c) = b(\tilde{b}(c)) \cdot c$. Then by (3.3), $\delta(c) \to 1$ as $c \to \infty$. Now write

$$F^{(c)}_{\tilde{b}(c)s(c)}(t) = P\{B(\tilde{b}(c))S(\tilde{b}(c)s(c)) \in M, b(\tilde{b}(c))T(\tilde{b}(c)s(c)) \le t\}$$

$$+ \Big(P\{B(\tilde{b}(c))S(\tilde{b}(c)s(c)) \in M, b(\tilde{b}(c))T(\tilde{b}(c)s(c)) \le \delta(c)t\}$$

$$- P\{B(\tilde{b}(c))S(\tilde{b}(c)s(c)) \in M, b(\tilde{b}(c))T(\tilde{b}(c)s(c)) \le t\}\Big),$$

where by (3.11) the first summand on the right-hand side of the equation above tends to $H_s(t)$. Furthermore, the absolute value of the difference in brackets on the right-hand side of that equation can be bounded from above by $P\{b(\tilde{b}(c))T(\tilde{b}(c)s(c)) \in I_c\}$, where $I_c =]\min\{\delta(c)t, t\}, \max\{\delta(c)t, t\}]$. Given $\varepsilon > 0$ arbitrary, choose $\delta > 0$ such that $P\{D(s) \in [t-\delta, t+\delta]\} < \varepsilon$. Since $\delta(c) \to 1$ as $c \to \infty$, there exists a $c_1 > 0$ such that $I_c \subset [t-\delta, t+\delta]$ for all $c > c_1$. Letting $M = \mathbb{R}^d$ in (3.11) gives $b(\tilde{b}(c))T(\tilde{b}(c)s(c)) \Rightarrow D(s)$ as $c \to \infty$ and then, using Proposition 1.2.13 of [18], we obtain

$$\limsup_{c\to\infty} P\{b(\tilde{b}(c))T(\tilde{b}(c)s(c)) \in I_c\}$$

$$\le \limsup_{c\to\infty} P\{b(\tilde{b}(c))T(\tilde{b}(c)s(c)) \in [t-\delta, t+\delta]\}$$

$$\le P\{D(s) \in [t-\delta, t+\delta]\} < \varepsilon.$$

Putting things together, we have shown

$$\lim_{c\to\infty} F^{(c)}_{\tilde{b}(c)s}(t) = H_s(t) \tag{3.12}$$

uniformly on compact subsets of $s \ge 0$.

Note that (2.1) implies that $\{N_t \ge n\} = \{T(n) \le t\}$. Then we get

$$P\{\tilde{B}(c)X(ct) \in M\}$$

$$= \sum_{n=0}^{\infty} P\{\tilde{B}(c)S(n) \in M, N_{ct} = n\}$$

$$= \sum_{n=0}^{\infty} \big[P\{\tilde{B}(c)S(n) \in M, N_{ct} \ge n\} - P\{\tilde{B}(c)S(n) \in M, N_{ct} \ge n+1\}\big]$$

$$= \sum_{n=0}^{\infty} \big[P\{\tilde{B}(c)S(n) \in M, T(n) \le ct\} - P\{\tilde{B}(c)S(n) \in M, T(n+1) \le ct\}\big].$$

Using the i.i.d. assumption on (Y_i, J_i) and letting ρ denote the distribution of J_1, we get

$$\begin{aligned}
&P\{\tilde{B}(c)S(n) \in M, T(n+1) \le ct\} \\
&\quad = \int_0^\infty P\{\tilde{B}(c)S(n) \in M, T(n) + \tau \le ct\}\, d\rho(\tau) \\
&\quad = \int_0^\infty P\{\tilde{B}(c)S(n) \in M, T(n) \le c(t-\tau)\}\, d(c^{-1}\rho)(\tau) \\
&\quad = F_n^{(c)} * (c^{-1}\rho)(t),
\end{aligned}$$

where $*$ denotes the usual convolution. Hence, we have shown that

$$P\{\tilde{B}(c)X(ct) \in M\} = \sum_{n=0}^{\infty} F_n^{(c)} * \big(\varepsilon_0 - (c^{-1}\rho)\big)(t). \tag{3.13}$$

Note that by Theorem 3.6 of [19] we have $\tilde{b}(c)^{-1}N_{ct} \Rightarrow E(t)$ as $c \to \infty$. Hence, given $\varepsilon > 0$, there exists a $s_0 > 0$ such that $P\{\tilde{b}(c)^{-1}N_{ct} \ge s_0\} < \varepsilon$ for all $c > 0$. Therefore,

$$\begin{aligned}
I_2^{(c)} &= \sum_{n=[\tilde{b}(c)s_0]+1}^{\infty} P\{\tilde{B}(c)S(n) \in M, N_{ct} = n\} \\
&\le \sum_{n=[\tilde{b}(c)s_0]+1}^{\infty} P\{N_{ct} = n\} \\
&\le P\{N_{ct} \ge \tilde{b}(c)s_0\} < \varepsilon
\end{aligned} \tag{3.14}$$

for all $c > 0$. Next we show that for some $s_1 \ge s_0$ we have

$$\left| \int_{s_1}^\infty \frac{\partial^\beta}{\partial t^\beta} H_s(t)\, ds \right| < \varepsilon. \tag{3.15}$$

Recall from [8], Theorem 4.10.2, that the distribution of (A, D) has a bounded C^∞ density $p(x, u)$ and let g_β denote the density of D, which is a bounded C^∞ function supported on $\mathbb{R}_+$. Note that by [24], page 109, we have

$$\frac{\partial^\beta}{\partial t^\beta} H_s(t) = \frac{1}{\Gamma(1-\beta)} \int_0^\infty \frac{\partial}{\partial t} H_s(t-\tau)\tau^{-\beta}\, d\tau,$$

where, using the operator stability of (A, D),

$$\begin{aligned}
\frac{\partial}{\partial t} H_s(t) &= \frac{\partial}{\partial t} P\big\{s^{(1/\beta)E} A \in M, s^{1/\beta} D \le t\big\} \\
&= s^{-1/\beta} \bigg(\frac{\partial}{\partial t} P\big\{s^{(1/\beta)E} A \in M, D \le \cdot\big\} \bigg)(s^{-1/\beta} t) \\
&= s^{-1/\beta} \int_{s^{-(1/\beta)E} M} p(x, s^{-1/\beta} t)\, dx \\
&\le s^{-1/\beta} g_\beta(s^{-1/\beta} t).
\end{aligned}$$

Then, for some positive constant $K > 0$, we have

$$\begin{aligned}\frac{\partial^\beta}{\partial t^\beta} H_s(t) &\le \frac{s^{-1/\beta}}{\Gamma(1-\beta)} \int_0^t g_\beta\big(s^{-1/\beta}(t-\tau)\big)\tau^{-\beta}\, d\tau \\ &\le K t^{1-\beta} s^{-1/\beta}.\end{aligned}$$

Hence, since $0 < \beta < 1$, (3.15) follows if s_1 is chosen large enough. Putting things together, in view of (3.13)–(3.15), it is enough to show that

$$\lim_{c\to\infty} \sum_{n=0}^{[\tilde b(c)s_1]} F_n^{(c)} * \big(\varepsilon_0 - (c^{-1}\rho)\big)(t) = \int_0^{s_1} \frac{\partial^\beta}{\partial t^\beta} H_s(t)\, ds. \tag{3.16}$$

To do so, let

$$\psi_c(s) = \frac{1}{\tilde b(c)} \sum_{n=0}^{\infty} (n+1) \mathbb{1}_{[n,n+1)}(s).$$

Note that for any fixed $s \ge 0$, we have $\psi_c(\tilde b(c)s) \to s$ as $c \to \infty$.

Note that (2.5) together with (3.3) implies that $c^{-1}T(\tilde b(c)) = b(\tilde b(c))T(\tilde b(c)) \times (b(\tilde b(c))/c) \Rightarrow D$ as $c \to \infty$. Then the continuity theorem for Laplace transforms (see, e.g., [6], page 433, Theorem 2a), implies

$$\big(\mathcal{L}(\rho)(c^{-1}\xi)\big)^{\tilde b(c)} \to \mathcal{L}(P_D)(\xi) \qquad \text{as } c \to \infty.$$

Using the fact that $\log(1+x) \sim x$ as $x \to 0$, together with (2.4), we get, as $c \to \infty$,

$$\begin{aligned}\mathcal{L}\big(\tilde b(c)\cdot\big((c^{-1}\rho) - \varepsilon_0\big)\big)(\xi) &\sim \tilde b(c)\log\big(\mathcal{L}(\rho)(c^{-1}\xi)\big) \\ &= \log\big(\big(\mathcal{L}(\rho)(c^{-1}\xi)\big)^{\tilde b(c)}\big) \\ &\to \log\big(\mathcal{L}(P_D)(\xi)\big) = -\xi^\beta.\end{aligned}$$

Hence, by taking Laplace transforms in (3.12), we also have uniformly on compact sets of $\{s \ge 0\}$, as $c \to \infty$,

$$\begin{aligned}&\mathcal{L}\big(F^{(c)}_{\tilde b(c)s} * \big[\tilde b(c)\cdot\big(\varepsilon_0 - (c^{-1}\rho)\big)\big]\big)(\xi) \\ &\quad = \mathcal{L}\big(F^{(c)}_{\tilde b(c)s}\big)(\xi)\cdot\mathcal{L}\big(\tilde b(c)\cdot\big(\varepsilon_0 - (c^{-1}\rho)\big)\big)(\xi) \\ &\quad \to \xi^\beta \mathcal{L}(H_s)(\xi) = \mathcal{L}\bigg(\frac{\partial^\beta}{\partial t^\beta} H_s(\cdot)\bigg)(\xi),\end{aligned}$$

and hence, using the continuity theorem for Laplace transforms again,

$$F^{(c)}_{\tilde b(c)s} * \big[\tilde b(c)\cdot\big(\varepsilon_0 - (c^{-1}\rho)\big)\big](t) \to \frac{\partial^\beta}{\partial t^\beta} H_s(t) \tag{3.17}$$

as $c \to \infty$, uniformly in $0 \le s \le s_1$.

Since

$$\begin{aligned}\sum_{n=0}^{[\tilde{b}(c)s_1]} F_n^{(c)} * [\varepsilon_0 - (c^{-1}\rho)](t) \\ &= \int_0^{[\tilde{b}(c)s_1]} F_s^{(c)} * \big[\tilde{b}(c)\cdot\big(\varepsilon_0 - (c^{-1}\rho)\big)\big](t)\, d\psi_c(s) \\ &= \int_0^{s_1} F_{\tilde{b}(c)s}^{(c)} * \big[\tilde{b}(c)\cdot\big(\varepsilon_0 - (c^{-1}\rho)\big)\big](t)\, d\psi_c\big(\tilde{b}(c)s\big),\end{aligned}$$

(3.16) follows from the uniform convergence in (3.17) together with the following argument: Assume that functions $f_n(t) \to f(t)$ uniformly on $[0, a]$, where f is continuous. Let λ_n be measures on $\mathbb{R}_+$ with $\lambda_n \to \lambda^1$ vaguely, where λ^1 is Lebesgue measure on $\mathbb{R}_+$. Then

$$\int_0^a f_n(t)\, d\lambda_n(t) \to \int_0^a f(t)\, dt \qquad \text{for } n \to \infty.$$

This concludes the proof. □

THEOREM 3.4. *Assume that* (Y_i, J_i) *are i.i.d.* $\mathbb{R}^d \times [0,\infty)$*-valued random vectors, and that* (2.3) *and* (2.4) *hold. Then for any fixed* $t > 0$, *we have*

$$\tilde{B}(c)X(ct) \Rightarrow A(E(t)) \qquad \text{as } c \to \infty, \tag{3.18}$$

where $A(t)$ *is the first coordinate of the limit in* (2.18) *and* $E(t)$ *is defined by* (3.1) *using the second coordinate of the limit in* (2.18).

PROOF. In view of Lemma 3.3, it suffices to show that for any Borel set $M \subset \mathbb{R}^d$ whose boundary has Lebesgue measure zero, we have

$$P\{A(E(t)) \in M\} = \int_0^\infty \frac{\partial^\beta}{\partial t^\beta} P\{A(s) \in M, D(s) \le t\}\, ds.$$

Note that for bounded C^∞ functions f on $[0, \infty)$ we have

$$\frac{\partial^\beta}{\partial t^\beta} f(t) = -\lim_{h\downarrow 0} \frac{1}{h} \int_0^\infty f(t-\tau)\, d\big(P_{D(h)} - \varepsilon_0\big)(\tau), \tag{3.19}$$

where $P_{D(h)}$ denotes the distribution of $D(h)$ and $\{D(t)\}_{t\ge 0}$ is the second component in (2.18), for example, a β-stable subordinator such that $\mathcal{L}(P_{D(t)})(s) = \exp(-t \cdot s^\beta)$. In fact, taking the Laplace transform of the integral on the right-hand side of (3.19) yields $(1/h)(\exp(-hs^\beta) - 1)\mathcal{L}f(s)$, which converges to $-s^\beta \mathcal{L}f(s)$ as $h \to 0$. Then the continuity theorem for Laplace transforms yields (3.19).

Hence, using the fact that $\{D(t)\}_{t\geq 0}$ has stationary and independent increments and $(A(s), D(s))$ has a bounded continuous density, together with (3.2), yields

$$
\begin{aligned}
\frac{\partial^\beta}{\partial t^\beta} P\{A(s)\in M, D(s)\leq t\} &= \frac{\partial^\beta}{\partial t^\beta} P\{A(s)\in M, D(s)< t\} \\
&= -\lim_{h\downarrow 0}\frac{1}{h}\int_0^\infty P\{A(s)\in M, D(s)<t-\tau\}\, d\big(P_{D(h)}-\varepsilon_0\big)(\tau) \\
&= -\lim_{h\downarrow 0}\frac{1}{h}\bigg[\int_0^\infty P\{A(s)\in M, D(s)<t-\tau\}\, dP_{D(s+h)-D(s)}(\tau) \\
&\qquad\qquad - P\{A(s)\in M, D(s)<t\}\bigg] \\
&= -\lim_{h\downarrow 0}\frac{1}{h}\big[P\{A(s)\in M, D(s+h)<t\} - P\{A(s)\in M, D(s)<t\}\big] \\
&= \lim_{h\downarrow 0}\frac{1}{h}\big[P\{A(s)\in M, E(t)>s\} - P\{A(s)\in M, E(t)>s+h\}\big] \\
&= \lim_{h\downarrow 0}\frac{1}{h} P\{A(s)\in M, s<E(t)\leq s+h\} \\
&= \lim_{h\downarrow 0}\frac{1}{h}\int_s^{s+h} P\{A(s)\in M|E(t)=u\}p_t(u)\, du \\
&= P\{A(s)\in M|E(t)=s\}p_t(s)
\end{aligned}
$$

for Lebesgue almost every $s\geq 0$, where p_t denotes the density of $E(t)$ (see [19]). Note that in the last step of the computation above, we have used Lebesgue's differentiation theorem; see, for example, [31], Theorem 7.16. Hence

$$
\begin{aligned}
P\{A(E(t))\in M\} &= \int_0^\infty P\{A(s)\in M|E(t)=s\}\, dP_{E(t)}(s) \\
&= \int_0^\infty P\{A(s)\in M|E(t)=s\}p_t(s)\, ds \\
&= \int_0^\infty \frac{\partial^\beta}{\partial t^\beta} P\{A(s)\in M, D(s)\leq t\}\, ds
\end{aligned}
$$

and the proof is complete. □

REMARK 3.5. Since by [19], Corollary 3.2, the density p_t of $P_{E(t)}$ is strictly positive on $(0,\infty)$, it follows from the proof of Lemma 3.4 above that

$$
P\{A(s)\in M|E(t)=s\} = \frac{1}{p_t(s)}\frac{\partial^\beta}{\partial t^\beta} P\{A(s)\in M, D(s)\leq t\}
$$

for almost all $s>0$ with respect to Lebesgue measure.

4. The limit distribution. In this section we analyze the structure of the distribution of $M(t) = A(E(t))$ obtained in Theorems 3.1 and 3.4. We present formulas for the density $h(x,t)$ of $M(t)$ and its FLT. It turns out that the FLT of $h(x,t)$ has a very simple form, which will enable us to compute various interesting examples of CTRW limit distributions in Section 5.

THEOREM 4.1. *For any fixed $t > 0$, the distribution of the limit $A(E(t))$ in* (3.18) *has the density*

$$h(x,t) = \int_0^\infty \frac{\partial^{\beta-1}}{\partial t^{\beta-1}} f_u(x,t)\,du, \tag{4.1}$$

where $f_u(x,t)$ is the Lebesgue density of the limit $(A(u), D(u))$ in (2.18).

PROOF. Recall from [8], Theorem 4.10.2, that the distribution of $(A(u), D(u))$ has a bounded C^∞ density $f_u(x,t)$ with respect to Lebesgue measure. Since $0 < \beta < 1$, $\partial^{\beta-1}/\partial t^{\beta-1}$ is a fractional integral of order $1-\beta$. Then in view of [24], page 94, we get from Lemma 3.3 or Corollary 3.4, using the fact that densities are nonnegative, and Tonelli's theorem that

$$\begin{aligned}
P\{M(t) \in S\} &= \int_0^\infty \frac{\partial^\beta}{\partial t^\beta} P\{A(u) \in S,\, D(u) \le t\}\,du \\
&= \int_0^\infty \frac{\partial^\beta}{\partial t^\beta}\left[\int_S \int_0^t f_u(x,\tau)\,d\tau\,dx\right] du \\
&= \int_0^\infty \frac{\partial^{\beta-1}}{\partial t^{\beta-1}} \int_S f_u(x,t)\,dx\,du \\
&= \int_S \left[\int_0^\infty \frac{\partial^{\beta-1}}{\partial t^{\beta-1}} f_u(x,t)\,du\right] dx,
\end{aligned}$$

which concludes the proof. □

Note that for h in (4.1) we have $\int_{\mathbb{R}^d \times \mathbb{R}_+} h(x,t)\,dx\,dt = \int_{\mathbb{R}_+} 1\,dt = \infty$, so the Fourier transform of $h(x,t)$ in both variables x, t is not defined. However, the following result gives the FLT of the function h in (4.1).

PROPOSITION 4.2. *For the function h in* (4.1) *the FLT $\mathcal{FL}(h)$ exists and*

$$\mathcal{FL}(h)(k,s) = s^{\beta-1} \int_0^\infty \mathcal{FL}(f_u)(k,s)\,du, \tag{4.2}$$

where $\mathcal{FL}(f_u)(k,s)$ is the FLT of the density f_u of $(A(u), D(u))$.

PROOF. Fix any $(k,s) \in \mathbb{R}^d \times (0,\infty)$ and note that for any $u \ge 0$, the FLT of the probability density $f_u(x,t)$ is well defined. Recall from [24], page 94, that for continuous functions g on $\mathbb{R}_+$ we have

$$\frac{\partial^{\beta-1}}{\partial t^{\beta-1}} g(t) = C_\beta \int_0^\infty g(t-\tau)\tau^{-\beta}\, d\tau,$$

where $C_\beta = 1/\Gamma(1-\beta) > 0$. Then by (4.1) we have

$$h(x,t) = C_\beta \int_0^\infty \int_0^t f_u(x, t-\tau)\tau^{-\beta}\, d\tau\, du.$$

Then by Tonelli's theorem, for any $x \in \mathbb{R}^d$ fixed, we obtain

$$\begin{aligned}
\mathcal{L}\big(h(x,\cdot)\big)(s) &= \int_0^\infty h(x,t)e^{-st}\, dt \\
&= \int_0^\infty \int_0^\infty e^{-st} C_\beta \int_0^t f_u(x,t-\tau)\tau^{-\beta}\, d\tau\, dt\, du \\
&= \int_0^\infty \mathcal{L}\bigg(\frac{\partial^{\beta-1}}{\partial t^{\beta-1}} f_u(x,t)\bigg)(s)\, du \\
&= s^{\beta-1} \int_0^\infty \mathcal{L}\big(f_u(x,\cdot)\big)(s)\, du.
\end{aligned} \tag{4.3}$$

Since $D(u)$ is a one-dimensional marginal of $(A(u), D(u))$, the density $g_\beta(t,u)$ of $D(u)$ is given by $g_\beta(t,u) = \int_{\mathbb{R}^d} f_u(x,t)\, dx$ and in view of (2.4), we have

$$e^{-us^\beta} = \int_0^\infty e^{-st} g_\beta(t,u)\, dt.$$

Hence, using Tonelli's theorem again,

$$\begin{aligned}
\int_{\mathbb{R}^d} \big|\mathcal{L}\big(h(x,\cdot)\big)(s)\big|\, dx &= s^{\beta-1} \int_0^\infty \int_0^\infty e^{-st} g_\beta(t,u)\, dt\, du \\
&= s^{\beta-1} \int_0^\infty e^{-us^\beta}\, du < \infty,
\end{aligned}$$

which implies that $\mathcal{F}\mathcal{L}(h)(k,s)$ is well defined as a Lebesgue integral. Moreover, (4.2) follows from (4.3) using Fubini's theorem. □

COROLLARY 4.3. *For the function* $\mathcal{F}\mathcal{L}(h)(k,s)$ *in* (4.2) *we have*

$$\mathcal{F}\mathcal{L}(h)(k,s) = \frac{s^{\beta-1}}{\psi(k,s)}, \tag{4.4}$$

where $\psi(k,s)$ *is the log-FLT of* (A, D).

PROOF. Recall from Lemma 2.1 that $\mathcal{F}\mathcal{L}(f_u)(k,s) = \exp(-u \cdot \psi(k,s))$. Then we can write (4.2) in the form

$$\mathcal{F}\mathcal{L}(h)(k,s) = s^{\beta-1} \int_0^\infty e^{-u\psi(k,s)}\, du, \tag{4.5}$$

where, by Proposition 4.2, the integral exists as a Lebesgue integral. If $\operatorname{Re}\psi(k, s) \le 0$, then $|\exp(-u\psi(k,s))| \ge 1$ for all $u \ge 0$, contradicting the existence of the integral in (4.5). Hence $\operatorname{Re}\psi(k,s) > 0$ and then (4.4) follows from (4.5) immediately. □

REMARK 4.4. It is well known that, under some regularity conditions, the log-characteristic function of an infinitely divisible distribution is the symbol of the pseudo-differential operator defined by the generator of the corresponding convolution semigroup; see [7] for details.

For a function $u \in C_0^\infty(\mathbb{R}^d \times]0,\infty[)$, the space of all C^∞ functions on $\mathbb{R}^d \times]0,\infty[$ with compact support and a *symbol* $\psi : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{C}$, we define a *pseudo-differential operator* $\psi(iD_x, \partial_t)$ by

$$\mathcal{F}\mathcal{L}\big(\psi(iD_x,\partial_t)u\big)(k,s) = \psi(k,s)\mathcal{F}\mathcal{L}(u)(k,s). \tag{4.6}$$

Since $\mathcal{F}\mathcal{L}(u)(k,s)$ is rapidly decreasing, it follows that if ψ does not grow too fast at infinity, the function $\psi(iD_x,\partial_t)u(x,t)$ is pointwise defined. Furthermore, it usually can be extended to larger spaces of functions (or even distributions), where the extension is also denoted by $\psi(iD_x,\partial_t)$. Some results in this direction are contained in a forthcoming article [2]. Since the distribution $\delta(x)t^{-\beta}/\Gamma(1-\beta)$, where $\delta(x)$ is the Dirac delta distribution, has FLT $s^{\beta-1}$, it follows that at least formally (4.4) can be written as

$$\psi(iD_x,\partial_t)h(x,t) = \delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)}. \tag{4.7}$$

This is made rigorous in [2]. The pseudo-differential equation (4.7) can be considered as a generalization of the fractional kinetic equation

$$\frac{\partial^\beta}{\partial t^\beta}h(x,t) = Lh(x,t) + \delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)} \tag{4.8}$$

introduced in [23, 34], where L is the generator of the continuous convolution semigroup generated by the distribution of A. It is shown in [19] and [1] that h is a solution of (4.8) if and only if $h(x,t)$ is the density of $M(t) = A(E(t))$, where A and D in (2.3) are independent. It follows from Theorem 2.2 that in this case $\psi(k,s) = \psi_1(k) + s^\beta$, where $E(e^{i\langle k,A\rangle}) = e^{-\psi_1(k)}$. Hence (4.7) reduces to (4.8) in this case.

5. Examples. In this section we discuss CTRW scaling limits that appear in the physics literature. The general assumptions and notation were laid out in Section 2. In short, a CTRW with jump sizes Y_i and waiting times J_i has a scaling limit of the form $M(t) = A(E(t))$, where $A(t)$ is a (operator) Lévy motion and $E(t)$ is an inverse or hitting time process for the stable subordinator $D(t)$. The random vectors (Y_i, J_i) are assumed i.i.d. with possible dependence between

Y_i and J_i. Since $M(t)$ represents the position of a randomly selected particle at time t, its density $h(x,t)$ describes relative concentration. Physicists are mainly interested in the tail behavior $P(\|M(t)\| > r)$ for r large, which describes the low concentrations, and the scaling rate $M(ct) \stackrel{d}{=} c^H M(t)$, which governs the speed at which a cloud of particles spreads. Using the methods of this article, we obtain some new results, and a simplified derivation of some old results. We also elucidate the connection between CTRW scaling limits and their governing pseudo-differential equations, which are useful as models for anomalous diffusion. For more information on the physical applications, see [16, 17].

EXAMPLE 5.1. Uncoupled CTRW lead to diffusion equations that incorporate fractional derivatives in both space and time. Metzler and Klafter [20] reviewed the basic theory and a diverse array of physical applications. If Y_i, J_i are independent, then, since $A(u)$ is operator-stable with $\mathbb{E}(e^{i\langle k, A(u)\rangle}) = e^{-u\psi_1(k)}$ and $D(u)$ is a stable subordinator with $\mathbb{E}(e^{-sD(u)}) = \exp\{-us^\beta\}$, the density $f_u(x,t)$ of $(A(u), D(u))$ has FLT

$$\mathbb{E}\big(e^{i\langle k, A(u)\rangle - sD(u)}\big) = e^{-u\psi(k,s)},$$

where $\psi(k,s) = s^\beta + \psi_1(k)$. Then Corollary 4.3 shows that the density $h(x,t)$ of the CTRW limit $M(t)$ is of the form

$$\mathcal{F}\mathcal{L}(h)(k,s) = \frac{s^{\beta-1}}{s^\beta + \psi_1(k)}. \tag{5.1}$$

Previous derivations of (5.1) involve taking limits in the Montroll–Weiss equation [21, 26], but Corollary 4.3 immediately gives the general form of that limit. Inverting the FLT on the right-hand side (see, e.g., [16]) yields

$$h(x,t) = \frac{t}{\beta} \int_0^\infty p(x,u) g_\beta(tu^{-1/\beta}) u^{-1/\beta - 1}\, du, \tag{5.2}$$

where $p(x,t)$ is the density of $A(t)$ and $g_\beta(t)$ is the density of D. Equation (5.2) also appeared in [12], Theorem 2, in the special case where A is normal. Equation (5.1) leads to $(s^\beta + \psi_1(k))\mathcal{F}\mathcal{L}(h)(k,s) = s^{\beta-1}$, and inverting this FLT gives

$$\left(\frac{\partial^\beta}{\partial t^\beta} + \psi_1(iD_x)\right) h(x,t) = \delta(x) \frac{t^{-\beta}}{\Gamma(1-\beta)}, \tag{5.3}$$

so that (5.2) solves the pseudo-differential equation (5.3), which is equivalent to the fractional Cauchy problem (4.8) with $Lh(x,t) = -\psi_1(iD_x)h(x,t)$; see [1]. Then the CTRW scaling limit $A(E(t))$ with density $h(x,t)$ is the stochastic solution to this fractional Cauchy problem. Since the density $p(x,t)$ of $A(t)$ solves the Cauchy problem $dp/dt = Lp$, the fractional time derivative subordinates the stochastic solution $A(t)$ to the inverse stable subordinator $E(t)$. If $A(t)$ is mean zero normal with variance $2t$, then (4.8) becomes

$$\frac{\partial^\beta}{\partial t^\beta}h(x,t) = \frac{\partial^2}{\partial x^2}h(x,t) + \delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)}, \tag{5.4}$$

whose stochastic solution $A(E(t))$ is self-similar with index $\beta/2$. Since $\beta < 1$, this is a model for subdiffusion, in which a cloud of particles spreads slower than the classical Brownian motion with self-similarity index 1/2. The β-fractional time derivative models particle sticking or trapping for a random waiting time $P(J_i > t) \approx t^{-\beta}$; see [16]. If $A(t)$ is stable, then the second derivative in (5.4) is replaced by a fractional derivative of order α (see [14]). In the general case where $A(t)$ is operator-stable, L is a mixture of fractional derivatives of different orders $0 < \alpha_i \le 2$ (see [15]). If $A(t)$ has exponent $(1/\beta)E$, then every eigenvalue of $(1/\beta)E$ has real part $1/\alpha_i$ for some i. While $A(ct) \stackrel{d}{=} c^{(1/\beta)E}A(t)$, the CTRW limit $M(ct) \stackrel{d}{=} c^E M(t)$ (see [19]), so that the fractional time derivative retards the rate of particle spreading in every x direction.

EXAMPLE 5.2. Shlesinger, Klafter and Wong [28] used the following CTRW example to show that diffusive behavior can also occur in the coupled case. Suppose D is a stable subordinator with $\mathbb{E}(e^{-sD}) = \exp\{-s^\beta\}$, and the conditional distribution of $A|D = t$ is normal with mean zero and variance $2t$. Then

$$\mathbb{E}(e^{ikA}) = \mathbb{E}\big(\mathbb{E}(e^{ikA}|D)\big) = \mathbb{E}\big(e^{-k^2 D}\big) = e^{-|k|^{2\beta}},$$

so that A is symmetric stable with index 2β. If we take (Y_i, J_i) i.i.d. with (A, D), then (2.3) holds with $A_n = n^{-1/(2\beta)}$ and $b_n = n^{-1/\beta}$, and the converse part of Theorem 2.2 with ω normal mean zero variance 2 and $E = 1/2$ shows that the operator-stable limit (A, D) has Lévy measure

$$\phi(dx, dt) = \frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{x^2}{4t}\right)dx\,\bar{\phi}(dt),$$

where $\bar{\phi}$ is the Lévy measure of D. Then

$$\begin{aligned}\psi(k,s) &= \int_0^\infty\int_{-\infty}^\infty\left(1 - e^{ikx}e^{-st} + \frac{ikx}{1+x^2}\right)\frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{x^2}{4t}\right)dx\,\bar{\phi}(dt)\\ &= \int_0^\infty (1 - e^{-t(s+k^2)})\bar{\phi}(dt)\\ &= (s+k^2)^\beta\end{aligned}$$

using the Lévy representation for the stable subordinator D. Then Corollary 4.3 shows that the density $h(x,t)$ of the CTRW limit $M(t)$ is of the form

$$\mathcal{F}\mathcal{L}(h)(k,s) = \frac{s^{\beta-1}}{(s+k^2)^\beta}. \tag{5.5}$$

Inverting the Laplace transform gives

$$\mathcal{F}(h)(k,t) = \int_0^t e^{-k^2 u} \frac{u^{\beta-1}}{\Gamma(\beta)} \frac{(t-u)^{-\beta}}{\Gamma(1-\beta)}\, du,$$

where we have used the formulas $\mathcal{L}t^{q-1} = s^{-q}/\Gamma(q)$ for $q > 0$, $\mathcal{L}[e^{-tc}g(t)] = \mathcal{L}(g)(s+c)$ and $\mathcal{L}(f * g)(t) = \mathcal{L}f(s)\mathcal{L}g(s)$. Finally we invert the Fourier transform to get

$$h(x,t) = \int_0^t \frac{1}{\sqrt{4\pi u}} \exp\left(-\frac{x^2}{4u}\right) \frac{u^{\beta-1}}{\Gamma(\beta)} \frac{(t-u)^{-\beta}}{\Gamma(1-\beta)}\, du, \tag{5.6}$$

which is the density of a random variable $(tB)^{1/2}Z$, where Z is mean zero normal with variance 2 and B has a Beta distribution independent of Z. From (5.5) we get $(s+k^2)^\beta \mathcal{F}\mathcal{L}(h)(k,s) = s^{\beta-1}$, which leads to the pseudo-differential equation

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)^\beta h(x,t) = \delta(x) \frac{t^{-\beta}}{\Gamma(1-\beta)} \tag{5.7}$$

with a coupled space–time-fractional derivative operator. Then (5.6) solves (5.7), and $M(t)$ is the stochastic solution to (5.7). Since $h(x,t)$ is the density of $t^{1/2} \cdot B^{1/2}Z$, it follows that $M(t) \stackrel{d}{=} t^{1/2}M(1)$, so that (5.7) models a coupled space–time diffusion that has the same scaling index as Brownian motion. To our knowledge, neither the exact form of the density (5.6) nor the governing equation (5.7) has appeared previously.

EXAMPLE 5.3. The following generalization of Example 5.2 also appeared in Shlesinger, Klafter and Wong [28]. Suppose D is a stable subordinator with $\mathbb{E}(e^{-sD}) = \exp\{-s^\beta\}$, and the conditional distribution of $Y|D = t$ is normal with mean zero and variance $2t^m$ for some $m > \beta$. Take (Y_i, J_i) i.i.d. with (Y, D). The converse part of Theorem 2.2 shows that (Y_i, J_i) belongs to the generalized domain of attraction of an operator stable limit (A, D) with Lévy measure

$$\phi(dy, dt) = \frac{1}{\sqrt{4\pi t^m}} \exp\left(-\frac{y^2}{4t^m}\right) dy\, \bar{\phi}(dt),$$

where $\bar{\phi}$ is the Lévy measure of D. The proof of that result with $\tilde{Y}_i = 0$, ω mean zero normal with variance 2 and $E = m/2$ so that $t^E\omega$ is mean zero normal with variance $2t^m$, shows that (2.8) holds with $B(n) = n^{-m/2\beta}$ and $b(n) = n^{-1/\beta}$. The limit (A, D) is a full operator-stable law on $\mathbb{R}^2$ with exponent $F = \mathrm{diag}(m/(2\beta), 1/\beta)$, and the restriction $m > \beta$ ensures that the eigenvalue $m/(2\beta) > 1/2$. Lemma 7.2.9 in [18] shows that A is operator-stable with exponent $m/(2\beta)$, and hence stable with index $2\beta/m$. In fact, since $\phi(dy, dt) =$

$\phi(-dy, dt)$, A is a symmetric stable law with index $2\beta/m$. The symmetry of ϕ with respect to y also implies that

$$\begin{aligned}\psi(k,s) &= \int_0^\infty \int_{-\infty}^\infty (1 - e^{ikx} e^{-st}) \frac{1}{\sqrt{4\pi t^m}} \exp\left(-\frac{y^2}{4t^m}\right) dy\, \bar{\phi}(dt) \\ &= \int_0^\infty (1 - e^{-ts - t^m k^2}) \bar{\phi}(dt)\end{aligned}$$

and then a simple computation using the fact that $c^\beta \cdot \bar{\phi}(dt) = \bar{\phi}(c^{-1} \cdot dt)$ shows that $c^\beta \psi(k,s) = \psi(c^{m/2}k, cs)$. Substituting into (4.4) yields $c^{-1}\mathcal{F}\mathcal{L}(h)(k,s) = \mathcal{F}\mathcal{L}(h)(c^{m/2}k, cs)$. Letting $q(k,t) = \mathbb{E}(e^{ikM(t)})$ leads to

$$\begin{aligned}c^{-1} \int_0^\infty e^{-st} q(k,t)\, dt &= c^{-1}\mathcal{F}\mathcal{L}(h)(k,s) \\ &= \mathcal{F}\mathcal{L}(h)(c^{m/2}k, cs) \\ &= \int_0^\infty e^{-sct} q(c^{m/2}k, t)\, dt \\ &= \int_0^\infty e^{-st} q(c^{m/2}k, c^{-1}t) c^{-1}\, dt\end{aligned}$$

showing that $q(k,t)$ and $q(c^{m/2}k, c^{-1}t)$ have the same Laplace transform for each fixed k, c. Then $q(k,t) = q(c^{m/2}k, c^{-1}t)$ and hence $M(t) \stackrel{d}{=} c^{m/2} M(c^{-1}t)$, so that $M(ct) \stackrel{d}{=} c^{m/2} M(t)$. In this case, a cloud of particles described by the random particle location $M(t)$ spreads at the rate $t^{m/2}$, which is subdiffusive when $\beta < m < 1$, diffusive when $m = 1$ (the case of Example 5.2) and superdiffusive when $m > 1$. Equation (41) in [28] shows that $\mathbb{E}(M(t)^2) \sim t^m$ and was proven using an asymptotic expansion of $\mathcal{F}\mathcal{L}h(k,t)$. Our approach is simpler, relying only on the scaling properties, and it also proves the scaling index suggested by (41) in [28].

EXAMPLE 5.4. Klafter, Blumen and Shlesinger [9] discussed the following tightly coupled CTRW model for anomalous diffusion, which Kotulski [11] called a *Lévy walk*. Suppose D is a stable subordinator with $\mathbb{E}(e^{-sD}) = \exp\{-s^\beta\}$, take U independent of D with $P(U = \pm 1) = 1/2$ and let $Y = UD^m$ for some $m > \beta/2$. Take (Y_i, J_i) i.i.d. with (Y, D). The converse part of Theorem 2.2 with $\omega(\pm 1) = 1/2$ and $E = m$ shows that (Y_i, J_i) belongs to the generalized domain of attraction of an operator-stable limit (A, D) with Lévy measure $\phi(dy, dt) = \varepsilon_{t^m}(dy)\bar{\phi}(dt)$. Then ϕ is concentrated on the set $\{(\pm t^m, t) : t > 0\}$ so that (A, D) is full. The marginal A is symmetric stable with index β/m. An argument similar to Example 5.3 shows that $c^\beta \psi(k,s) = \psi(c^m k, cs)$, which leads to $c^{-1}\mathcal{F}\mathcal{L}(h)(k,s) = \mathcal{F}\mathcal{L}(h)(c^m k, cs)$ and hence $M(ct) \stackrel{d}{=} c^m M(t)$. This model is subdiffusive when $m < 1/2$, diffusive when $m = 1/2$ and superdiffusive when

$m > 1/2$. Our approach identifies the scaling index suggested by the result in Table I of [9] that $\mathbb{E}(M(t)^2) \sim t^{2m}$ using only the scaling properties of (A, D).

EXAMPLE 5.5. Kotulski [10] considered a CTRW closely related to Example 5.4. Suppose D is a stable subordinator with $\mathbb{E}(e^{-sD}) = \exp\{-s^\beta\}$ and let $A = D$. If we take (Y_i, J_i) i.i.d. with (A, D), then the converse part of Theorem 2.2 with $\omega = \varepsilon_1$ and $E = 1$ shows that (Y_i, J_i) belongs to the generalized domain of attraction of an infinitely divisible limit (A, D) with Lévy measure $\phi(dy, dt) = \varepsilon_t(dy)\bar{\phi}(dt)$. This limit is not full on $\mathbb{R}^2$ since ϕ is supported on $\{(t,t) : t > 0\}$. An easy computation shows that $\psi(k, s) = (s - ik)^\beta$ so that

$$\mathcal{F}\mathcal{L}h(k, s) = \frac{s^{\beta-1}}{(s - ik)^\beta}.$$

Inverting as in Example 5.2 we get

$$h(x, t) = \frac{x^{\beta-1}}{\Gamma(\beta)} \frac{(t - x)^{-\beta}}{\Gamma(1 - \beta)},$$

which agrees with the result in Kotulski [10] when $t = 1$. Now the CTRW limit $M(t) \stackrel{d}{=} tB$, where B has a Beta distribution, and the pseudo-differential equation (4.7) becomes

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)^\beta h(x, t) = \delta(x) \frac{t^{-\beta}}{\Gamma(1 - \beta)}. \tag{5.8}$$

The scaling index and governing equation for this model have not been mentioned previously.

EXAMPLE 5.6. Suppose that ν is the probability distribution on $\mathbb{R}^+$ such that $\int_0^\infty e^{-st}\nu(dt) = \exp(-s^\beta)$. Let ω be a symmetric [i.e., $\omega(dy) = \omega(-dy)$] t^E operator-stable distribution, so that $\int e^{i\langle k,y\rangle} t^E \omega(dy) = \exp(-t\psi_0(k))$, where $\psi_0(k)$ is the log-characteristic function of ω. Take (Y_i, J_i) i.i.d. with $\mu(dy, dt) = t^E\omega(dy)\nu(dt)$. Then the converse part of Theorem 2.2 shows that (2.3) holds with $A_n = n^{-(1/\beta)E}$ and $b_n = n^{-1/\beta}$, and the operator-stable limit (A, D) has Lévy measure $\phi(dy, dt) = t^E\omega(dy)\bar{\phi}(dt)$, where $\bar{\phi}(dt)$ is the Lévy measure of the stable subordinator ν. Then

$$\begin{aligned}
\psi(k, s) &= \int_0^\infty \int_{\mathbb{R}^d} \left(1 - e^{i\langle k,y\rangle} e^{-st} + \frac{i\langle k, y\rangle}{1 + \|y\|^2}\right) t^E \omega(dy)\bar{\phi}(dt) \\
&= \int_0^\infty (1 - e^{-t\psi_0(k)} e^{-st})\bar{\phi}(dt) \\
&= \big(s + \psi_0(k)\big)^\beta
\end{aligned}$$

using the Lévy representation of ν. If $L = -\psi_0(iD_x)$, then the pseudo-differential equation (4.7) becomes

$$\left(\frac{\partial}{\partial t} - L\right)^\beta h(x,t) = \delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)} \tag{5.9}$$

and the CTRW limit $M(t)$ is the stochastic solution to (5.9). As in Example 5.2, we can invert the FLT $\mathcal{F}\mathcal{L}h(k,s) = s^{\beta-1}/(s+\psi_0(k))^\beta$ to get

$$h(x,t) = \int_0^t p(x,u)\frac{u^{\beta-1}}{\Gamma(\beta)}\frac{(t-u)^{-\beta}}{\Gamma(1-\beta)}\,du, \tag{5.10}$$

where $p(x,t)$ is the density of $t^E\omega$. If we let $\{Z(t)\}$ be an operator Lévy motion such that $Z(t)$ has distribution $t^E\omega$, then the CTRW scaling limit $M(t)$ with density $h(x,t)$ is identically distributed with $Z(tB)$, where B has a Beta distribution independent of $\{Z(t)\}$, and a simple conditioning argument shows that $M(t) \stackrel{d}{=} t^E M(1)$. If ω has no normal component, then the eigenvalues of E all have real part exceeding $1/2$, and (5.9) models d-dimensional anomalous superdiffusion, in which a cloud of particles spreads faster than the classical diffusion model predicts. In this case, Theorem 3.1 in [13] shows that, since $\mathbb{E}(B)$ exists, the random vector $M(t)$ belongs to the generalized domain of attraction of $Z(t)$ and hence both random vectors have essentially the same tail behavior. For example, if $\psi_0(k) = \|k\|^\alpha$, so that ω is spherically symmetric α-stable, then $M(t) \stackrel{d}{=} t^{1/\alpha}M(1)$ and $P(\|M(t)\| > r) = r^{-\alpha}L(r)$ for some slowly varying function $L(r)$. Also $A(t)$ is a symmetric stable Lévy motion with index $\alpha\beta$ so that $A(t) \stackrel{d}{=} t^{1/(\alpha\beta)}A(1)$ and $P(\|A(t)\| > r) = O(r^{-\alpha\beta})$. Hence the effect of subordinating $A(t)$ is to lighten the tails and slow the spreading rate. This is in contrast to the uncoupled CTRW limit in Example 5.1, where the subordinated process $A(E(t))$ spreads slower, but has the same tail behavior as $A(t)$. Coupled CTRW limits of this type provide a very flexible model for anomalous diffusion, which has not been considered previously.

6. Summary. CRTW are useful in physics as a model for anomalous diffusion, in which a cloud of particles spreads in a different manner than the classical diffusion model predicts. The CTRW model is a simple random walk of particle jumps with i.i.d. waiting times between jumps. Our focus is on the large time behavior of CTRW, represented by a scaling limit of the stochastic process. Infinite mean waiting times cause the CTRW scaling limit to differ from that of the underlying simple random walk. In this article, we have computed CTRW limits with infinite mean waiting times in the coupled case, where the waiting times and jump sizes are dependent. The effect of long waiting times is to subordinate the scaling limit $A(t)$ of the underlying random walk to an inverse stable subordinator $E(t)$. Due to the coupling of jump sizes and waiting times, these two processes are not independent. As a general rule, this coupled subordination thins the probability tails as well as retarding the spread. A simple formula involving fractional derivatives describes the probability distribution of the CTRW scaling limit $A(E(t))$, allowing us to compute the probability densities of this process in cases of practical interest. We conjecture that, as in the uncoupled case, these densities solve certain pseudo-differential equations that may be useful in modeling coupled anomalous diffusion processes in physics. That conjecture will be pursued elsewhere, since its resolution is outside the scope of probability theory.

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P. BECKER-KERN
H.-P. SCHEFFLER
FACHBEREICH MATHEMATIK
UNIVERSITY OF DORTMUND
44221 DORTMUND
GERMANY
E-MAIL: pbk@math.uni-dortmund.de
hps@math.uni-dortmund.de
URL: www.mathematik.uni-dortmund.de/lsiv/becker-kern/becker-kern.html
www.mathematik.uni-dortmund.de/lsiv/scheffler/scheffler.html

M. M. MEERSCHAERT
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NEVADA
RENO, NEVADA 89557
USA
E-MAIL: mcubed@unr.edu
URL: unr.edu/homepage/mcubed/