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LIMIT DISTRIBUTIONS OF STUDENTIZED MEANS

BY G. P. CHISTYAKOV[1] AND F. GÖTZE[2]

Institute for Low Temperature Physics and Engineering and University of Bielefeld

Let $X, X_j, j \in \mathbb{N}$, be independent, identically distributed random variables with probability distribution F. It is shown that Student's statistic of the sample $\{X_j\}_{j=1}^n$ has a limit distribution G such that $G(\{-1, 1\}) \neq 1$, if and only if: (1) X is in the domain of attraction of a stable law with some exponent $0 < \alpha \leq 2$; (2) $\mathbf{E}X = 0$ if $1 < \alpha \leq 2$; (3) if $\alpha = 1$, then X is in the domain of attraction of Cauchy's law and Feller's condition holds: $\lim_{n\to\infty} n\mathbf{E}\sin(X/a_n)$ exists and is finite, where a_n is the infimum of all $x > 0$ such that $nx^{-2}(1 + \int_{(-x,x)} y^2 F\{dy\}) \leq 1$. If $G(\{-1, 1\}) = 1$, then Student's statistic of the sample $\{X_j\}_{j=1}^n$ has a limit distribution if and only if $\mathbf{P}(|X| > x), x > 0$, is a slowly varying function at $+\infty$.

1. Introduction and results. Throughout this paper $X, X_j, j \in \mathbb{N}$, denote independent, identically distributed random variables (i.i.d.) with a distribution function $F(x)$. Furthermore, let

$$S_n = \sum_{j=1}^n X_j, \qquad V_n^2 = \sum_{j=1}^n X_j^2, \qquad n \in \mathbb{N}. \tag{1.1}$$

The quotient S_n/V_n may be considered as a self-normalized sum. For those ω where $V_n(\omega) = 0$ and hence $S_n(\omega) = 0$, we define the quotient $S_n(\omega)/V_n(\omega)$ to be zero. In terms of S_n/V_n we can write Student's statistic as

$$T_n = \frac{S_n/V_n}{\sqrt{(n - (S_n/V_n)^2)/(n-1)}}.$$

If T_n or S_n/V_n has a limiting distribution, so does the other and both coincide [see Efron (1969)].

A fundamental problem in the theory of limit distributions for identically distributed summands [see Gnedenko and Kolmogorov (1968)] was the identification of limit distributions of normalized sums

$$Z_n = \frac{X_1 + \cdots + X_n}{B_n} - A_n$$

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for suitably chosen real constants $B_n > 0$ and A_n. Another problem was the description of necessary and sufficient conditions for the distribution function $F(x)$ of the X_j such that the distributions of Z_n converge to a limit. It was proved that the stable distributions are the only nondegenerate limit distributions of normalized sums Z_n. Furthermore, if the distributions of Z_n converge to such a distribution, $F(x)$ has to be in the domain of attraction of a stable distribution. See Feller (1971), Gnedenko and Kolmogorov (1968) and Ibragimov and Linnik (1971).

Here we consider limit distributions of Studentized means. The aim of this article is to describe the class of distribution functions $F(x)$ such that T_n or, equivalently, the self-normalized sum S_n/V_n, has a limiting distribution. To formulate our main results we introduce some notation. Denote by $\mathcal{L}(Z)$ the probability distribution of a random variable Z. Following Feller [(1971), page 579], denote by a_n the infimum of all $x > 0$ such that $nx^{-2}(1 + \mathbf{E}X^2 \times \mathbf{I}\{|X| < x\}) \le 1$. Here and in the sequel we shall denote the indicator function of a set S by $\mathbf{I}\{S\}$.

THEOREM 1.1. *Let X, X_j, $j \in \mathbb{N}$, denote i.i.d. random variables. The self-normalized sums*

$$S_n/V_n \text{ converge weakly as } n \to \infty \text{ to a random variable } Z \tag{1.2}$$

such that

$$\mathbf{P}(|Z| = 1) \neq 1, \tag{1.3}$$

if and only if:

(i) *X is in the domain of attraction of a stable law with some exponent $\alpha \in (0, 2]$*;

(ii) *$\mathbf{E}X = 0$ if $1 < \alpha \le 2$*;

(iii) *if $\alpha = 1$, then X is in the domain of attraction of Cauchy's law and Feller's condition, that is*,

$$\lim_{n\to\infty} n\mathbf{E}\sin(X/a_n) \qquad \text{exists and is finite}, \tag{1.4}$$

holds.

In the following we shall call a distribution of Z nondegenerate if (1.3) holds and degenerate otherwise.

In the symmetric case Theorem 1.1 may be reformulated as:

COROLLARY 1.1. *Assume that the i.i.d. random variables X, X_j, $j \in \mathbb{N}$, are symmetric, that is, $\mathcal{L}(-X) = \mathcal{L}(X)$. The self-normalized sums S_n/V_n converge weakly to a nondegenerate limit Z if and only if X is in the domain of attraction of a stable law.*

THEOREM 1.2. *Let X, X_j, $j \in \mathbb{N}$, denote i.i.d. random variables. The self-normalized sums S_n/V_n converge weakly to a degenerate limit Z if and only if* $\mathbf{P}(|X| > x)$, $x > 0$, *is a slowly varying function at* $+\infty$.

To prove the "only if" part of Theorem 1.1 we establish the following two results.

THEOREM 1.3. *Assume that the self-normalized sums S_n/V_n converge weakly to a nondegenerate non-Gaussian limit. Then X is in the domain of attraction of a stable law with exponent $\alpha \in (0, 2)$. If $\alpha = 1$, then X is in the domain of attraction of Cauchy's law.*

THEOREM 1.4. *The self-normalized sums S_n/V_n converge weakly to a Gaussian distribution if and only if X is in the domain of attraction of a normal law and* $\mathbf{E}X = 0$.

The condition that Z is nondegenerate in Theorems 1.1 and 1.3 is essential. Using Darling's (1952) argument, Logan, Mallows, Rice and Shepp (1973) (LMRS for short) proved in particular that for distributions with extremely heavy tails like

$$\mathbf{P}(X > x) \sim \frac{r}{\log x}, \qquad \mathbf{P}(X < -x) \sim \frac{l}{\log x}, \qquad x \to +\infty,$$

where $r, l > 0$, the distribution of S_n/V_n degenerates asymptotically to the two point law with mass $r/(r+l)$ and $l/(r+l)$ at 1 and -1, respectively. Theorem 1.2 shows that under assumption (1.2) of convergence of S_n/V_n the random variable Z is degenerate only for distributions with an extremely heavy tail

$$\mathbf{P}(X < -x) + \mathbf{P}(X > x) = \frac{1}{h(x)}, \qquad x > 0,$$

where $h(x)$ is a slowly varying function at $+\infty$ such that $h(x) \uparrow +\infty$ as $x \to +\infty$.

It is well known [see Efron (1969)] that, for $x \geq 0$,

$$\mathbf{P}(T_n \geq x) = \mathbf{P}\left(\frac{S_n}{V_n} \geq \left(\frac{nx^2}{n + x^2 - 1}\right)^{1/2}\right).$$

It follows from this result that the limiting distributions for Student's statistic and for the self-normalized sum S_n/V_n coincide. Therefore the statements of Theorems 1.1–1.4 remain valid for Student's statistic.

The line of research leading to our results starts perhaps with Efron (1969), who studied the limiting behavior of Student's statistic and, equivalently, of self-normalized sums in some nonstandard cases. More strictly it actually starts with the conjecture of LMRS, stating in particular that "S_n/V_n is asymptotically normal if (and perhaps only if) X is in the domain of attraction of the normal law" (and X is centered). And in addition "It seems worthy of conjecture that the only possible nontrivial limiting distributions of S_n/V_n are those obtained when X_j follows a stable law." In the symmetric case the "if" part of the first conjecture follows rather easily from basic principles (Raikov's theorem), as was noticed, among others, by Maller (1981). In the symmetric case the parenthetical "only if" part was proved by Griffin and Mason (1991). In the nonsymmetric case the first conjecture of LMRS was proved by Giné, Götze and Mason (1997) assuming more restrictively that S_n/V_n is asymptotically *standard* normal.

Our results are connected with the first and second statements of LMRS. Theorem 1.4 shows that the first statement holds. Theorems 1.1 and 1.2 show that the second statement also holds if one interprets nontrivial limit distributions as those distributions not concentrating at the points $+1$ and -1.

For other important aspects of the asymptotic distribution of self-normalized sums we refer to LePage, Woodroof and Zinn (1981), Csörgő (1989), Bentkus and Götze (1996), Shao (1997, 1998, 1999) and Wang and Jing (1999).

We prove Theorems 1.1–1.4 studying the behavior of a special family of probability measures $\{\mu_n\}_{n=1}^{\infty}$. To introduce this family we define $B_n > 0$ for every $n \in \mathbb{N}$ uniquely determined by the relation

$$n\mathbf{E}\frac{X^2}{B_n^2 + X^2} = 1. \tag{1.5}$$

It is easy to see that $\{B_n\}_{n=1}^{\infty}$ is an increasing sequence which converges to $+\infty$. We then introduce the probability measure

$$\mu_n(A) := n\mathbf{E}\mathbf{I}\{X \in B_n A\}\frac{X^2}{B_n^2 + X^2}, \tag{1.6}$$

where A is Borel set in $\mathbb{R}$ and $B_n A := \{B_n x : x \in A\}$.

Let $\varepsilon \in (0, 1]$ denote

$$\lambda(\varepsilon) := \liminf_{n\to\infty} n\mathbf{P}(|X| > \varepsilon B_n). \tag{1.7}$$

Assume that S_n/V_n converges weakly and the following "non-CLT" condition holds:

$$\lambda(\varepsilon) > 0 \qquad \text{for some } \varepsilon \in (0, 1]. \tag{1.8}$$

The measures $\{\mu_n\}_{n=1}^{\infty}$ will be used in the following transforms. Assuming that (1.2) holds, we consider the function

$$f(t) := \lim_{n\to\infty} \int_0^{\infty} (\mathbf{E}e^{it\tau S_n/V_n} - 1)e^{-\tau^2}\,\frac{d\tau}{\tau}, \qquad t \in \mathbb{R}, \tag{1.9}$$

and, using results of Giné, Götze and Mason (1997) on bounds for absolute moments of stochastically bounded self-normalized sums, we establish that

$f(t)$ admits the representation

$$f(t) = \lim_{n\to\infty} \int_0^\infty \exp\Big\{\lambda \int_{\mathbb{R}} (e^{-\tau^2 x^2} - 1)\, d\nu_n\Big\} \times \Big(\exp\Big\{\lambda \int_{\mathbb{R}} (e^{it\tau x} - 1) e^{-\tau^2 x^2}\, d\nu_n\Big\} - 1\Big) \frac{d\tau}{\tau} \tag{1.10}$$

for $t \in \mathbb{R}$ and $\lambda > 0$, where $d\nu_n := (1 + x^2)x^{-2}\, d\mu_n$. Note that the left-hand side of (1.10) does not depend on λ.

This fact about (1.10) turns out to be an essential characteristic of Lévy's measure ν of stable distributions which is the limit of the sequence ν_n. Here this measure satisfies the relations $\lambda\nu(\tau x, \infty) = \nu(\lambda^{-1/\alpha}\tau x, \infty)$ for $x > 0$ and $\lambda\nu(-\infty, \tau x) = \nu(-\infty, \lambda^{-1/\alpha}\tau x)$ for $x < 0$ with some $\alpha \in (0, 2)$. The change of variable $\tau = r\lambda^{1/\alpha}$ with respect to the scale invariant measure $\frac{d\tau}{\tau}$ motivates why in this case the right-hand side of (1.10) should not depend on $\lambda > 0$.

Assuming (1.2) and the "non-CLT" condition (1.8) we shall try to invert the scale transform in τ. To this end we apply a Laplace transform in λ to both sides of (1.10). We thus obtain a relation between the Laplace transform $f(t)/z$ of $f(t)$ and some limiting functions of z on the right-hand side of (1.10). These functions are limits of Cauchy type integrals, defined by means of a vague limit μ of some subsequence of $\{\mu_n\}$. Recall that $\{\mu_n\}$ has a vague limit μ, with $\mu(\mathbb{R}) \le 1$, provided

$$\lim_{n\to\infty} \int_{\mathbb{R}} q\, d\mu_n = \int_{\mathbb{R}} q\, d\mu$$

for any continuous real-valued function q with a compact support [see Loève (1963)].

Using results of complex analysis we prove in Lemmas 3.2 and 3.20 that such a vague limit μ has a very special structure and is not concentrated at zero. Then we consider two cases.

In the first case the sequence $\{\mu_n\}$ has the vague limit $\mu \equiv 0$. Here we conclude that $\mathbf{P}(|X| > x)$, $x > 0$, is a slowly varying function at $+\infty$ and, using Darling's result [Darling (1952)], we establish that $\mathbf{P}(|Z| = 1) = 1$.

In the second case $\{\mu_n\}$ has not the vague zero limit. We show, using the fundamental Lemma 3.2, that in this case $\{\mu_n\}$ even has a weak limit point μ such that

$$\int_{(x,\infty)} \frac{1+u^2}{u^2}\, d\mu = \frac{c_1(\alpha)}{\alpha x^\alpha}, \qquad x > 0,$$

$$\int_{(-\infty,x)} \frac{1+u^2}{u^2}\, d\mu = \frac{c_2(\alpha)}{\alpha |x|^\alpha}, \qquad x < 0,$$

for some $\alpha \in (0, 2)$, and constants $c_j(\alpha) \ge 0$, $j = 1, 2$, satisfy $B(1 - \alpha/2, \alpha/2) \times (c_1(\alpha) + c_2(\alpha)) = 2$, where $B(x, y)$ denotes the beta function. In addition,

$c_1(1) = c_2(1)$. Hence X is in the domain of attraction of a stable law with exponent $\alpha \in (0, 2)$. If $\alpha = 1$, then X is in the domain of attraction of Cauchy's law. An outline of the complex analytic characterization process in more detail is given in the paragraphs preceding Lemma 3.2.

Furthermore we show, assuming weak convergence of S_n/V_n, that the following "CLT" condition, that is,

$$\lambda(\varepsilon) = 0 \qquad \text{for any } \varepsilon \in (0, 1], \tag{1.11}$$

holds if and only if $\lim_{n\to\infty} \mathcal{L}(S_n/V_n) = N(0, 1)$. For comparison Giné, Götze and Mason (1997) proved that the sequence of self-normalized sums S_n/V_n is asymptotically *standard* normal if and only if X is in the domain of attraction of a normal law and $\mathbf{E}X = 0$.

The statements of the "only if" part of Theorem 1.2 and the statements of Theorems 1.3 and 1.4 easily follow from the previous arguments.

Thus Theorem 1.4 sharpens the result of Giné, Götze and Mason (1997). Theorem 1.4 shows that S_n/V_n is asymptotically normal if and only if S_n/V_n is asymptotically *standard* normal.

LMRS proved, among other things, that if X is in the domain of attraction of an α-stable law, $0 < \alpha < 1$, then the sequence of self-normalized sums converges in distribution to a limit D_α, which is sub-Gaussian. This result holds in the case $1 < \alpha < 2$ under the condition $\mathbf{E}X = 0$ and fails if $\mathbf{E}X \neq 0$. In the case $\alpha = 1$ the random variable X has to be in the domain of attraction of Cauchy's law and satisfy Feller's condition (1.4) in order to have a limiting distribution.

In view of these results, we note that in Theorem 1.1 it is necessary that $\mathbf{E}X = 0$ holds if $\mathbf{E}|X| < \infty$, and X should be in the domain of attraction of Cauchy's law and satisfy Feller's condition (1.4) in the case $\alpha = 1$.

The "if" part of Theorem 1.1 follows from the results of LMRS as well.

As described in LMRS the class of limiting distributions for $\alpha \in (0, 2)$ does *not* contain Gaussian distributions.

The "if" part of Theorem 1.2 follows from Darling's result [Darling (1952)].

Using the arguments above we deduce Theorems 1.5 and 1.6 which emphasize the importance of conditions (1.8) and (1.11) for our results. Conditions (1.8) and (1.11) allow to establish under assumptions (1.2) and (1.3) when X belongs to the domain of attraction of a stable law with exponent $\alpha \in (0, 2)$ and when X belongs to the domain of attraction of a normal law.

THEOREM 1.5. *Assume that* (1.2) *and* (1.3) *hold. Then the random variable X is in the domain of attraction of a stable law with exponent $\alpha \in (0, 2)$ if and only if the* "*non-CLT*" *condition* (1.8) *holds.*

THEOREM 1.6. *Assume that* (1.2) *and* (1.3) *hold. Then the random variable X is in the domain of attraction of a normal law and* $\mathbf{E}X = 0$ *if and only if the* "*CLT*" *condition* (1.11) *holds.*

In the symmetric case LMRS showed that the limiting distributions D_α depend on the parameter $\alpha, 0 < \alpha \le 2$, only and $D_\alpha(\{1\}) = 0$. Moreover, if $\alpha_1 \ne \alpha_2$, then $D_{\alpha_1} \ne D_{\alpha_2}$. Using these results and Theorems 1.3 and 1.4, we arrive at the following statement.

THEOREM 1.7. *Let X be symmetric. Then*

$$\lim_{n\to\infty} \mathcal{L}(S_n/V_n) = D_\alpha, \qquad \alpha \in (0, 2], \tag{1.12}$$

if and only if X is in the domain of attraction of an α-stable symmetric law.

This result remains valid in the nonsymmetric case but with some obvious restrictions. We omit a formulation of a result in this case.

Most results in this paper refer to the self-normalized sums S_n/V_n [with S_n and V_n as in (1.1)] from a sequence of i.i.d. random variables X_j distributed like X. We implicitly assume this set-up throughout and avoid repeating it in every statement. The notation c will be used throughout for absolute positive constants, the notation $c(\alpha), c_1(\alpha), c_2(\alpha), \dots$ for real constants depending on α. By $c, c(\alpha)$ we denote different constants in different (or even in the same formulas). The symbols $c_1(\alpha), \dots$ are applied for explicit constants.

Section 2 is devoted to the proofs of Theorems 1.1–1.7. Section 3 contains proofs of auxiliary results from probability theory and complex analysis. In the Appendix we give the proofs of auxiliary results which are connected with well-known results from probability theory.

2. Limit distributions of self-normalized sums. To prove Theorems 1.1–1.7 we need some preliminary results. The first is the following well-known result on domains of attraction [see Ibragimov and Linnik (1971), pages 76–79].

LEMMA 2.1. *A distribution function $F(x)$ belongs to the domain of attraction of a stable law with exponent $\alpha, 0 < \alpha < 2$, if and only if for some choice of positive constants b_n such that $b_n \uparrow \infty, n \to \infty$,*

$$\begin{aligned} nF(b_n x) &\to c_1(-x)^{-\alpha}, \qquad && x < 0,\\ n\big(1 - F(b_n x)\big) &\to c_2 x^{-\alpha}, && x > 0, \end{aligned} \tag{2.1}$$

as $n \to \infty$, where c_1 and c_2 are constants with $c_1, c_2 \ge 0, c_1 + c_2 > 0$. If $c_1 = c_2$, then the stable law is symmetric.

LEMMA 2.2. *Assume that for some choice of positive constants b_n such that $b_n \uparrow \infty, n \to \infty$,*

$$nW_F(b_n x) := n\big(1 - F(b_n x) + F(-b_n x)\big) \to c_3, \qquad x > 0, \tag{2.2}$$

as $n \to \infty$, where $F(x)$ is a distribution function, c_3 is a positive constant. Then $W_F(x),\ x > 0$, is a slowly varying function at $+\infty$.

PROOF. It suffices to prove that, for every $k > 0$,

$$\lim_{y \to +\infty} \frac{W_F(y)}{W_F(ky)} = 1. \tag{2.3}$$

Fix $x > 0$ and, for large $y > 0$, take n so that $b_n x \le y \le b_{n+1} x$. Then

$$\frac{(n+1)W_F(b_{n+1}x)}{nW_F(b_n kx)} \frac{n}{n+1} \le \frac{W_F(y)}{W_F(ky)} \le \frac{nW_F(b_n x)}{(n+1)W_F(b_{n+1}kx)} \frac{n+1}{n}.$$

As $y \to \infty$, $n \to \infty$, and (2.2) therefore implies (2.3). The lemma is proved. □

We need the following result which is a consequence of a result by Darling (1952) and proved in the Appendix.

LEMMA 2.3. *If* $\mathbf{P}(|X| > x), x > 0$, *is a slowly varying function at* $+\infty$, *then* $|S_n|/V_n \to 1$ *as* $n \to \infty$ *almost surely.*

The following two lemmas are stated in LMRS (1973). See also Csörgő and Horváth (1988). For convenience we shall supply brief proofs in the Appendix.

LEMMA 2.4. (i) *Assume that* X *is in the domain of attraction of a stable law with exponent* $\alpha, 0 < \alpha < 1$, *then the sequence* $S_n/V_n, n \in \mathbb{N}$, *of self-normalized sums converges in distribution to a limit.*

(ii) *Assume that* X *is in the domain of attraction of a stable law with exponent* $\alpha, 1 < \alpha < 2$, *then the sequence* $S_n/V_n, n \in \mathbb{N}$, *converges in distribution to a limit if* $\mathbf{E}X = 0$ *and fails to converge if* $\mathbf{E}X \neq 0$.

(iii) *Assume that* X *is in the domain of attraction of Cauchy's law. Then* S_n/V_n, $n \in \mathbb{N}$, *converges in distribution to a limit if and only if Feller's condition* (1.4) *is satisfied.*

(iv) *Let* X *satisfy one of the assumptions of* (i)–(iii) *of the lemma. Then the limiting distributions of the sequence* S_n/V_n, $n \in \mathbb{N}$, *of self-normalized sums are not Gaussian and have no mass at the points* $+1$ *and* -1.

REMARK 2.1. It is easy to see that if $X = a + Y$, where $a \in \mathbb{R}$ and the random variable Y has Cauchy's distribution, then Feller's condition (1.4) is satisfied.

LEMMA 2.5. *Assume that* X *is in the domain of attraction of an* α-*stable law,* $0 < \alpha < 2$, *and let* X *be symmetric. Then*

$$\lim_{n \to \infty} \mathcal{L}(S_n/V_n) = D_\alpha,$$

where D_α *is not a normal distribution and* $D_\alpha(\{1\}) = 0$. *In addition* $D_{\alpha_1} \neq D_{\alpha_2}$ *if* $0 < \alpha_1 < \alpha_2 < 2$.

We also need the following result which is due to Giné, Götze and Mason (1997).

LEMMA 2.6. *The following two statements are equivalent*:

(a) *X is in the domain of attraction of a normal law and* $\mathbf{E}X = 0$;
(b) $\lim_{n\to\infty} \mathcal{L}(S_n/V_n) = N(0,1)$.

We prove the next two auxiliary lemmas in Section 3.

LEMMA 2.7. *Assume that S_n/V_n converges weakly and the "non-CLT" condition* (1.8) *holds*. *Then the following alternatives hold. We have either "very heavy tails," that is*,

$$\text{(2.4)} \qquad \textit{for every } x > 0, \qquad n\big(1 - F(B_n x) + F(-B_n x)\big) \to 1, \qquad n \to \infty;$$

or "stable tails," that is, *there exists an $\alpha \in (0,2)$ such that, for $n \to \infty$*,

$$\text{(2.5)} \qquad \begin{aligned} n\big(1 - F(B_n x)\big) &\to c_1(\alpha)\alpha^{-1}x^{-\alpha}, \qquad & x > 0, \\ nF(B_n x) &\to c_2(\alpha)\alpha^{-1}|x|^{-\alpha}, \qquad & x < 0. \end{aligned}$$

Here the constants $c_j(\alpha)$ satisfy $c_j(\alpha) \ge 0$, $j = 1,2$, and

$$\text{(2.6)} \qquad B(1-\alpha/2, \alpha/2)\big(c_1(\alpha) + c_2(\alpha)\big) = 2,$$

where $B(x,y)$ denotes the beta function. *In addition*, $c_1(1) = c_2(1)$.

In the case (2.4) *S_n/V_n has a degenerate limit*. *In the case* (2.5) *X is in the domain of attraction of a stable law with $\alpha \in (0,2)$ and S_n/V_n has a nondegenerate limit*.

LEMMA 2.8. *Assume that S_n/V_n converges weakly and the "CLT" condition* (1.11) *holds*. *Then*

$$\text{(2.7)} \qquad \lim_{n\to\infty} \mathcal{L}(S_n/V_n) = N(0,1)$$

and X is in the domain of attraction of a normal law and $\mathbf{E}X = 0$.

Now we shall prove Theorems 1.1–1.7 with the help of the preceding lemmas.

PROOF OF THEOREM 1.3. Assume that S_n/V_n converges weakly to a nongenerate non-Gaussian limit. Assuming the "CLT" condition (1.11) leads to a contradiction, using Lemma 2.8. Hence the "non-CLT" (1.8) holds and applying Lemma 2.7 the alternative of "very heavy tails" would lead to a degenerate limit. The remaining alternative of "stable tails" (2.5) shows that X is in the domain of attraction of a stable law with exponent $\alpha \in (0,2)$. By Lemma 2.1, if $\alpha = 1$, then X is in the domain of attraction of Cauchy's law. $\square$

PROOF OF THEOREM 1.4. Assume that S_n/V_n converges weakly to a Gaussian random variable Z. We shall show that the "CLT" condition (1.11) holds. Assume, to the contrary, that (1.11) does not hold. Then the "non-CLT" condition (1.8) holds. Applying Lemma 2.7 the alternative of "very heavy tails" (2.4) would lead to degenerate limit contradicting the assumption on Z. The remaining alternative (2.5) says that X has to be in the domain of attraction of a stable law with exponent $\alpha \in (0, 2)$. In addition if $\alpha = 1$, then X is in the domain of attraction of Cauchy's law. Lemma 2.4(iv) implies that Z cannot have a normal distribution, contradicting again our assumption. Hence, (2.7) is not satisfied and the "CLT" condition holds. By Lemma 2.8, we see that X has to be in the domain of attraction of a normal law and $\mathbf{E}X = 0$.

Let X be in the domain of attraction of a normal law and assume that $\mathbf{E}X = 0$. Then, by Lemma 2.6, S_n/V_n converges weakly to a standard normal random variable Z, thus proving Theorem 1.4. □

PROOF OF THEOREM 1.1. By Theorems 1.3 and 1.4, it follows that if S_n/V_n converges weakly to a nondegenerate limit, then X is in the domain of attraction of a stable law with some exponent $\alpha \in (0, 2]$ and if $\alpha = 1$, then X is in the domain of attraction of Cauchy's law. This yields, by Lemma 2.4(i)–(iii), the "only if" part of the theorem. The statement "if" part of the theorem follows immediately from Lemma 2.4(i)–(iii) as well. □

PROOF OF COROLLARY 1.1. If X is a symmetric random variable and belongs to the domain of attraction of a stable law, then it is obvious that $\mathbf{E}X = 0$ for $1 < \alpha \leq 2$ and Feller's condition (1.4) for $\alpha = 1$ is satisfied. Therefore the statement of Corollary 1.1 is an immediate consequence of Theorem 1.1. □

PROOF OF THEOREM 1.2. The "if" part of the theorem follows from Lemma 2.3. In order to prove the "only if" part of the theorem we note that the "non-CLT" condition (1.8) holds. Indeed, assume, to the contrary, that the "CLT" condition (1.11) holds. Then, by Lemma 2.8, Z is a standard normal random variable, a contradiction. Assuming the alternative of "stable tails," (2.5) shows that X is in the domain of attraction of a stable law with exponent $\alpha \in (0, 2)$ and hence by Lemma 2.4 the limit of S_n/V_n is not degenerate, a contradiction. Hence the alternative of "very heavy tails" (2.4) of Lemma 2.7 holds and by Lemma 2.2 we arrive at the statement of the "only if" part of the theorem. □

PROOF OF THEOREM 1.5. In order to prove the "if" part of the theorem we note that if the "non-CLT" condition (1.8) holds and the limit of S_n/V_n is nondegenerate, the alternative of "very heavy tails" (2.4) of Lemma 2.7 cannot hold. The remaining alternative of "stable tails" (2.5) implies that X is in the domain of attraction of a stable law with exponent $\alpha \in (0, 2)$.

As for the "only if" part of the theorem, let X be in the domain of attraction of a stable law with exponent $\alpha \in (0,2)$ and assume, to the contrary, that the "CLT" condition (1.11) holds. Then, by Lemma 2.8, X has to be in the domain of attraction of a normal law and $\mathbf{E}X = 0$, a contradiction which proves Theorem 1.5. □

PROOF OF THEOREM 1.6. If the "CLT" condition (1.11) holds, then, by Lemma 2.8, X is in the domain of attraction of a normal law and $\mathbf{E}X = 0$. If X is in the domain of attraction of a normal law and $\mathbf{E}X = 0$, then (1.11) holds. Indeed, assume, to the contrary, that (1.8) holds. Then we conclude that, by (1.3), Lemmas 2.2 and 2.3, the alternative of "stable tails" (2.5) of Lemma 2.7 is true. In view of Lemma 2.1 we see that X is in the domain of attraction of a stable law with exponent $\alpha \in (0,2)$, a contradiction which proves Theorem 1.6. □

PROOF OF THEOREM 1.7. Assume that (1.12) holds. In view of Lemma 2.5, we have $D_\alpha(\{1\}) = 0$. By Theorems 1.3 and 1.4, X has to be in the domain of attraction of a stable symmetric law with exponent $\alpha \in (0,2]$. Since, by Lemmas 2.5 and 2.6, the exponent α in the definition of D_α coincides with the exponent of the stable symmetric law, it follows that the exponent of the stable symmetric law is equal to α.

Now let X be in the domain of attraction of an α-stable symmetric law. By Lemmas 2.5 and 2.6, (1.12) holds, thus proving Theorem 1.7. □

3. Proofs of Lemmas 2.7 and 2.8. In order to prove Lemma 2.7 we need some preliminary results.

The following lemma is due to Helly [see Loève (1963), page 179] and shows that the family $\{\mu_n\}$, defined in (1.6), contains *a vaguely convergent subsequence*.

LEMMA 3.1. *The family $\{\mu_n\}_{n=1}^{\infty}$ is compact in the vague topology.*

We recall that (at least here) a set is *compact* for a notion of convergence if every infinite sequence in the set contains a subsequence which converges in this notion of convergence.

Consider the family of the probability measures $\{\mu_n\}$ which we introduced in (1.6). The following important Lemma 3.2 shows that under the condition (1.2) any subsequence $\{\mu_{n'}\} \subset \{\mu_n\}$, having a vague limit μ which is not zero and is not concentrated at zero, must have a very special structure. Let us describe now the idea of the proof of this lemma and Lemma 2.7.

Outline of the proof of Lemma 3.2. Let $\widehat{g}(t,\tau;\mu)$ denote the limit of

$$\widehat{g}(t,\tau;\mu_n) = \int_{\mathbb{R}} (1 - e^{-\tau^2 x^2}) \frac{1+x^2}{x^2}\, d\mu_n - \int_{\mathbb{R}} \left(e^{it\tau x} - 1 - \frac{it\tau x}{1+x^2} \right) e^{-\tau^2 x^2} \frac{1+x^2}{x^2}\, d\mu_n$$

for the vaguely convergent subsequence of $\{\mu_n\}$. Here, for simplicity, we denote all such subsequences by $\{\mu_{n'}\}$. Then we have, for $t \in [-1, 1]$ and $u > 0$, by Laplace transform of (1.10),

$$-\frac{f(t)}{u} = \lim_{n'\to\infty}\Bigl(\int_0^\delta + \int_\delta^1 + \int_1^\infty\Bigr) \tag{3.01}$$
$$\Bigl(\frac{1}{u+\widehat{g}(0,\tau;\mu_{n'})} - \frac{1}{u+\widehat{g}(t,\tau;\mu_{n'}) - ib(\tau;\mu_{n'})t\tau}\Bigr)\frac{d\tau}{\tau}$$
$$= \lim_{n'\to\infty}\bigl(f_{n',1}(u;t) + f_{n',2}(u;t) + f_{n',3}(u;t)\bigr),$$

where the positive parameter δ is sufficiently small and $b(\tau;\mu_n) = \int_{\mathbb{R}} x^{-1} \times e^{-\tau^2x^2}\, d\mu_n$. For every fixed $t \in [-\delta^4, \delta^4]$ and all $z \in D_2 \setminus \overline{D}_1$ (see Figure 1) we may select a subsequence of $\{n'\}$ such that $f_{n',1}(z;t)$ and $f_{n',3}(z;t)$ have limits $f_1(z;t)$ and $f_3(z;t)$, respectively, which are regular in $D_2 \setminus \overline{D}_1$. On the other hand, note that

$$\lim_{n'\to\infty} f_{n',2}(u;t) = f_2(u;t)$$
$$= \int_\delta^1 \Bigl(\frac{1}{u+\widehat{g}(0,\tau;\mu)} - \frac{1}{u+\widehat{g}(t,\tau;\mu) - ib(\tau;\mu)t\tau}\Bigr)\frac{d\tau}{\tau}$$
$$:= 2\pi i\bigl(\widehat{f_2}(u,0;\mu) - \widehat{f_2}(u,t;\mu)\bigr),$$

where $b(\tau;\mu)$ is a real-valued differentiable function. From this formula it follows that the functions $\widehat{f_2}(z,0;\mu)$ and $\widehat{f_2}(z,t;\mu)$ are regular for all complex $z \notin \hat{\gamma}_0$

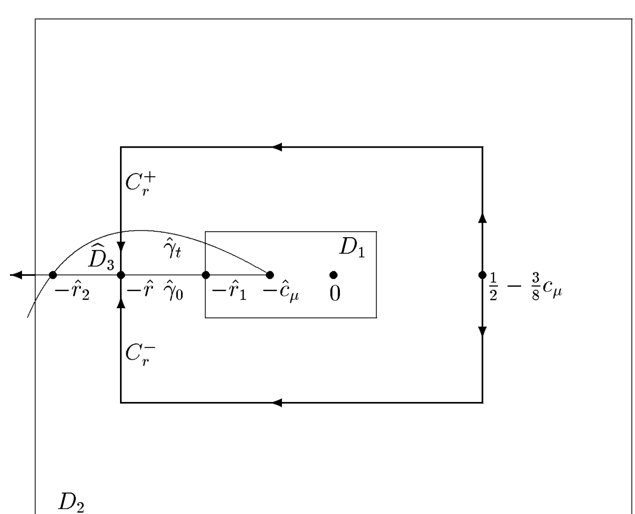

FIG. 1. *Functional identity for φ.*

and $z \notin \hat{\gamma}_t$, respectively, where $\hat{\gamma}_t$ is a curve with equation $z = -\widehat{g}(t, \tau; \mu) + ib(\tau; \mu)t\tau, \tau \in [\delta, 1]$. The function $f_2(z; t)$, being the difference of Cauchy type integrals, has, by Sokhotski's equations [see (3.41)], jumps when crossing the curves $\hat{\gamma}_t$ or $\hat{\gamma}_0$. But since $f(t)/z$ is regular for all $z \in \mathbb{C} \setminus \{0\}$, this leads to a relation for these jumps in (3.39). Considering this relation as a differential equation in τ we finally arrive at the desired scale behavior of $\widehat{g}(t, \tau; \mu)$ in τ for fixed t in (3.47) and (3.48) which allows us to get the special structure (3.1) and (3.2) for μ.

Outline of the proof of Lemma 2.7. We show assuming (1.2) that, for a subsequence $\{\mu_{n'}\} \subset \{\mu_n\}$ with a vague limit $\mu \not\equiv 0$, we have, for $t \in \mathbb{R}$ and $u > 0$,

$$\widehat{g}(t, u; \mu) - 1 + \mu(\mathbb{R}) - ib(u; \mu)tu = u^\alpha \rho(t, \alpha; \mu), \tag{3.02}$$

for some $\alpha \in (0, 2]$, where the function $\rho(t, \alpha; \mu)$ is explicitly defined for $\alpha \in (0, 2)$ in (3.64)–(3.66) and for $\alpha = 2$ in (3.83).

Note that from (3.01) it follows

$$-\frac{f(t)}{z} = f_1(z; t) + f_2(z; t) + f_3(z; t), \qquad z \in D_2 \setminus \overline{D}_1. \tag{3.03}$$

Using the formula (3.02), we now analyze (3.03). Integrating both sides of (3.03) along the curve C_r (see Figure 2), we note that the integrals of the first and third summands on the right-hand side of (3.03) are equal to zero. We evaluate the integral of the Cauchy type integrals $f_2(u; t)$ directly, using analytic continuation methods and the explicit form of the curves $\hat{\gamma}_0$ and $\hat{\gamma}_t$. In this way we

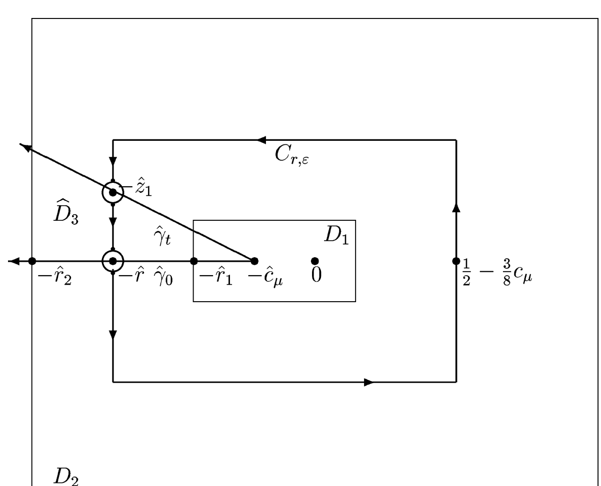

FIG. 2.

arrive at (3.86)

$$f(t) = -\frac{1}{\alpha}\log\frac{\rho(t,\alpha;\mu)}{\rho(0,\alpha;\mu)}, \qquad t \in \mathbb{R}.$$

The last formula allows us to complete the proof of Lemma 2.7. Indeed, assume that the conditions (1.2) and (1.8) hold and $\{\mu_n\}$ does not converge to $\mu \equiv 0$. Then, using the preceding formula in the case $\alpha = 2$ and the result of Giné, Götze and Mason (1997), we prove in Lemma 3.20, that all vague limit points $\{\mu\}$ of the sequence $\{\mu_n\}$ are not concentrated at zero. Using again the preceding formula, we prove that $\{\mu_n\}$ has a weak limit μ. Then, by Lemma 3.2, μ satisfy the relations (3.1) and (3.2). Recalling the definition of the measures μ_n we arrive at assertion (2.5). Then, by Lemmas 2.1 and 2.4, we conclude that X is in the domain of attraction of a stable law with exponent $\alpha \in (0,2)$ and S_n/V_n has a nondegenerate limit.

It remains to consider the case when $\{\mu_n\}$ has the vague limit $\mu \equiv 0$. By the definition of the measures μ_n we obtain assertion (2.4) of Lemma 2.7. Then, using Lemmas 2.2 and 2.3, we establish that S_n/V_n has a degenerate limit.

LEMMA 3.2. *Assume that* (1.2) *holds and that a subsequence* $\{\mu_{n'}\}_{n'=1}^{\infty}$ *of* $\{\mu_n\}_{n=1}^{\infty}$ *has a vague limit* μ, $0 < \mu(\mathbb{R}) \le 1$. *If* μ *is not concentrated at zero, then there exists* $\alpha \in (0,2)$ *such that*

$$\int_{(x,\infty)} \frac{1+u^2}{u^2}\,d\mu = \mu(\mathbb{R})c_1(\alpha)\alpha^{-1}x^{-\alpha}, \qquad x > 0, \tag{3.1}$$

$$\int_{(-\infty,x)} \frac{1+u^2}{u^2}\,d\mu = \mu(\mathbb{R})c_2(\alpha)\alpha^{-1}|x|^{-\alpha}, \qquad x < 0, \tag{3.2}$$

where the constants $c_j(\alpha)$ *satisfy* $c_j(\alpha) \ge 0$, $j = 1,2$, *and* (2.6).

PROOF. We shall split the proof of this lemma into several steps.

Introduce the functions

$$\begin{aligned} g(t,\tau;\mu_n) &:= \int_{\mathbb{R}} (1 - e^{-\tau^2x^2})\frac{1+x^2}{x^2}\,d\mu_n \\ &\quad + \int_{\mathbb{R}} (1 - e^{it\tau x})e^{-\tau^2x^2}\frac{1+x^2}{x^2}\,d\mu_n, \qquad t \in \mathbb{R}, \tau > 0, n \in \mathbb{N}. \end{aligned}$$

Define the function $g(t,\tau;\mu)$ similarly, replacing μ_n by μ. Denote as well

$$M := \max\left\{1, \sup_n \mathbf{E}\frac{|S_n|}{V_n}\right\}.$$

Giné, Götze and Mason (1997) proved that M is finite whenever the sequence $\{\frac{S_n}{V_n}\}$ is stochastically bounded. Hence, using the condition (1.2), M is finite.

Recall the definition of the function $f(t) := \lim_{n\to\infty}\int_0^\infty (\mathbf{E}e^{it\tau S_n/V_n} - 1) \times e^{-\tau^2}\frac{d\tau}{\tau}$, $t \in \mathbb{R}$, introduced in (1.9). We shall prove the following lemma.

LEMMA 3.3. *Equation* (1.10) *holds*, *that is*,

$$f(t) = \lim_{n\to\infty} \int_0^\infty \big(e^{-\lambda g(t,\tau;\mu_n)} - e^{-\lambda g(0,\tau;\mu_n)}\big) \frac{d\tau}{\tau} \tag{3.3}$$

for $t \in \mathbb{R}$, $\lambda > 0$ *and furthermore*,

$$\int_0^\infty \big|e^{-\lambda g(t,\tau;\mu_n)} - e^{-\lambda g(0,\tau;\mu_n)}\big| \frac{d\tau}{\tau} \le M|t| \tag{3.4}$$

holds for $t \in \mathbb{R}$, $\lambda > 0$, $n \in \mathbb{N}$.

PROOF. To prove the lemma we use the Poissonization of S_n/V_n in n, motivated by the hope that a subsequence of $U_n := (S_n/q_n, V_n^2/q_n^2)$, for some normalizing q_n, converges to an infinitely divisible distribution. In that case it is technically simpler to study the limiting distribution (and the Lévy measure) after the Poissonization of U_n.

Let therefore W_n denote a random variable which has a standard Poisson distribution with expectation $n\lambda$, where $\lambda > 0$, and let W_n and $X_1, X_2, \dots$ be independent. It is not difficult to see that

$$\begin{aligned} \mathbf{E}\exp\{itZ\} &= \lim_{k\to\infty} \mathbf{E}\exp\{itS_k/V_k\} \\ &= \lim_{n\to\infty} \mathbf{E}\exp\{itS_{W_n}/V_{W_n}\}, \qquad t \in \mathbb{R}. \end{aligned} \tag{3.5}$$

Note that

$$\mathbf{E}\exp\{it\tau S_{W_n}/V_{W_n}\} = e^{-n\lambda} \sum_{m=0}^\infty \frac{(n\lambda)^m}{m!} \mathbf{E}\exp\{it\tau S_m/V_m\}, \qquad t \in \mathbb{R}, \tau > 0.$$

We have, for $t \in \mathbb{R}$, $\tau > 0$,

$$\begin{aligned} &\tau^{-1}e^{-\tau^2}\big|\mathbf{E}\exp\{it\tau S_{W_n}/V_{W_n}\} - 1\big| \\ &\quad\le \tau^{-1}e^{-\tau^2}e^{-n\lambda} \sum_{m=0}^\infty \frac{(n\lambda)^m}{m!} |\mathbf{E}\exp\{it\tau S_m/V_m\} - 1| \\ &\quad\le \sup_m \mathbf{E}\frac{|S_m|}{V_m}|t|e^{-\tau^2} = M|t|e^{-\tau^2}. \end{aligned} \tag{3.6}$$

Following Giné, Götze and Mason (1997) we use the elementary identity

$$\int_0^\infty e^{-(ua)^2}\,du = \sqrt{\pi}/(2a),$$

with $a = V_n > 0$, to obtain

$$\mathbf{E}\frac{|S_m|}{V_m} = \frac{2}{\sqrt{\pi}}\mathbf{E}\int_0^\infty |S_m|e^{-u^2V_m^2}\,du.$$

Because of the $0/0 = 0$ convention used for S_n/V_n, this identity holds even though $V_m(\omega) = 0$ for some ω. With the help of the monotone convergence theorem, and Fubini's theorem, we deduce the following inequalities, which hold for all $t \in \mathbb{R}$, $\lambda > 0$ and $n \in \mathbb{N}$, using the estimate $|e^{it} - 1| \le |t|$, $t \in \mathbb{R}$:

$$\begin{aligned}
M|t| &\ge e^{-n\lambda} \sum_{m=0}^{\infty} \frac{(n\lambda)^m}{m!} |t| \mathbf{E} \frac{|S_m|}{V_m} \\
&\ge e^{-n\lambda} \sum_{m=0}^{\infty} \frac{(n\lambda)^m}{m!} |t| \mathbf{E} \int_0^\infty \frac{|S_m|}{B_n} e^{-\tau^2 V_m^2/B_n^2} \, d\tau \\
&\ge e^{-n\lambda} \sum_{m=0}^{\infty} \frac{(n\lambda)^m}{m!} \int_0^\infty \left| \mathbf{E} e^{it\tau S_m/B_n - \tau^2 V_m^2/B_n^2} - \mathbf{E} e^{-\tau^2 V_m^2/B_n^2} \right| \frac{d\tau}{\tau} \\
&= e^{-n\lambda} \int_0^\infty \sum_{m=0}^{\infty} \frac{(n\lambda)^m}{m!} \left| \mathbf{E} e^{it\tau S_m/B_n - \tau^2 V_m^2/B_n^2} - \mathbf{E} e^{-\tau^2 V_m^2/B_n^2} \right| \frac{d\tau}{\tau} \\
&\ge \int_0^\infty e^{-n\lambda} \left| \sum_{m=0}^{\infty} \frac{(n\lambda)^m}{m!} \mathbf{E} \big(e^{it\tau S_m/B_n - \tau^2 V_m^2/B_n^2} - e^{-\tau^2 V_m^2/B_n^2} \big) \right| \frac{d\tau}{\tau} \\
&= \int_0^\infty \left| e^{-\lambda g(t,\tau;\mu_n)} - e^{-\lambda g(0,\tau;\mu_n)} \right| \frac{d\tau}{\tau},
\end{aligned} \tag{3.7}$$

thus proving (3.4). Using (3.5)–(3.7), and the dominated convergence theorem, we write

$$\begin{aligned}
f(t) &= \int_0^\infty \lim_{k\to\infty} \big(\mathbf{E} e^{it\tau S_k/V_k} - 1 \big) e^{-\tau^2} \frac{d\tau}{\tau} \\
&= \int_0^\infty \lim_{n\to\infty} \big(\mathbf{E} e^{it\tau S_{W_n}/V_{W_n}} - 1 \big) e^{-\tau^2} \frac{d\tau}{\tau} \\
&= \lim_{n\to\infty} \int_0^\infty \big(\mathbf{E} e^{it\tau S_{W_n}/V_{W_n}} - 1 \big) e^{-\tau^2} \frac{d\tau}{\tau} \\
&= \lim_{n\to\infty} \int_0^\infty e^{-n\lambda} \sum_{m=0}^{\infty} \frac{(n\lambda)^m}{m!} \mathbf{E} \big(e^{it\tau (S_m/B_n)/(V_m/B_n) - \tau^2} - e^{-\tau^2} \big) \frac{d\tau}{\tau} \\
&= \lim_{n\to\infty} e^{-n\lambda} \sum_{m=0}^{\infty} \frac{(n\lambda)^m}{m!} \int_0^\infty \mathbf{E} \big(e^{it\tau S_m/B_n - \tau^2 V_m^2/B_n^2} - e^{-\tau^2 V_m^2/B_n^2} \big) \frac{d\tau}{\tau} \\
&= \lim_{n\to\infty} \int_0^\infty e^{-n\lambda} \Big(\exp\{ n\lambda \mathbf{E} e^{it\tau X/B_n - \tau^2 X^2/B_n^2} \} - \exp\{ n\lambda \mathbf{E} e^{-\tau^2 X^2/B_n^2} \} \Big) \frac{d\tau}{\tau} \\
&= \lim_{n\to\infty} \int_0^\infty \big(e^{-\lambda g(t,\tau;\mu_n)} - e^{-\lambda g(0,\tau;\mu_n)} \big) \frac{d\tau}{\tau}
\end{aligned}$$

for any $t \in \mathbb{R}$ and $\lambda > 0$, which proves (3.3). □

LEMMA 3.4. *For $u > 0$, $\tau > 0$, and $t \in \mathbb{R}$, the functions*

$$K_n(u,\tau;t) := \frac{1}{u+g(0,\tau;\mu_n)} - \frac{1}{u+g(t,\tau;\mu_n)}, \qquad n=1,\dots, \tag{3.8}$$

are well defined and for $u > 0$, $t \in \mathbb{R}$, $n \in \mathbb{N}$,

$$\int_0^\infty |K_n(u,\tau;t)|\,\frac{d\tau}{\tau} \le \frac{M|t|}{u}. \tag{3.9}$$

PROOF. Since $\operatorname{Re} g(t,\tau;\mu_n) \ge 0, \tau > 0, t \in \mathbb{R}$, the functions $K_n(u,\tau;t)$, $n \in \mathbb{N}$, are well defined for $u > 0, \tau > 0$ and $t \in \mathbb{R}$. For $t \in \mathbb{R}$ and $u > 0$, we deduce from (3.4), with the help of Fubini's theorem,

$$\begin{aligned}
&\int_0^\infty |K_n(u,\tau;t)|\,\frac{d\tau}{\tau}\\
&\quad= \int_0^\infty \frac{|g(t,\tau;\mu_n)-g(0,\tau;\mu_n)|}{(u+g(0,\tau;\mu_n))|u+g(t,\tau;\mu_n)|}\,\frac{d\tau}{\tau}\\
&\quad= \int_0^\infty \left|\int_0^\infty e^{-\lambda u}\big(e^{-\lambda g(0,\tau;\mu_n)} - e^{-\lambda g(t,\tau;\mu_n)}\big)\,d\lambda\right| \frac{d\tau}{\tau}\\
&\quad\le \int_0^\infty e^{-\lambda u}\int_0^\infty \big|e^{-\lambda g(0,\tau;\mu_n)} - e^{-\lambda g(t,\tau;\mu_n)}\big|\,\frac{d\tau}{\tau}\,d\lambda \le \frac{M|t|}{u},
\end{aligned}$$

thus proving (3.9). □

LEMMA 3.5. *The functions*

$$f_n(u;t) := \int_0^\infty K_n(u,\tau;t)\,\frac{d\tau}{\tau}, \qquad n=1,\dots, \tag{3.10}$$

are well defined for $u > 0$ and $t \in \mathbb{R}$ and for such u, t,

$$-\frac{f(t)}{u} = \lim_{n\to\infty} f_n(u;t). \tag{3.11}$$

PROOF. The functions $f_n(u;t), n \in \mathbb{N}$, are well defined for $u > 0$ and $t \in \mathbb{R}$, by (3.9). Apply Laplace transform in λ to both sides of (3.3). Using (3.4) we conclude with the help of the dominated convergence theorem and Fubini's theorem, that, for all $u > 0$,

$$\begin{aligned}
-\frac{f(t)}{u} &= \int_0^\infty e^{-u\lambda} \lim_{n\to\infty}\int_0^\infty \big(e^{-\lambda g(0,\tau;\mu_n)} - e^{-\lambda g(t,\tau;\mu_n)}\big)\,\frac{d\tau}{\tau}\,d\lambda\\
&= \lim_{n\to\infty}\int_0^\infty e^{-u\lambda}\int_0^\infty \big(e^{-\lambda g(0,\tau;\mu_n)} - e^{-\lambda g(t,\tau;\mu_n)}\big)\,\frac{d\tau}{\tau}\,d\lambda\\
&= \lim_{n\to\infty}\int_0^\infty \frac{1}{\tau}\int_0^\infty e^{-u\lambda}\big(e^{-\lambda g(0,\tau;\mu_n)} - e^{-\lambda g(t,\tau;\mu_n)}\big)\,d\lambda\,d\tau\\
&= \lim_{n\to\infty}\int_0^\infty K_n(u,\tau;t)\,\frac{d\tau}{\tau} := \lim_{n\to\infty} f_n(u;t),
\end{aligned}$$

thus proving (3.11). □

Write

$$b(\mu_n) = \int_{\mathbb{R}} \frac{1}{x}\, d\mu_n \quad \text{and} \quad b(\tau;\mu_n) = \int_{\mathbb{R}} \frac{1}{x} e^{-\tau^2 x^2}\, d\mu_n, \qquad n \in \mathbb{N}.$$

Recall that $0 < \mu(\mathbb{R}) \le 1$. In the sequel we denote $c_\mu := \mu(\mathbb{R})$ and $\hat{c}_\mu := 1 - \mu(\mathbb{R})$.

LEMMA 3.6. *The following inequalities hold*:

$$\frac{c_\mu}{2}\min\{\tau^2, 1\} \le g(0,\tau;\mu) \le 2c_\mu\tau + 2\mu\big(\mathbb{R} \setminus [-1/\sqrt{\tau}, 1/\sqrt{\tau}\,]\big), \tag{3.12}$$

where the left-hand side of (3.12) *holds for* $\tau > 0$ *and the right-hand side of* (3.12) *holds for* $\tau \in (0,1]$.

For all $n \in \mathbb{N}$,

$$0 \le \operatorname{Re} g(t,\tau;\mu_n) - g(0,\tau;\mu_n) \le 6|t|, \qquad \tau \in (0,1],\ 0 < |t| \le 1, \tag{3.13}$$

$$|\operatorname{Im} g(t,\tau;\mu_n)| \le \big(6 + |b(\tau;\mu_n)|\big)|t|^{1/2}, \qquad \tau \in (0,1], 0 < |t| \le 1. \tag{3.14}$$

In addition we have

$$\frac{d}{d\tau} g(0,\tau;\mu) > 0, \qquad \tau > 0. \tag{3.15}$$

For any $0 < \delta < 1$, *there exists a positive number* $\eta = \eta(\delta,\mu)$ (*depending on* δ *and* μ) *such that*

$$\frac{d}{d\tau} \operatorname{Re} g(t,\tau;\mu) > 0, \qquad 0 < \delta \le \tau \le 1, \qquad -\eta \le t \le \eta. \tag{3.16}$$

PROOF. Since $(1 - e^{-y})/y$ is a decreasing function on $(0,\infty)$, we have by the definition of $g(t,\tau;\mu)$:

$$\begin{aligned} g(0,\tau;\mu) &\ge (1 - e^{-1})\bigg(\tau^2 \int_{\mathbb{R}} (1+x^2)\mathbf{I}\{|x| \le 1/\tau\}\, d\mu \\ &\qquad + \int_{\mathbb{R}} \frac{1+x^2}{x^2}\mathbf{I}\{|x| > 1/\tau\}\, d\mu\bigg) \\ &\ge \frac{c_\mu}{2}\min\{\tau^2, 1\}, \qquad \tau > 0,\ n \in \mathbb{N}. \end{aligned} \tag{3.17}$$

Applying the inequality $1 - e^{-y} \le y$ for all $y \ge 0$, we obtain, for $\tau \in (0,1]$ and $n \in \mathbb{N}$,

$$\begin{aligned} g(0,\tau;\mu) &\le \tau^2 \int_{\mathbb{R}} (1+x^2)\mathbf{I}\big\{|x| \le 1/\sqrt{\tau}\big\}\, d\mu \\ &\qquad + \int_{\mathbb{R}} \frac{1+x^2}{x^2}\mathbf{I}\big\{|x| > 1/\sqrt{\tau}\big\}\, d\mu \\ &\le 2c_\mu\tau + 2\mu\big(\mathbb{R} \setminus [-1/\sqrt{\tau}, 1/\sqrt{\tau}\,]\big). \end{aligned} \tag{3.18}$$

The relation (3.12) follows from (3.17) and (3.18).

For $\tau \in (0, 1]$, $0 < |t| \le 1$ and $n \in \mathbb{N}$ we get, using $1 - \cos y \le y^2/2$, $y \in \mathbb{R}$,

$$\begin{aligned}\operatorname{Re} & g(t, \tau; \mu_n) - g(0, \tau; \mu_n) \\ &\le \tau^2 t^2 \int_{\mathbb{R}} (1 + x^2)\mathbf{I}\{|x| \le 1/(\tau\sqrt{|t|})\}\, d\mu_n \\ &\quad + 2e^{-1/|t|} \int_{\mathbb{R}} \frac{1 + x^2}{x^2}\mathbf{I}\{|x| > 1/(\tau\sqrt{|t|})\}\, d\mu_n \\ &\le 2|t| + 4e^{-1/|t|} \le 6|t|,\end{aligned}$$

which proves (3.13).

Using the estimates $|\sin y - y| \le |y|^3/6$ and $|\sin y/y| \le 1$ for $y \in \mathbb{R}$, one concludes for τ, t and n as above,

$$\begin{aligned}|\operatorname{Im} & g(t, \tau; \mu_n)| \\ &\le \tau|tb(\tau; \mu_n)| + |\operatorname{Im} g(t, \tau; \mu_n) + \tau tb(\tau; \mu_n)| \\ &\le \tau|tb(\tau; \mu_n)| + \int_{\mathbb{R}} |\sin(t\tau x) - t\tau x|\frac{1 + x^2}{x^2}\mathbf{I}\{|x| \le 1/(\tau\sqrt{|t|})\}\, d\mu_n \\ &\quad + \tau|t| \int_{\mathbb{R}} |x|\mathbf{I}\{|x| \le 1/(\tau\sqrt{|t|})\}\, d\mu_n \\ &\quad + 2e^{-1/|t|} \int_{\mathbb{R}} \frac{1 + x^2}{x^2}\mathbf{I}\{|x| > 1/(\tau\sqrt{|t|})\}\, d\mu_n \\ &\le (2 + |b(\tau; \mu_n)|)|t|^{1/2} + 4e^{-1/|t|} \le (6 + |b(\tau; \mu_n)|)|t|^{1/2},\end{aligned}$$

which proves (3.14).

Assertion (3.15) is obvious. Using again $|\sin y| \le |y|$, $y \in \mathbb{R}$, we see that

$$\begin{aligned}\frac{d}{d\tau}\operatorname{Re} g(t, \tau; \mu) &= 2\tau \int_{\mathbb{R}} \Big(1 - 2\sin^2\Big(\frac{t\tau x}{2}\Big) + \frac{t\sin(t\tau x)}{2\tau x}\Big)(1 + x^2)e^{-\tau^2 x^2}\, d\mu \\ &\ge 2\tau\Big\{\int_{\mathbb{R}} (1 + x^2)e^{-x^2}\, d\mu - t^2 \int_{\mathbb{R}} (1 + x^2)^2 e^{-\delta^2 x^2}\, d\mu\Big\} > 0\end{aligned}$$

for $0 < \delta \le \tau \le 1$, $-\eta \le t \le \eta$, where $\eta(\delta, \mu)$ is a sufficiently small positive number which depends on δ and μ. Hence (3.16) and the lemma are proved. □

Writing

$$\widehat{g}(t, \tau; \mu_{n'}) := g(t, \tau; \mu_{n'}) + ib(\tau; \mu_{n'})t\tau,$$

it is easy to see with the help of the Helly–Bray lemma [see Loève (1963), page 180], that

$$\lim_{n'\to\infty} \widehat{g}(t,\tau;\mu_{n'}) = \widehat{g}(t,\tau;\mu)$$

$$:= \hat{c}_\mu + \int_{\mathbb{R}} (1-e^{-\tau^2x^2})\frac{1+x^2}{x^2}\,d\mu \tag{3.19}$$

$$- \int_{\mathbb{R}} \left(e^{it\tau x} - 1 - \frac{it\tau x}{1+x^2}\right)e^{-\tau^2x^2}\frac{1+x^2}{x^2}\,d\mu$$

for $t \in \mathbb{R}$ and $\tau > 0$. Note that the function $\widehat{g}(t,\tau;\mu)$ is infinitely differentiable on the real line with respect to t for every fixed $\tau > 0$. It is a regular function of the complex variable τ in the domain $\operatorname{Re}\tau^2 > 0$ for every fixed real t. In addition it is easy to see that $|\widehat{g}(t,\tau;\mu)| \le c$ for $t \in [-1,1]$, $0 < \tau \le 1$.

LEMMA 3.7. *We have*

$$\limsup_{n'\to\infty} |b(\mu_{n'})| < \infty. \tag{3.20}$$

PROOF. Suppose, to the contrary, that (3.20) does not hold. Then there exists a subsequence $\{n''\} \subset \{n'\}$ such that $|b(\mu_{n''})| \to \infty$ as $n'' \to \infty$. Note that

$$b(\tau;\mu_n) = b(\mu_n) - \int_{\mathbb{R}} \frac{1}{x}(1-e^{-\tau^2x^2})\,d\mu_n,$$

where, for all $0 < \tau \le 1$ and $n \in \mathbb{N}$,

$$\left|\int_{\mathbb{R}} \frac{1}{x}(1-e^{-\tau^2x^2})\,d\mu_n\right| \le \tau^2 \int_{\mathbb{R}} |x|\mathbf{I}\{|x|\le 1\}\,d\mu_n + \int_{\mathbb{R}} \mathbf{I}\{|x|>1\}\,d\mu_n \le 1.$$

Hence

$$|b(\mu_n)| - 1 \le |b(\tau;\mu_n)| \le |b(\mu_n)| + 1 \tag{3.21}$$

for all $0 < \tau \le 1$ and $n \in \mathbb{N}$. In addition, we conclude from (3.9) that

$$\int_0^1 \frac{|\operatorname{Im}\widehat{g}(t,\tau;\mu_{n''}) - b(\tau;\mu_{n''})t\tau|}{(u+g(0,\tau;\mu_{n''}))|u+\widehat{g}(t,\tau;\mu_{n''}) - ib(\tau;\mu_{n''})t\tau|}\,\frac{d\tau}{\tau} \le \frac{M|t|}{u}$$

for all real t and $u > 0$. Choose $t = 1$, $u = 1$ and apply Fatou's theorem. By (3.21), $|b(\tau;\mu_{n''})| \ge |b(\mu_{n''})| - 1$ tends to infinity as n'' tends to infinity for $0 < \tau \le 1$. Hence we obtain $\int_0^1 (1/\tau)\,d\tau \le cM$, a contradiction. □

Choose a subsequence $\{n''\} \subset \{n'\}$ such that

$$\lim_{n''\to\infty} b(\mu_{n''}) = \limsup_{n\to\infty} b(\mu_n) := b(\mu).$$

By Lemma 3.7, the parameter $b(\mu)$ is finite. Hence

$$\lim_{n''\to\infty} b(\tau;\mu_{n''}) = b(\mu) - \int_{\mathbb{R}} \frac{1}{x}\big(1-e^{-\tau^2x^2}\big)\,d\mu := b(\tau;\mu) \tag{3.22}$$

and $|b(\tau;\mu)| \le |b(\mu)|+1$, $0<\tau\le 1$. In addition it is easy to see that $b(\tau;\mu)$ is a regular function of the complex variable τ in the domain $\operatorname{Re}\tau^2>0$.

Let $0<\delta<1$. In the sequel we assume that the parameter δ is sufficiently small. Fix $t\in[-1,1]$ and, recalling (3.10), represent $f_n(u;t)$ in the form, for $u>0$ and $t\in\mathbb{R}$,

$$\begin{aligned} f_n(u;t) &= \Big(\int_0^\delta + \int_\delta^1 + \int_1^\infty\Big) K_n(u,\tau;t)\,\frac{d\tau}{\tau} \\ &:= f_{n,1}(u;t) + f_{n,2}(u;t) + f_{n,3}(u;t). \end{aligned} \tag{3.23}$$

Introduce the following domains in the complex plane $\mathbb{C}$ (compare Figure 1):

$$D_1 := \big\{z\in\mathbb{C}: -\hat{c}_\mu - g(0,\delta;\mu) - \delta < \operatorname{Re} z < \delta,\ |\operatorname{Im} z| < 4\big(16+|b(\mu)|\big)\delta\big\}$$

and

$$D_2 := \big\{z\in\mathbb{C}: |\operatorname{Re} z| < 1-3c_\mu/4,\ |\operatorname{Im} z| < 1-3c_\mu/4\big\}.$$

By (3.12), $g(0,\delta;\mu)$ tends to zero as δ tends to zero.

For a domain D, satisfying the condition $\mathbb{R}_+\cap D\neq\varnothing$, call a function $q(u)$, $u\in\mathbb{R}_+\cap D$ regular in D, if there is a regular function of the complex variable z that coincides with $q(u)$ for $z=u$, $u\in\mathbb{R}_+\cap D$. We denote this function for $z\in D$ by $q(z)$.

In the sequel we denote the closure of a domain D by $\overline{D}$.

LEMMA 3.8. *For sufficiently large $n''\ge n_0$ and $t\in[-\delta^4,\delta^4]$ the function $f_{n'',1}(u;t)$ is regular in the domain $\mathbb{C}\setminus\overline{D}_1$ and admits the estimate*

$$|f_{n'',1}(z;t)| \le cM\delta^{-2}, \qquad z\in\mathbb{C}\setminus\overline{D}_1. \tag{3.24}$$

For $n''\ge n_0$ and $t\in[-\delta^4,\delta^4]$ the function $f_{n'',3}(u;t)$ is regular in the domain D_2 and

$$|f_{n'',3}(z;t)| \le cMc_\mu^{-2}, \qquad z\in D_2. \tag{3.25}$$

PROOF. In order to prove the first assertion of the lemma we shall estimate the modulus of the kernel $K_{n''}(z,\tau;t)$ for $z\in\mathbb{C}\setminus\overline{D}_1$, $0<\tau\le\delta$, and $t\in[-\delta^4,\delta^4]$. For the values of the parameters τ, t considered above we observe, by (3.13), (3.14), (3.20), (3.21) and the definition of $b(\mu)$, that

$$|g(t,\tau;\mu_n) - g(0,\tau;\mu_n)| \le \big(16+|b(\mu)|\big)\delta^2 \tag{3.26}$$

for sufficiently large $n\ge n_0$. In addition, by the Helly–Bray lemma, we have, for $n\ge n_0$,

$$|g(0,a;\mu_n) - \hat{c}_\mu - g(0,a;\mu)| \le \delta^2 \qquad \text{for } a=\delta \text{ and } a=1. \tag{3.27}$$

It follows from (3.26) and (3.27) with $a = \delta$,

$$|z + g(t, \tau; \mu_{n''})| \geq c\delta, \qquad n'' \geq n_0, \tag{3.28}$$

for $z \in \mathbb{C} \setminus \overline{D}_1$, $0 < \tau \leq \delta$, and $t \in [-\delta^4, \delta^4]$. Therefore, recalling (3.8), we obtain, for the same z, τ, t, n'',

$$|K_{n''}(z, \tau; t)| \leq \frac{c}{\delta^2} |g(t, \tau; \mu_{n''}) - g(0, \tau; \mu_{n''})|. \tag{3.29}$$

On the other hand, choosing in (3.9) $u = 1$ and noting that, by (3.20) and (3.21),

$$|1 + g(t, \tau; \mu_{n''})| = |1 + \widehat{g}(t, \tau; \mu_{n''}) - ib(\tau; \mu_{n''})t\tau| \leq c$$

for τ, t as above, we easily conclude that

$$\int_0^\delta |g(t, \tau; \mu_{n''}) - g(0, \tau; \mu_{n''})| \frac{d\tau}{\tau} \leq cM, \qquad t \in [-\delta^4, \delta^4], n'' \geq n_0. \tag{3.30}$$

The estimates (3.28)–(3.30) together imply that the integral $\int_0^\delta K_{n''}(z, \tau; t) \frac{d\tau}{\tau}$ is a regular function in the domain $\mathbb{C} \setminus \overline{D}_1$. Denote this integral for $z \in \mathbb{C} \setminus \overline{D}_1$ by $f_{n'',1}(z; t)$ again. Thus, we proved that the function $f_{n'',1}(u; t)$, with $n'' \geq n_0$ and $t \in [-\delta^4, \delta^4]$, is regular in $z \in \mathbb{C} \setminus \overline{D}_1$. It remains to prove (3.24). From (3.29) and (3.30) it follows, for $z \in \mathbb{C} \setminus \overline{D}_1$,

$$\begin{aligned} |f_{n'',1}(z; t)| &\leq \int_0^\delta |K_{n''}(z, \tau; t)| \frac{d\tau}{\tau} \\ &\leq \frac{c}{\delta^2} \int_0^\delta |g(t, \tau; \mu_{n''}) - g(0, \tau; \mu_{n''})| \frac{d\tau}{\tau} \leq cM\delta^{-2}, \end{aligned}$$

as stated.

In the second step let us estimate $|K_{n''}(z, \tau; t)|$ for $z \in D_2$, $\tau \geq 1$ and $t \in [-1, 1]$. Using (3.12) and (3.27), with $a = 1$, and the inequality $g'_\tau(0, \tau; \mu_n) > 0$, $\tau \geq 1$, we deduce, for these values of τ and t,

$$\begin{aligned} \operatorname{Re} g(t, \tau; \mu_n) &\geq g(0, \tau; \mu_n) \geq g(0, 1; \mu_n) \geq 1 - c_\mu + g(0, 1; \mu) - \delta^2 \\ &\geq 1 - c_\mu/2 - \delta^2 \geq 1 - 5c_\mu/8. \end{aligned} \tag{3.31}$$

From this inequality it is not difficult to conclude that, for the values of z, τ and t considered above,

$$|z + g(t, \tau; \mu_n)| \geq c c_\mu |g(t, \tau; \mu_n)|. \tag{3.32}$$

By the definition of $K_{n''}(z, \tau; t)$ and (3.32), we note that

$$|K_{n''}(z, \tau; t)| \leq \frac{c}{c_\mu^2} \frac{|g(t, \tau; \mu_{n''}) - g(0, \tau; \mu_{n''})|}{g(0, \tau; \mu_{n''})|g(t, \tau; \mu_{n''})|}. \tag{3.33}$$

Choosing in (3.9) $u = 1$ and using (3.31), we obtain

$$\int_1^\infty \frac{|g(t, \tau; \mu_{n''}) - g(0, \tau; \mu_{n''})|}{g(0, \tau; \mu_{n''})|g(t, \tau; \mu_{n''})|} \frac{d\tau}{\tau} \leq cM, \qquad t \in [-\delta^4, \delta^4]. \tag{3.34}$$

The estimates (3.32)–(3.34) imply that the integral $\int_1^\infty K_{n''}(z,\tau;t)\,\frac{d\tau}{\tau}$ is a regular function in D_2. Denote this integral for $z \in D_2$ by $f_{n'',3}(z;t)$ again. It is easy to see from (3.33) and (3.34) that

$$|f_{n'',3}(z;t)| \le \int_1^\infty |K_{n''}(z,\tau;t)| \le \frac{cM}{c_\mu^2}, \qquad z \in D_2, t \in [-\delta^4, \delta^4], n'' \ge n_0.$$

Thus (3.25) is proved. The lemma is proved completely. □

LEMMA 3.9. *For every fixed* $t \in [-\delta^4, \delta^4]$, *there exists* $\{n_1\} \subseteq \{n''\}$ *such that* $\{f_{n_1,1}(z;t)\}$ *and* $\{f_{n_1,3}(z;t)\}$ *converge uniformly in the interior of the domain* $D_2 \setminus \overline{D}_1$, *to regular functions* $f_1(z;t)$ *and* $f_3(z;t)$, *respectively.*

To prove this lemma we need the following well-known condensation principle for regular functions [see Goluzin (1969), page 15].

LEMMA 3.10. *Let* $\{f_n(z)\}, n = 1, 2, \dots,$ *denote a sequence of functions that are regular in a domain* B. *Suppose that the sequence is uniformly bounded in the interior of* B. *Then this sequence* $\{f_n(z)\}$ *contains a subsequence that converges uniformly in the interior of* B *to a regular function.*

PROOF OF LEMMA 3.9. The estimates (3.24) and (3.25) show that we may apply Lemma 3.10 to the families of regular functions $\{f_{n'',1}(z;t)\}$, $\{f_{n'',3}(z;t)\}$. Hence there exists a subsequence $\{n_1\} \subseteq \{n''\}$ such that $\{f_{n_1,1}(z;t)\}$ and $\{f_{n_1,3}(z;t)\}$ converge uniformly in the interior of $D_2 \setminus \overline{D}_1$ to regular functions $f_1(z;t)$ and $f_3(z;t)$, respectively. □

In view of (3.11), (3.23) and Lemma 3.9, we see that the following relation holds for $z \in [2\delta, 1 - 4c_\mu/5]$:

$$-\frac{f(t)}{z} = f_1(z;t) + f_2(z;t) + f_3(z;t) \tag{3.35}$$

with $f_2(z;t) := \lim_{n_1\to\infty} f_{2,n_1}(z;t)$. Hence $f_2(z;t)$ admits an analytic continuation in $D_2 \setminus \overline{D}_1$. Denote again this continuation by $f_2(z;t)$. On the other hand, by the dominated convergence theorem, for $z \in [2\delta, 1 - 4c_\mu/5]$,

$$\begin{aligned}\frac{1}{2\pi i}\lim_{n_1\to\infty} f_{2,n_1}(z;t) &= \frac{1}{2\pi i}\int_\delta^1 \frac{1}{z + \widehat{g}(0,\tau;\mu)}\,\frac{d\tau}{\tau} \\ &\quad - \frac{1}{2\pi i}\int_\delta^1 \frac{1}{z + \widehat{g}(t,\tau;\mu) - ib(\tau;\mu)t\tau}\,\frac{d\tau}{\tau} \\ &:= \widehat{f_2}(z,0;\mu) - \widehat{f_2}(z,t;\mu).\end{aligned} \tag{3.36}$$

See the definition of $\widehat{g}(t,\tau;\mu)$ and $b(\tau;\mu)$ in (3.19) and (3.22), respectively.

In the sequel we need to work with simple smooth closed or open curves. Recall that a continuous curve γ is said to be *smooth* if there is at least one representation $z = \lambda(u)$, $a \le u \le b$, such that $\lambda(u)$ has a continuous nonvanishing derivative $\lambda'(u)$ at every point of the interval $[a, b]$. The curve γ is called a *simple open* curve if $\lambda(a) \ne \lambda(b)$ and if, in addition, $\lambda(u)$ is one-to-one on $a \le u \le b$. The curve γ is called a *simple closed* curve if $\lambda(a) = \lambda(b)$ and $\lambda(u)$ is one-to-one on $a \le u < b$. Denote by $\gamma + \zeta$, where $\zeta \in \mathbb{C}$, the curve γ shifted by ζ. We shall regard γ as being traversed in the direction of increasing u.

Consider a family $\{\gamma_t\}$, $t \in [-t_0, t_0]$, of curves with equations $z = -\widehat{g}(t, \tau; \mu) + \hat{c}_\mu + ib(\tau; \mu)t\tau$, $\delta \le \tau \le 1$. In the sequel we assume that the parameter $t_0 = t_0(\delta, \mu) > 0$ is sufficiently small. It follows from (3.12), (3.15) and (3.16) that γ_t, $t \in [-t_0, t_0]$, are simple smooth open curves in $\mathbb{C}$ such that γ_t intersects every vertical line $\operatorname{Re} z = x$, $-c_\mu/4 \le x \le -(g(0, \delta; \mu) + \delta)$, at one point only. The curve γ_t tends to γ_0 as $t \to 0$. Since $\widehat{g}(t, \tau; \mu) - ib(\tau; \mu)t\tau$, as a function of τ, is regular on $(0, \infty)$, in the case where $\operatorname{Im} \widehat{g}(t, \tau; \mu) - b(\tau; \mu)t\tau \not\equiv 0$ for $\tau > 0$ the curve γ_t may intersect γ_0 in a finite number of points only. By $\hat{\gamma}_t$ we denote the curve $\gamma_t - \hat{c}_\mu$. Recalling the definition of D_1, D_2 and (3.12), we see that, for sufficiently small $t \in [-t_0, t_0]$, $\hat{\gamma}_t$ intersects the left sides of the rectangles D_1 and D_2. See Figure 1.

The integrals in (3.36) are regular functions in $\mathbb{C} \setminus \hat{\gamma}_0$, $\mathbb{C} \setminus \hat{\gamma}_t$, respectively, therefore the functions $\widehat{f_2}(z - \hat{c}_\mu, 0; \mu)$ and $\widehat{f_2}(z - \hat{c}_\mu, t; \mu)$ are regular in $\mathbb{C} \setminus \gamma_0$, $\mathbb{C} \setminus \gamma_t$, respectively. Hence, if $\tau \mapsto \operatorname{Im} \widehat{g}(t, \tau; \mu) - b(\tau; \mu)t\tau \not\equiv 0$ for $\tau > 0$, the function $\widehat{f_2}(z, 0; \mu) - \widehat{f_2}(z, t; \mu)$ is regular in the domain $(D_2 \setminus \overline{D}_1) \setminus \overline{D}_0$, where $\overline{D}_0$ is the closure of the open set $D_0 \subset \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ which consists of points $z \in D_2 \setminus \overline{D}_1$ between the curves $\hat{\gamma}_0$ and $\hat{\gamma}_t$. If $\tau \mapsto \operatorname{Im} \widehat{g}(t, \tau; \mu) - b(\tau; \mu)t\tau \equiv 0$ for $\tau > 0$, then $D_0 := \varnothing$ and $\overline{D}_0 := [-(1 - 3c_\mu/4), -(\hat{c}_\mu + g(0, \delta; \mu) + \delta)]$. In addition, we note that $\widehat{f_2}(z, 0; \mu) - \widehat{f_2}(z, t; \mu) = f_2(z; t)$ for $z \in (D_2 \setminus \overline{D}_1) \setminus \overline{D}_0$.

For fixed $t \in [-t_0, t_0]$ and for $\delta \le \tau \le 1$ denote by $\varphi(\zeta, t; \mu)$ the inverse function of $\zeta = -\widehat{g}(t, \tau; \mu) + \hat{c}_\mu + ib(\tau; \mu)t\tau$. The function $\varphi(\zeta, t; \mu)$ is well defined by the strict monotonicity of the function $\operatorname{Re} g(t, \tau; \mu)$ on the segment $[\delta, 1]$. [See (3.15) and (3.16).] It is differentiable with respect to ζ for $\zeta \in \gamma_t$. In addition, $\zeta = -\widehat{g}(t, \tau; \mu) + \hat{c}_\mu + ib(\tau; \mu)t\tau$ is a one-to-one mapping of the segment $[\delta, 1]$ onto the curve γ_t. Therefore, changing variables, we have the following representation for the functions $\widehat{f_2}(z, 0; \mu)$ and $\widehat{f_2}(z, t; \mu)$:

$$
\begin{aligned}
\widehat{f_2}(z, 0; \mu) &= \frac{1}{2\pi i} \int_{\gamma_0} \frac{\varphi'(\zeta, 0; \mu)}{\varphi(\zeta, 0; \mu)} \frac{d\zeta}{z + \hat{c}_\mu - \zeta}, \\
\widehat{f_2}(z, t; \mu) &= \frac{1}{2\pi i} \int_{\gamma_t} \frac{\varphi'(\zeta, t; \mu)}{\varphi(\zeta, t; \mu)} \frac{d\zeta}{z + \hat{c}_\mu - \zeta}.
\end{aligned} \tag{3.37}
$$

Since $\operatorname{Re} g(t, \delta; \mu) < g(0, \delta; \mu) + \delta < c_\mu/4$ for sufficiently small $\delta > 0$ and $t \in [-t_0, t_0]$, and, by (3.12), $\operatorname{Re} g(t, 1; \mu) \ge g(0, 1; \mu) \ge c_\mu/2$, we see that there exists $\tau_1 \in (\delta, 1)$ such that

$$\operatorname{Re} g(t, \tau_1; \mu) = r_1 \qquad \text{where } r_1 = r_1(\mu) := g(0, \delta; \mu) + \delta.$$

If $\tau \mapsto \operatorname{Im} \widehat{g}(t, \tau; \mu) - b(\tau; \mu)t\tau \not\equiv 0$, $\tau > 0$, then we may assume, without loss of generality, that there exists a point $\tau^* \in (\tau_1, 1)$ such that

$$-1 + 4c_\mu/5 < \operatorname{Im} \widehat{g}(t, \tau; \mu) - b(\tau; \mu)t\tau < 0 \qquad \text{for } \tau_1 < \tau < \tau^*.$$

[The case where $0 < \operatorname{Im} \widehat{g}(t, \tau; \mu) - b(\tau; \mu)t\tau < 1 - 4c_\mu/5$ for $\tau_1 < \tau < \tau^*$ may be treated in a similar way.] Denote by τ_2 the maximal point of $(\tau_1, 1]$, satisfying this condition. Let

$$r_2 = r_2(t, \mu) := \min\{\operatorname{Re} g(t, \tau_2; \mu), c_\mu/4\}.$$

Then we define a bounded domain D_3 in the following way. It consists of the points $z \in \mathbb{C}$ of the strip $\{-r_2 < \operatorname{Re} z < -r_1\}$ between the curves γ_0 and γ_t. Compare Figure 1. It is not difficult to see by the strict monotonicity of $\operatorname{Re} g(t, \tau; \mu)$ (Lemma 3.6) that $D_3 \neq \varnothing$. Note that D_3 depends on t and μ. Below we shall use the notation $\hat{r}_1 = r_1 + \hat{c}_\mu$, $\hat{r}_2 = r_2 + \hat{c}_\mu$ and $\widehat{D}_3 = \{z \in \mathbb{C} : z + \hat{c}_\mu \in D_3\}$.

If $\tau \mapsto \operatorname{Im} \widehat{g}(t, \tau; \mu) - b(\tau; \mu)t\tau \equiv 0$, $\tau > 0$, then $D_3 := \varnothing$ and we introduce r_1 in the same way as before and $r_2 := c_\mu/4$. We also assume that $\overline{D}_3 := [-c_\mu/4, -r_1]$. It is easy to see that the last case holds if X is a symmetric random variable.

Note that $\zeta(z) := -\widehat{g}(t, z; \mu) + \hat{c}_\mu + ib(z; \mu)tz$ is regular function of the complex variable z in the domain $\operatorname{Re} z^2 > 0$ for every fixed real t. In addition $|\zeta(z)| \leq c_1(\delta, \mu)$ and $|\frac{d}{dz}\zeta(z)| \geq c_2(\delta, \mu)$ for $z \in \mathbb{C}$ such that $\delta/2 \leq \operatorname{Re} z \leq 3/2$, $|\operatorname{Im} z| \leq c_3(\delta, \mu)$ and for sufficiently small $|t| \leq t_0(\delta, \mu)$. Here $c_j(\delta, \mu) > 0$, $j = 1, 2, 3$, and depend on δ and μ only. Therefore the inverse function $\varphi(\zeta, t; \mu)$ of the function $\zeta = \zeta(z)$ is regular on the closure of the domain D_3 and is infinitely differentiable with respect to t for $t \in [-t_0, t_0]$.

Consider the rectangle

$$R(r, \mu) := \left\{z \in \mathbb{C} : -(\hat{c}_\mu + r) < \operatorname{Re} z < \tfrac{1}{2} - \tfrac{3}{8}c_\mu,\ |\operatorname{Im} z| < 1 - \tfrac{4}{5}c_\mu\right\}, \tag{3.38}$$

where $r_1 < r < r_2$. Denote by C_r^+ and C_r^- continuous curves which are the parts of the boundary of the rectangle $R(r, \mu)$ in the half-planes $\operatorname{Im} z > 0$ and $\operatorname{Im} z < 0$, respectively, joining the point $1/2 - 3c_\mu/8$ to the point $-\hat{r} := -(\hat{c}_\mu + r)$. The limiting values of the regular function $f_2(z; t)$ as z tends to the point $-\hat{r}$ along the curves C_r^+ and C_r^- should coincide.

Figure 1 illustrates this construction. Using (3.36), (3.37) and the behavior of the Cauchy type integral on the integration curve, we express these limiting values by the densities φ'/φ, with the parameters 0 and t. Since the limiting values should coincide, we obtain a functional equation for these densities. Our next aim is to establish the following lemma.

LEMMA 3.11. *For every fixed $t \in [-t_0, t_0]$, we have*

$$\frac{\varphi'(-r,0;\mu)}{\varphi(-r,0;\mu)} - \frac{\varphi'(-r,t;\mu)}{\varphi(-r,t;\mu)} = 0 \qquad \text{for } r_1 < r < r_2. \tag{3.39}$$

We prove this lemma using properties of the Cauchy type integrals

$$\Phi(z) = \frac{1}{2\pi i}\int_\gamma \frac{\varphi(\tau)}{\tau - z}\,d\tau, \tag{3.40}$$

where $\varphi(\tau)$ satisfies a Hölder condition. We assume as well that the curve γ is closed. The positive direction of the curve γ is usually that for which the domain within the contour γ is on the left. In the case of an open curve we may supplement it by an arbitrary curve so that it becomes closed, defining on the additional curve $\varphi(\tau) = 0$.

We shall denote the limiting values of the analytic functions $\Phi(z)$ when z tends to a point t of the curve γ from the inside by $\Phi^+(t)$ and from the outside by $\Phi^-(t)$. (For an open curve this corresponds to the limiting values from the left and from the right.) To emphasize the direction of passing to the limit we shall accordingly write $z \to t^+$ or $z \to t^-$.

The following lemma deals with the Cauchy type integrals [see Gakhov (1966), page 25].

LEMMA 3.12. *Let γ denote a smooth curve* (*closed or open*) *and $\varphi(\tau)$ a function on the curve*, *which satisfies Hölder's condition. Then the Cauchy type integral* (3.40) *has limiting values $\Phi^+(t)$, $\Phi^-(t)$ at all points of the curve γ not coinciding with its ends. On approaching the curve from the left or from the right along an arbitrary path, these limiting values are expressed by the density of the integral $\varphi(t)$ and the singular integral $\Phi(t)$ by means of Sokhotski's equations*

$$\begin{aligned} \Phi^+(t) &= \frac{1}{2}\varphi(t) + \frac{1}{2\pi i}\int_\gamma \frac{\varphi(\tau)}{\tau - t}\,d\tau, \\ \Phi^-(t) &= -\frac{1}{2}\varphi(t) + \frac{1}{2\pi i}\int_\gamma \frac{\varphi(\tau)}{\tau - t}\,d\tau, \end{aligned} \tag{3.41}$$

where the singular integral $\int_\gamma \frac{\varphi(\tau)}{\tau - t}\,d\tau$ is understood in the sense of Cauchy's principal value.

Subtracting and adding the formulae (3.41) we obtain the following two equivalent equations:

$$\Phi^+(t) - \Phi^-(t) = \varphi(t), \qquad \Phi^+(t) + \Phi^-(t) = \frac{1}{\pi i}\int_\gamma \frac{\varphi(\tau)}{\tau - t}\,d\tau. \tag{3.42}$$

PROOF OF LEMMA 3.11. We shall prove Lemma 3.11 in the case $D_3 \neq \varnothing$. In the case $D_3 = \varnothing$ one can prove this lemma in the same way. Let $z_0 + \hat{c}_\mu \in \gamma_t$ and $\operatorname{Re} z_0 = -\hat{r} = -(\hat{c}_\mu + r), r_1 < r < r_2$. Taking into account that $f_2(z;t)$ and $\widehat{f_2}(z,0;\mu)$ are regular at z_0 and hence $f_2(z_0;t) = \lim_{z\to z_0} f_2(z;t)$

and $\widehat{f_2}(z_0, 0; \mu) = \lim_{z\to z_0} \widehat{f_2}(z, 0; \mu)$, we conclude from (3.36), (3.37) and Lemma 3.12,

$$\begin{aligned}\frac{1}{2\pi i} f_2(z_0; t) &= \widehat{f_2}(z_0, 0; \mu) - \lim_{z\to z_0^-} \widehat{f_2}(z, t; \mu)\\ &= \widehat{f_2}(z_0, 0; \mu) - \frac{1}{2}\frac{\varphi'(z_0 + \hat{c}_\mu, t; \mu)}{\varphi(z_0 + \hat{c}_\mu, t; \mu)}\\ &\quad + \frac{1}{2\pi i}\int_{\gamma_t} \frac{\varphi'(\zeta, t; \mu)}{\varphi(\zeta, t; \mu)} \frac{d\zeta}{\zeta - (z_0 + \hat{c}_\mu)}.\end{aligned}$$

On the other hand, we similarly obtain from this equation

$$\begin{aligned}\frac{1}{2\pi i} f_2(z_0; t) &= \widehat{f_2}(z_0, 0; \mu) + \frac{1}{2}\frac{\varphi'(z_0 + \hat{c}_\mu, t; \mu)}{\varphi(z_0 + \hat{c}_\mu, t; \mu)}\\ &\quad + \frac{1}{2\pi i}\int_{\gamma_t} \frac{\varphi'(\zeta, t; \mu)}{\varphi(\zeta, t; \mu)} \frac{d\zeta}{\zeta - (z_0 + \hat{c}_\mu)} - \frac{\varphi'(z_0 + \hat{c}_\mu, t; \mu)}{\varphi(z_0 + \hat{c}_\mu, t; \mu)}\\ &= \widehat{f_2}(z_0, 0; \mu) - \lim_{z\to z_0^+} \widehat{f_2}(z, t; \mu) - \frac{\varphi'(z_0 + \hat{c}_\mu, t; \mu)}{\varphi(z_0 + \hat{c}_\mu, t; \mu)}.\end{aligned}$$

In the last two formulas we passed to the limit along the curve C_r^+. In the first equation we used limiting values from the right and in the second one from the left. The integrals here are singular. These formulae show that the function $\varphi'(z, t; \mu)/\varphi(z, t; \mu)$, $z \in \gamma_t$, admits an analytic continuation to the domain D_3 by the formula

$$\begin{aligned}\frac{\varphi'(z, t; \mu)}{\varphi(z, t; \mu)} &= -\frac{1}{2\pi i} f_2(z - \hat{c}_\mu; t) + \widehat{f_2}(z - \hat{c}_\mu, 0; \mu)\\ &\quad + \frac{1}{2\pi i}\int_{\gamma_t} \frac{\varphi'(\zeta, t; \mu)}{\varphi(\zeta, t; \mu)} \frac{d\zeta}{\zeta - z}.\end{aligned} \tag{3.43}$$

By Lemma 3.12, passing to the limit along the curve C_r^+, we have

$$\lim_{z\to(-\hat{r})^-} \widehat{f_2}(z, 0; \mu) = \frac{1}{2}\frac{\varphi'(-r, 0; \mu)}{\varphi(-r, 0; \mu)} - \frac{1}{2\pi i}\int_{\gamma_0} \frac{\varphi'(\zeta, 0; \mu)}{\varphi(\zeta, 0; \mu)} \frac{d\zeta}{\zeta + r}, \tag{3.44}$$

where the integral on the right-hand side of this formula is singular. We obtain from (3.43) and (3.44), for $r_1 < r < r_2$,

$$\begin{aligned}\frac{1}{2\pi i} f_2(-\hat{r}; t) &= -\frac{\varphi'(-r, t; \mu)}{\varphi(-r, t; \mu)} + \frac{1}{2}\frac{\varphi'(-r, 0; \mu)}{\varphi(-r, 0; \mu)}\\ &\quad - \frac{1}{2\pi i}\int_{\gamma_0} \frac{\varphi'(\zeta, 0; \mu)}{\varphi(\zeta, 0; \mu)} \frac{d\zeta}{\zeta + r}\\ &\quad + \frac{1}{2\pi i}\int_{\gamma_t} \frac{\varphi'(\zeta, t; \mu)}{\varphi(\zeta, t; \mu)} \frac{d\zeta}{\zeta + r}.\end{aligned} \tag{3.45}$$

On the other hand, by Lemma 3.12,

$$\lim_{z\to(-\hat r)^+}\widehat{f_2}(z,0;\mu)=-\frac{1}{2}\frac{\varphi'(-r,0;\mu)}{\varphi(-r,0;\mu)}-\frac{1}{2\pi i}\int_{\gamma_0}\frac{\varphi'(\zeta,0;\mu)}{\varphi(\zeta,0;\mu)}\frac{d\zeta}{\zeta+r},$$

where the integral on the right-hand side of this formula is singular. Therefore, passing to the limit along the curve C_r^-, we get the formula

$$\begin{aligned}\frac{1}{2\pi i}f_2(-\hat r;t)&=\lim_{z\to(-\hat r)^+}\widehat{f_2}(z,0;\mu)-\widehat{f_2}(-\hat r,t;\mu)\\&=-\frac{1}{2}\frac{\varphi'(-r,0;\mu)}{\varphi(-r,0;\mu)}-\frac{1}{2\pi i}\int_{\gamma_0}\frac{\varphi'(\zeta,0;\mu)}{\varphi(\zeta,0;\mu)}\frac{d\zeta}{\zeta+r}\\&\quad+\frac{1}{2\pi i}\int_{\gamma_t}\frac{\varphi'(\zeta,t;\mu)}{\varphi(\zeta,t;\mu)}\frac{d\zeta}{\zeta+r}.\end{aligned}\tag{3.46}$$

Comparing (3.45) and (3.46) we arrive at (3.39). □

Now we can conclude from Lemma 3.11 that the following assertion holds.

LEMMA 3.13. *There exist positive constants A_1, A_2 and a real constant A_3 such that, for $u>0$,*

$$\frac{d}{du}g(0,u;\mu)=A_1u^{1-A_2},\tag{3.47}$$

$$i\frac{d^3}{dt^3}\widehat{g}(t,u;\mu)\Big|_{t=0}=A_3u^{2-A_2}.\tag{3.48}$$

PROOF. Integrating both sides of (3.39) over the interval (u,r_2) with a parameter $u\in(r_1,r_2)$, we obtain

$$\varphi(-u,0;\mu)=q(t)\varphi(-u,t;\mu),\qquad u\in(r_1,r_2),\tag{3.49}$$

where for sufficiently small $|t|$, $|t|\le t_0$, $q(t)$ is complex-valued infinitely differentiable function such that $q(0)=1$. Choosing in (3.49)

$$u=\widehat{g}(t,\zeta;\mu)-\hat c_\mu-ib(\zeta;\mu)t\zeta,\qquad u\in(r_1,r_2),\tag{3.50}$$

we have

$$\varphi\big(-\widehat{g}(t,\zeta;\mu)+\hat c_\mu+ib(\zeta;\mu)t\zeta,0;\mu\big)=q(t)\zeta$$

and finally obtain

$$\widehat{g}(t,\zeta;\mu)-\hat c_\mu-ib(\zeta;\mu)t\zeta=g\big(0,q(t)\zeta;\mu\big)\tag{3.51}$$

for ζ satisfying (3.50) and fixed $t\in[-t_0,t_0]$. The functions $\widehat{g}(t,z;\mu)$, $b(z;\mu)$ and $g(0,z;\mu)$ are regular functions of z for $\operatorname{Re}z^2>0$. Since equality (3.51) holds on a continuous curve in the domain $\operatorname{Re}z^2>0$, we conclude that

$$\widehat{g}(t,u;\mu)-\hat c_\mu-ib(u;\mu)tu=g\big(0,q(t)u;\mu\big)\tag{3.52}$$

holds for all $u > 0$ and $t \in [-t_0, t_0]$. The functions on both sides of (3.52) are infinitely differentiable with respect to t. Differentiating relation (3.52) with respect to t and choosing $t = 0$, we arrive at the equations, for $u > 0$,

$$\frac{d}{dt}\widehat{g}(t, u; \mu)\Big|_{t=0} = ib(u; \mu)u + q'(0)g'(0, u; \mu)u, \tag{3.53}$$

$$\frac{d^2}{dt^2}\widehat{g}(t, u; \mu)\Big|_{t=0} = q^{(2)}(0)g'(0, u; \mu)u + \big(q'(0)\big)^2 g^{(2)}(0, u; \mu)u^2, \tag{3.54}$$

$$\begin{aligned}\frac{d^3}{dt^3}\widehat{g}(t, u; \mu)\Big|_{t=0} &= q^{(3)}(0)g'(0, u; \mu)u + 3q'(0)q^{(2)}(0)g^{(2)}(0, u; \mu)u^2 \\ &\quad + \big(q'(0)\big)^3 g^{(3)}(0, u; \mu)u^3,\end{aligned} \tag{3.55}$$

$$\begin{aligned}\frac{d^4}{dt^4}\widehat{g}(t, u; \mu)\Big|_{t=0} &= q^{(4)}(0)g'(0, u; \mu)u \\ &\quad + \big(4q'(0)q^{(3)}(0) + 3\big(q^{(2)}(0)\big)^2\big)g^{(2)}(0, u; \mu)u^2 \\ &\quad + 3\big(q'(0)\big)^2 q^{(2)}(0)g^{(3)}(0, u; \mu)u^3 \\ &\quad + \big(q'(0)^4\big)g^{(4)}(0, u; \mu)u^4,\end{aligned} \tag{3.56}$$

where in (3.53)–(3.56) the functions $g^{(j)}(0, u; \mu)$, $j = 1, 2, 3, 4$, denote the derivatives of $g(0, u; \mu)$ with respect to u.

Assume first $q'(0) \neq 0$. Since, by the definition of $\widehat{g}(t, u; \mu)$,

$$\frac{d^2}{dt^2}\widehat{g}(t, u; \mu)\Big|_{t=0} = \frac{1}{2}ug'(0, u; \mu), \qquad u > 0, \tag{3.57}$$

we conclude, by (3.54),

$$\frac{g^{(2)}(0, u; \mu)}{g'(0, u; \mu)} = \frac{A}{u}, \qquad u > 0, \tag{3.58}$$

where A is a real-valued constant. Since the measure $\mu \not\equiv 0$ is not concentrated at zero, solving this differential equation we arrive at

$$g'(0, u; \mu) = A_1 u^{1-A_2} \qquad \text{for } u > 0, \tag{3.59}$$

where A_1, A_2 denote positive constants.

Now assume $q'(0) = 0$. From (3.54) and (3.57) it follows that $q^{(2)}(0) = 1/2$. Since, by definition of $\widehat{g}(t, u; \mu)$,

$$4\,\frac{d^4}{dt^4}\widehat{g}(t, u; \mu)\Big|_{t=0} = u^2 g^{(2)}(0, u; \mu) - ug'(0, u; \mu), \qquad u > 0,$$

we conclude, using (3.56), that equation (3.58) holds. Therefore in the case $q'(0) = 0$ the function $g'(0, u; \mu)$ satisfies (3.59) as well. Relation (3.48) immediately follows from (3.47) and (3.55). The lemma is proved. □

Now we complete the proof of Lemma 3.2.

Rewrite relation (3.47) in the form

$$\int_{\mathbb{R}} (1 + x^2) e^{-u^2 x^2}\, d\mu = \tfrac{1}{2} A_1 u^{-A_2}, \qquad u > 0, \tag{3.60}$$

where A_1, A_2 are the same constants as in (3.47). We note from (3.60) that $\mu(\{0\}) = 0$. Denote by $\mu(x)$, $x \in \mathbb{R}$, the distribution function of the measure μ and rewrite (3.60) in the form

$$\int_0^\infty (1 + x^2) e^{-u^2 x^2}\, d\big(\mu(x) - \mu(-x)\big) = \tfrac{1}{2} A_1 u^{-A_2}, \qquad u > 0.$$

On the other hand, it easy to see that

$$\frac{A_1}{\Gamma(A_2/2)} \int_0^\infty x^{A_2 - 1} e^{-u^2 x^2}\, dx = \frac{1}{2} A_1 u^{-A_2}, \qquad u > 0.$$

Taking into account that μ is a finite measure and that distinct probability distributions have distinct Laplace transforms, we conclude from the last two relations that $0 < A_2 < 2$ and

$$\frac{x^2 + 1}{c_\mu x^2}\, d\big(\mu(x) - \mu(-x)\big) = c_3(\alpha) x^{-1-\alpha}\, dx, \qquad x > 0, \tag{3.61}$$

where $0 < \alpha = 2 - A_2 < 2$, $c_3(\alpha) > 0$. Since μ / c_μ is a probability measure, $c_3(\alpha)$ satisfies the relation

$$c_3(\alpha) \int_0^\infty \frac{x^{1-\alpha}}{1 + x^2}\, dx = \frac{1}{2} c_3(\alpha) B(1 - \alpha/2, \alpha/2) = 1. \tag{3.62}$$

From (3.48) we conclude

$$\int_{\mathbb{R}} x(1 + x^2) e^{-u^2 x^2}\, d\mu = -A_3 u^{-1-A_2}, \qquad u > 0,$$

where A_2, A_3 are the same constants as in (3.48). This relation implies

$$\frac{1 + x^2}{c_\mu x^2}\, d\big(\mu(x) + \mu(-x)\big) = c_4(\alpha) x^{-1-\alpha}\, dx, \qquad x > 0, \tag{3.63}$$

where the parameter α is the same as in (3.61) and $c_4(\alpha)$ denotes a real constant such that $c_3(\alpha) \pm c_4(\alpha) \geq 0$.

Assertions (3.1) and (3.2) of the lemma with $c_1(\alpha) = (c_3(\alpha) + c_4(\alpha))/2$ and $c_2(\alpha) = (c_3(\alpha) - c_4(\alpha))/2$ immediately follow from (3.61)–(3.63). Thus Lemma 3.2 is proved completely. □

LEMMA 3.14. *Assume that the conditions of Lemma* 3.2 *are satisfied. Then, for $t \in \mathbb{R}$ and $u > 0$, we have*

$$\begin{aligned}
&\widehat{g}(t,u;\mu) - \hat{c}_\mu - ib(u;\mu)tu \\
&= u^\alpha c_\mu\Big(c_3(\alpha)\int_0^\infty (1-e^{-x^2})\,\frac{dx}{x^{\alpha+1}} \\
&\qquad - c_1(\alpha)\int_0^\infty (e^{itx}-1)e^{-x^2}\,\frac{dx}{x^{\alpha+1}} \\
&\qquad - c_2(\alpha)\int_{-\infty}^0 (e^{itx}-1)e^{-x^2}\,\frac{dx}{|x|^{\alpha+1}}\Big) \\
&:= u^\alpha \rho(t,\alpha;\mu) \qquad \text{if } \alpha < 1,
\end{aligned} \tag{3.64}$$

$$\begin{aligned}
&\widehat{g}(t,u;\mu) - \hat{c}_\mu - ib(u;\mu)tu \\
&= u^\alpha c_\mu\Big(\int_0^\infty (1-e^{-x^2})\frac{c_3(\alpha)+ic_4(\alpha)tx}{x^{\alpha+1}}\,dx \\
&\qquad - c_1(\alpha)\int_0^\infty (e^{itx}-1-itx)e^{-x^2}\,\frac{dx}{x^{\alpha+1}} \\
&\qquad - c_2(\alpha)\int_{-\infty}^0 (e^{itx}-1-itx)e^{-x^2}\,\frac{dx}{|x|^{\alpha+1}}\Big) \\
&:= u^\alpha \rho(t,\alpha;\mu) \qquad \text{if } \alpha > 1
\end{aligned} \tag{3.65}$$

and

$$\begin{aligned}
&\widehat{g}(t,u;\mu) - \hat{c}_\mu - ib(u;\mu)tu \\
&= u\Big(c_3(1)c_\mu\int_0^\infty (1-e^{-x^2})\,\frac{dx}{x^2} \\
&\qquad - c_3(1)c_\mu\int_0^\infty (\cos(tx)-1)e^{-x^2}\,\frac{dx}{x^2} - ib(\mu)t\Big) \\
&:= u\rho(t,1;\mu) \qquad \text{if } \alpha = 1,
\end{aligned} \tag{3.66}$$

where the parameter α and the constants $c_1(\alpha)$, $c_2(\alpha)$ are defined in (3.1) *and* (3.2) *and the constants $c_3(\alpha)$, $c_4(\alpha)$ are equal to $c_1(\alpha)+c_2(\alpha)$, $c_1(\alpha)-c_2(\alpha)$, respectively.*

PROOF. In the first step let us prove (3.64). We use the notation of Lemmas 3.2–3.13. From (3.61), (3.63) and (3.53) it follows that, for $0 < \alpha < 1$,

$$\begin{aligned} &-iu^{\alpha} c_4(\alpha) c_{\mu} \int_0^{\infty} e^{-x^2} \frac{dx}{x^{\alpha}} \\ &\quad + iuc_4(\alpha) c_{\mu} \int_0^{\infty} e^{-u^2x^2} \frac{dx}{(1+x^2)x^{\alpha}} - ib(u;\mu)u \\ &= 2q'(0) u^{\alpha} c_3(\alpha) c_{\mu} \int_0^{\infty} x^{-\alpha+1} e^{-x^2}\, dx, \qquad u > 0. \end{aligned} \tag{3.67}$$

As we established earlier in the proof of Lemma 3.2, $|b(u;\mu)| \le |b(\mu)| + 1$, $0 < u \le 1$, therefore it follows from (3.67) that

$$c_4(\alpha) c_{\mu} \int_0^{\infty} e^{-u^2x^2} \frac{dx}{(1+x^2)x^{\alpha}} - b(u;\mu) = 0, \qquad u > 0.$$

In view of the definition of the function $\widehat{g}(t,u;\mu)$ and (3.61), (3.63), this proves assertion (3.64) of the lemma.

Let us prove (3.65). In the case $\alpha > 1$ rewrite (3.53) in the form

$$\begin{aligned} &-iuc_4(\alpha) c_{\mu} \int_0^{\infty} \frac{x^{2-\alpha}}{1+x^2} e^{-u^2x^2}\, dx - iub(u;\mu) \\ &= 2q'(0) u^{\alpha} c_3(\alpha) c_{\mu} \int_0^{\infty} x^{-\alpha+1} e^{-x^2}\, dx \end{aligned} \tag{3.68}$$

for $u > 0$. It is not difficult to see that the left-hand side of (3.68) can be represented in the form

$$\begin{aligned} &-iu\left(c_4(\alpha) c_{\mu} \int_0^{\infty} \frac{x^{2-\alpha}}{1+x^2}\, dx + b(\mu)\right) \\ &\quad + iu^{\alpha} c_4(\alpha) c_{\mu} \int_0^{\infty} (1 - e^{-x^2}) \frac{dx}{x^{\alpha}} + r(u), \qquad u > 0, \end{aligned} \tag{3.69}$$

where $|r(u)| \le c(\alpha) c_{\mu} u^2$, $u \in (0, 1]$. Thus we conclude from (3.68) that the first term in (3.69) is equal to zero. Using this information we easily arrive at assertion (3.65) of the lemma.

It remains to establish (3.66). In the case $\alpha = 1$ (3.53) has the form

$$\begin{aligned} &-iuc_4(1) c_{\mu} \int_0^{\infty} \frac{x}{1+x^2} e^{-u^2x^2}\, dx - iub(u;\mu) \\ &= 2q'(0) uc_3(1) c_{\mu} \int_0^{\infty} e^{-x^2}\, dx, \qquad u > 0. \end{aligned} \tag{3.70}$$

Since $|b(u;\mu)| \le |b(\mu)| + 1$ for all $u > 0$ and the integral on the left-hand side of (3.70) tends to infinity as u tends to zero, we obtain that $c_4(1) = 0$. Thus we get assertion (3.66) of the lemma, which completes the proof of Lemma 3.14. □

By the definition of $\rho(t, \alpha; \mu)$, the function $\rho(t, \alpha; \mu)$, as a function of t, admits an analytic continuation in $\mathbb{C}$ as an entire function for any $0 < \alpha < 2$. In addition the following lemma holds.

LEMMA 3.15. *Let $\alpha \in (0, 2)$. If $c_3(\alpha) > 0$, then*

$$c_5(\alpha, \mu) y^{-2-\alpha} e^{y^2/4} \le \max\{-\rho(-iy, \alpha; \mu), -\rho(iy, \alpha; \mu)\} \le c_6(\alpha, \mu) e^{y^2/4}$$

for sufficiently large $y \ge y_0 > 0$, where $c_5(\alpha, \mu) > 0$, $c_6(\alpha, \mu) > 0$. If $c_1(\alpha) > 0$, then the preceding inequality remains valid for the function $-\rho(-iy, \alpha; \mu)$.

This lemma follows by straightforward bounds.

In order to complete the proof of Lemma 2.7 we need the following auxiliary results as well.

LEMMA 3.16. *Assume that S_n/V_n converges weakly to Z, with $\mathbf{P}(|Z| = 1) = 0$, and that (1.8) holds. Then $\{\mu_n\}_{n=1}^{\infty}$ is a tight family.*

PROOF. Let $\varepsilon \in (0, 1)$ such that $\lambda(\varepsilon) > 0$. [See the definition of $\lambda(\varepsilon)$ in (1.7).] Such an ε exists by (1.8). Note that, by (1.5),

$$n\mathbf{P}(|X| > \varepsilon B_n) \le (1 + \varepsilon^2)/\varepsilon^2 := \varepsilon_1, \qquad n \in \mathbb{N}. \tag{3.71}$$

This estimate immediately yields $\lambda(\varepsilon) \le \varepsilon_1$. Let $X_{j,n}$, $j = 1, \dots, n$, denote independent random variables such that

$$\mathbf{P}(X_{j,n} \in A) = \mathbf{P}(X_j \in A \,|\, |X_j| \le \varepsilon B_n) \tag{3.72}$$

for all Borel sets A in $\mathbb{R}$. We assume as well that the random vectors $(X_1, \dots, X_n)$ and $(X_{1,n}, \dots, X_{n,n})$ are independent. Consider the random variables

$$\begin{aligned} \mathcal{S}_{n-1} &:= \frac{1}{\varepsilon B_n} \sum_{j=1}^{n-1} X_{j,n}, \\ \mathcal{V}_{n-1}^2 &:= \frac{1}{(\varepsilon B_n)^2} \sum_{j=1}^{n-1} X_{j,n}^2, \\ N_n &:= \sum_{j=1}^{n} \mathbf{I}\{|X_j| > \varepsilon B_n\}, \end{aligned} \tag{3.73}$$

and denote $I_\eta = [1 - \eta, 1 + \eta]$, $\eta > 0$. We have, for all $\eta > 0$,

$$\begin{aligned} \Delta_n(\eta) &:= \mathbf{P}\left(\frac{|S_n|}{V_n} \in [1 - \eta, 1 + \eta]\right) = \sum_{j=0}^{n} \mathbf{P}(N_n = j)\mathbf{P}\left(\frac{|S_n|}{V_n} \in I_\eta \Big| N_n = j\right) \\ &\ge \mathbf{P}(N_n = 1)\mathbf{P}\left(\frac{|\mathcal{S}_{n-1} + X_n/(\varepsilon B_n)|}{\sqrt{\mathcal{V}_{n-1}^2 + (X_n/(\varepsilon B_n))^2}} \in I_\eta \Big| |X_n| > \varepsilon B_n\right) \\ &:= \mathbf{P}(N_n = 1) p_n. \end{aligned}$$

By (1.2) and $\mathbf{P}(|Z|=1)=0$, the limit

$$\Delta(\eta) := \lim_{n\to\infty} \Delta_n(\eta) = \lim_{n\to\infty} \mathbf{P}\Big(\frac{|S_n|}{V_n} \in [1-\eta, 1+\eta]\Big)$$

exists and $\Delta(\eta) \to 0$ as $\eta \to 0$ for η such that $\mathbf{P}(|Z| = 1 \pm \eta) = 0$. Taking into account (3.71) we see, that, for sufficiently large $n \geq n_0$,

$$\begin{aligned}\mathbf{P}(N_n = 1) &= n\mathbf{P}(|X| > \varepsilon B_n)\big(1 - \mathbf{P}(|X| > \varepsilon B_n)\big)^{n-1} \\ &\geq n\mathbf{P}(|X| > \varepsilon B_n)\exp\{-2n\mathbf{P}(|X| > \varepsilon B_n)\} \geq \frac{\lambda(\varepsilon)}{2}\exp\{-2\varepsilon_1\}\end{aligned}$$

holds and we obtain the relation

$$\Delta(\eta) \geq \limsup_{n\to\infty} \mathbf{P}(N_n = 1)\, p_n \geq \frac{\lambda(\varepsilon)}{2} e^{-2\varepsilon_1} \limsup_{n\to\infty} p_n. \tag{3.74}$$

For $N > 1$, consider the events

$$C_n = \{\mathcal{V}_{n-1} \leq N\} \quad \text{and} \quad D_n = \{|\mathcal{S}_{n-1}| \leq N^2\}, \qquad n = 2, 3, \ldots.$$

First we note, for sufficiently large n,

$$\begin{aligned}\mathbf{P}(C_n^c) &\leq \frac{\mathbf{E}\mathcal{V}_{n-1}^2}{N^2} = \frac{n-1}{N^2}\mathbf{E}\Big(\frac{X_{1,n}}{\varepsilon B_n}\Big)^2 \\ &= \frac{n-1}{N^2\mathbf{P}(|X| \leq \varepsilon B_n)}\mathbf{E}\Big(\frac{X}{\varepsilon B_n}\Big)^2 \mathbf{I}\{|X| \leq \varepsilon B_n\} \\ &\leq \frac{2\mu_n(\mathbb{R})}{(N\varepsilon)^2\mathbf{P}(|X| \leq \varepsilon B_n)} \\ &= \frac{2}{(N\varepsilon)^2\mathbf{P}(|X| \leq \varepsilon B_n)} \\ &\leq \frac{4}{(N\varepsilon)^2}.\end{aligned} \tag{3.75}$$

Here we have used the inequality $\mathbf{P}(|X| \leq \varepsilon B_n) \geq 1/2$, which holds for sufficiently large n. From (3.75) it follows that

$$\limsup_{n\to\infty} \mathbf{P}(\mathcal{V}_{n-1} > N) \leq \frac{4}{(N\varepsilon)^2}. \tag{3.76}$$

Hence we get, using (3.76),

$$\begin{aligned}
&\limsup_{n\to\infty} \mathbf{P}(D_n^c)\\
&\quad\le \limsup_{n\to\infty} \mathbf{P}(|\mathcal{S}_{n-1}| > N^2, \mathcal{V}_{n-1} \le N) + \limsup_{n\to\infty} \mathbf{P}(\mathcal{V}_{n-1} > N)\\
&\quad\le \limsup_{n\to\infty} \mathbf{P}\Big(\frac{|\mathcal{S}_{n-1}|}{\mathcal{V}_{n-1}} > N\Big) + \frac{4}{(N\varepsilon)^2}\\
&\quad\le \limsup_{n\to\infty} \big(\mathbf{P}(|X| \le \varepsilon B_n)\big)^{-n} \mathbf{P}\Big(\frac{|S_{n-1}|}{V_{n-1}} > N\Big) + \frac{4}{(N\varepsilon)^2}\\
&\quad\le e^{2\varepsilon_1} \limsup_{n\to\infty} \mathbf{P}\Big(\frac{|S_{n-1}|}{V_{n-1}} > N\Big) + \frac{4}{(N\varepsilon)^2}.
\end{aligned} \tag{3.77}$$

We obtain from (3.76) and (3.77)

$$\limsup_{n\to\infty} \mathbf{P}\big((C_n \cap D_n)^c\big) \le \limsup_{n\to\infty} e^{2\varepsilon_1} \mathbf{P}\Big(\frac{|S_{n-1}|}{V_{n-1}} > N\Big) + \frac{8}{(N\varepsilon)^2}. \tag{3.78}$$

It is easy to see that, for sufficiently small positive η, $\eta \le \eta_0$,

$$\begin{aligned}
&\big\{|X_n| > 2\varepsilon B_n N^2/\eta\big\} \cap (C_n \cap D_n)\\
&\qquad \subset \left\{ \frac{|\mathcal{S}_{n-1} + X_n/(\varepsilon B_n)|}{\sqrt{\mathcal{V}_{n-1}^2 + (X_n/(\varepsilon B_n))^2}} \in I_\eta \right\} \cap (C_n \cap D_n).
\end{aligned}$$

Hence, by (3.74), we arrive at the lower bound

$$\Delta(\eta) \ge \frac{\lambda(\varepsilon)}{2} e^{-2\varepsilon_1} \limsup_{n\to\infty} \mathbf{P}\Big(\Big\{\frac{|X_n|}{\varepsilon B_n} > 2\frac{N^2}{\eta}\Big\} \cap (C_n \cap D_n) \Big| \frac{|X_n|}{\varepsilon B_n} > 1\Big). \tag{3.79}$$

On the other hand, using the independence of the events $C_n \cap D_n$ and $\{|X_n| > \varepsilon B_n\}$ and (3.78), we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} \mathbf{P}\Big((C_n \cap D_n)^c \Big| \frac{|X_n|}{\varepsilon B_n} > 1\Big)\\
&\qquad \le e^{2\varepsilon_1} \limsup_{n\to\infty} \mathbf{P}\Big(\frac{|S_{n-1}|}{V_{n-1}} > N\Big) + \frac{8}{(N\varepsilon)^2}.
\end{aligned} \tag{3.80}$$

Now we conclude from (3.79) and (3.80) that

$$\begin{aligned}
&\frac{\lambda(\varepsilon)}{2} \varepsilon^{-2\varepsilon_1} \limsup_{n\to\infty} \mathbf{P}\Big(\frac{|X|}{\varepsilon B_n} > 2\frac{N^2}{\eta} \Big| \frac{|X|}{\varepsilon B_n} > 1\Big)\\
&\qquad \le \Delta(\eta) + \frac{\lambda(\varepsilon)}{2} \limsup_{n\to\infty} \mathbf{P}\Big(\frac{|S_{n-1}|}{V_{n-1}} > N\Big) + \frac{4\lambda(\varepsilon)}{(N\varepsilon)^2}.
\end{aligned} \tag{3.81}$$

It remains to note that, by (1.6) and (3.71),

$$\mathbf{P}\biggl(\frac{|X|}{\varepsilon B_n} > 2\frac{N^2}{\eta}\Big|\frac{|X|}{\varepsilon B_n} > 1\biggr) = \frac{n\mathbf{P}(|X| > 2\varepsilon B_n N^2/\eta)}{n\mathbf{P}(|X| > \varepsilon B_n)} \geq \frac{\mu_n(\mathbb{R} \setminus [-2\varepsilon N^2/\eta, 2\varepsilon N^2/\eta])}{2\varepsilon_1}$$

for sufficiently large n. Finally, by (3.81), we conclude that

$$\frac{\lambda(\varepsilon)}{4\varepsilon_1} e^{-2\varepsilon_1} \limsup_{n\to\infty} \mu_n(\mathbb{R} \setminus [-2\varepsilon N^2/\eta, 2\varepsilon N^2/\eta]) \leq \Delta(\eta) + \frac{\lambda(\varepsilon)}{2} \limsup_{n\to\infty} \mathbf{P}\biggl(\frac{|S_{n-1}|}{V_{n-1}} > N\biggr) + \frac{4\lambda(\varepsilon)}{(N\varepsilon)^2}$$

for all $0 < \eta \leq \eta_0$ and $N > 1$. The assertion of the lemma immediately follows from this relation. □

Formulating and proving the following four lemmas we shall use the notation of Lemmas 3.2–3.11.

LEMMA 3.17. *Assume that the convergence* (1.2) *holds and that there is a subsequence* $\{n'\}$ *of* $\mathbb{N}$ *such that the family* $\{\mu_{n'}\}$ *has a vague limit* μ_0, *with* $c_{\mu_0} := \mu_0(\mathbb{R}) > 0$, *which is concentrated at zero. Then, for some subsequence* $\{n_1'\} \subset \{n'\}$,

$$\lim_{n_1'\to\infty} \int_{\mathbb{R}} \frac{1}{x}\, d\mu_{n_1'} = 0 \tag{3.82}$$

and

$$\widehat{g}(t,\tau;\mu_0) - 1 + c_{\mu_0} - ib(\tau;\mu_0)t\tau = c_{\mu_0}\tau^2(1+t^2/2), \qquad \tau > 0, t \in \mathbb{R}. \tag{3.83}$$

PROOF. It is not difficult to see that the arguments which we employed in the proofs of Lemmas 3.3–3.11 hold in the case where the limiting measure $\mu_0 \not\equiv 0$ is concentrated at zero. Repeating these arguments, we have, for the measure μ_0 and $t \in \mathbb{R}$, $\tau > 0$,

$$\widehat{g}(t,\tau;\mu_0) - \hat{c}_{\mu_0} - ib(\tau;\mu_0)t\tau = c_{\mu_0}\tau^2(1+t^2/2) - ib(\mu_0)t\tau, \tag{3.84}$$

where $\hat{c}_{\mu_0} := 1 - c_{\mu_0}$ and the parameter $b(\mu_0)$ is finite. Let us prove that $b(\mu_0) = 0$. Suppose, to the contrary, that $b(\mu_0) \neq 0$. Without loss of generality we may assume that $b(\mu_0) > 0$. Taking into account (3.84), we see that in the case where $\mu = \mu_0$ the curves γ_t, $t \geq 0$, with equations $z = -c_{\mu_0}\tau^2(1+t^2/2) + ib(\mu_0)t\tau$, $\delta \leq \tau \leq 1$, are simple such that γ_t intersects every vertical line $\operatorname{Re} z = x$, $-c_{\mu_0}(1+t^2/2) < x < -c_{\mu_0}\delta^2(1+t^2/2)$, at one point only.

Since $b(\mu_0) > 0$, then $\operatorname{Im} \widehat{g}(t,\tau;\mu_0) - b(\tau;\mu_0)t\tau = -b(\mu_0)t\tau < 0$ for $\tau > 0, t > 0$. Consider δ and t such that $0 < 10^4\delta \le \delta_1 := \min\{c_{\mu_0}, (b(\mu_0))^2\}$ and $0 < t \le t_1 := \min\{1, 1/(5b(\mu_0))\}$, respectively. For such δ and t we define the domain D_3 in the case where $\mu = \mu_0$ in the same way as in the proof of Lemma 3.2. It is not difficult to see that, for these values of δ, t,

$$r_1 = \delta + c_{\mu_0}\delta^2, \qquad r_2 = \min\Big\{c_{\mu_0}(1+t^2/2), \frac{c_{\mu_0}}{4}\Big\} = \frac{c_{\mu_0}}{4}$$

and $r_1 < c_{\mu_0}/8 = r_2/2$. Hence the domain $D_3 \ne \varnothing$ and $2r_1 < r_2$. In addition there is $t \in (0, t_1)$ such that $2r_1 = (b(\mu_0)t)^2/(c_{\mu_0}(4+2t^2))$.

Repeating the arguments which we employed in the proof of (3.43), we conclude that (3.43), with $\mu = \mu_0$, holds in D_3 for fixed $0 < t \le t_1$, where the parameter δ satisfies the condition $0 < \delta \le 10^{-4}\delta_1$. We see from (3.84) that the inverse function $\varphi(z,t;\mu_0)$ of the function $z = -\widehat{g}(t,\tau;\mu_0) + \hat{c}_{\mu_0} + ib(\tau;\mu_0)t\tau$, $\delta \le \tau \le 1$, has the form, for $z \in \gamma_t$ with fixed $t \ge 0$,

$$\varphi(z,t;\mu_0) = \frac{ib(\mu_0)t}{c_{\mu_0}(2+t^2)} - i\sqrt{\Big(\frac{b(\mu_0)t}{c_{\mu_0}(2+t^2)}\Big)^2 + \frac{2z}{c_{\mu_0}(2+t^2)}}. \tag{3.85}$$

Here the function $\sqrt{z}$ is regular in the complex plane, cut along the negative imaginary semiaxis, selecting the branch with $\sqrt{1} = 1$. We note from (3.85) that $\varphi(z,t;\mu_0)$, $z \in \gamma_t$, admits an analytic continuation to the domain D_3 by the formula (3.85) for every fixed $0 < t \le t_1$. We obtain from (3.43) and (3.44), with $\varphi(\zeta,0;\mu_0)$, $\zeta \in \gamma_0$, and $\varphi(\zeta,t;\mu_0)$, $\zeta \in \gamma_t$, from (3.85), that the function $\varphi'(z,t;\mu_0)/\varphi(z,t;\mu_0)$ in (3.43) has a finite limit as $z \in D_3$ and z tends to $-2r_1$. On the other hand, we conclude from (3.85) that, for $z \in D_3$,

$$\frac{\varphi'(z,t;\mu_0)}{\varphi(z,t;\mu_0)} = -\frac{i}{\sqrt{(b(\mu_0)t)^2 + c_{\mu_0}z(4+2t^2)}} \times \frac{c_{\mu_0}(2+t^2)}{ib(\mu_0)t - i\sqrt{(b(\mu_0)t)^2 + c_{\mu_0}z(4+2t^2)}}.$$

This relation implies that if $b(\mu_0) \ne 0$, then $\varphi'(z,t;\mu_0)/\varphi(z,t;\mu_0) \to \infty$ as $z \in D_3$ and $z \to -(b(\mu_0)t)^2/(c_{\mu_0}(4+2t^2))$. Choosing here $t \in (0,t_1)$ such that $2r_1 = (b(\mu_0)t)^2/(c_{\mu_0}(4+2t^2))$ we arrive at a contradiction. Hence $b(\mu_0) = 0$ and (3.83) follows from (3.84). Assertion (3.82) follows from the definition of $b(\mu_0)$. The lemma is proved. $\square$

In order to formulate the next lemma denote $\rho(t,2;\mu) = c_\mu(1+t^2/2)$. Recall that the functions $\rho(t,\alpha;\mu)$, $\alpha \in (0,2)$, are defined in Lemma 3.14.

LEMMA 3.18. *Assume that S_n/V_n converges weakly, and a subsequence $\{\mu_{n'}\} \subset \{\mu_n\}$ converges vaguely to some measure μ with $\mu(\mathbb{R}) > 0$. Then there exists $\alpha \in (0, 2]$ such that the function $f(t)$ of the form* (1.9) *may be written as*

$$f(t) = -\frac{1}{\alpha}\log\frac{\rho(t,\alpha;\mu)}{\rho(0,\alpha;\mu)}, \qquad t \in \mathbb{R}, \tag{3.86}$$

where $\alpha \in (0, 2)$ if μ is not concentrated at zero and $\alpha = 2$ otherwise.

In (3.86) and in the sequel we take the principal branch of the logarithm of considered functions.

PROOF OF LEMMA 3.18. We use the arguments which we employed in the proof of Lemma 3.2. Return to (3.35). This relation, as it is easy to see, holds for $z \in D_2 \setminus \overline{D}_1$. Denote by C_r, $r \in (r_1, r_2)$, the closed curve which is the boundary of the rectangle $R(r, \mu)$ [see (3.38) and Figure 2]. The positive direction of the curve C_r is that for which the domain $R(r, \mu)$ is on the left. Then we obtain from (3.35) the following relation:

$$\begin{aligned}-\frac{1}{2\pi i}\int_{C_r}\frac{f(t)}{z}\,dz &= \frac{1}{2\pi i}\int_{C_r} f_1(z;t)\,dz + \frac{1}{2\pi i}\int_{C_r} f_2(z;t)\,dz\\ &\quad + \frac{1}{2\pi i}\int_{C_r} f_3(z;t)\,dz.\end{aligned} \tag{3.87}$$

Since $g'_\tau(0, \tau; \mu_n) > 0$, $\tau > 0$, $n \in \mathbb{N}$, we see from (3.26) and (3.27), with $a = \delta$, that, for $n_1 \ge n_0$, $t \in [-t_0, t_0]$, $0 < \tau \le \delta$, the value $-g(t, \tau; \mu_{n_1})$ lies in the domain D_1. Recall that, by the assumption, $\delta > 0$ is a sufficiently small but fixed parameter. Then, recalling (3.24), (3.29) and (3.30), by the dominated convergence theorem, Fubini's theorem and the relation

$$\int_{C_r}(z-w)^{-1}\,dz = 2\pi i\zeta, \tag{3.88}$$

where $\zeta = 1$ if $w \in \mathbb{C}$, $w \in R(r, \mu)$, $\zeta = 0$ if $w \notin \overline{R(r, \mu)}$ and $\zeta = 1/2$ if $w \in C_r$ and w is not a vertex of the rectangle $R(r, \mu)$, we have

$$\begin{aligned}&\frac{1}{2\pi i}\int_{C_r}\lim_{n_1\to\infty} f_{1,n_1}(z;t)\,dz\\ &\quad = \lim_{n_1\to\infty}\frac{1}{2\pi i}\int_{C_r} f_{1,n_1}(z;t)\,dz\\ &\quad = \lim_{n_1\to\infty}\int_0^\delta \frac{1}{2\pi i}\int_{C_r}\left(\frac{1}{z+g(0,\tau;\mu_{n_1})} - \frac{1}{z+g(t,\tau;\mu_{n_1})}\right)\frac{d\tau}{\tau} = 0.\end{aligned}$$

It is not difficult to conclude from Lemma 3.8 and from the proof of Lemma 3.10 that the function $f_3(z; t)$ is regular in the domain D_2. Therefore, by Cauchy's theorem, the third summand on the right-hand side of (3.87) is equal to zero. Hence we can rewrite (3.87) in the form

$$-f(t) = \frac{1}{2\pi i}\int_{C_r} f_2(z;t)\,dz, \qquad r \in (r_1, r_2). \tag{3.89}$$

By Lemma 3.14, $\widehat{g}(t,\tau;\mu) - \hat{c}_\mu - ib(\tau;\mu)t\tau$ has the form (3.64)–(3.66) in the case where μ is not concentrated at zero. By Lemma 3.17, this function has the form (3.83) in the case, where μ is concentrated at zero. Without loss of generality we may assume in these formulae that $\operatorname{Im}\rho(t,\alpha;\mu) \not\equiv 0$ and $\operatorname{Im}\rho(t,\alpha;\mu) < 0$. In the case where $\operatorname{Im}\rho(t,\alpha;\mu) \equiv 0$ we shall argue in the same way. Recalling the definition of the function $f_2(z;t)$, we see that

$$\begin{aligned}\frac{1}{2\pi i}&f_2(z;t)\\ &= \frac{1}{2\pi i}\int_\delta^1\Big(\frac{1}{z+\widehat{g}(0,\tau;\mu)} - \frac{1}{z+\widehat{g}(t,\tau;\mu)-ib(\tau;\mu)t\tau}\Big)\frac{d\tau}{\tau}\end{aligned} \tag{3.90}$$

for $z \in C_r$ such that $z + \hat{c}_\mu$ does not belong to the closure of D_3, where in our case, for sufficiently small $t \in [-t_0, t_0]$,

$$D_3 = \{z \in \mathbb{C} : \pi + \arg\rho(t,\alpha;\mu) < \arg z < \pi\} \cap \{z \in \mathbb{C} : -c_\mu/4 < \operatorname{Re} z < -r_1\}.$$

Using the argument which we employed in the proof of Lemma 3.11 [see (3.43)], we have, for $z \in C_r$ such that $\operatorname{Re} z = -\hat{r}$ and $z + \hat{c}_\mu \in D_3$,

$$\begin{aligned}\frac{1}{2\pi i}&f_2(z;t)\\ &= -\frac{\varphi'(z+\hat{c}_\mu,t;\mu)}{\varphi(z+\hat{c}_\mu,t;\mu)}\\ &\quad + \frac{1}{2\pi i}\int_\delta^1\Big(\frac{1}{z+\widehat{g}(0,\tau;\mu)} - \frac{1}{z+\widehat{g}(t,\tau;\mu)-ib(\tau;\mu)t\tau}\Big)\frac{d\tau}{\tau}.\end{aligned} \tag{3.91}$$

Here the function

$$\varphi(z,t;\mu) = e^{-i\pi/\alpha}\big(z/\rho(t,\alpha;\mu)\big)^{1/\alpha}, \tag{3.92}$$

with $\alpha \in (0,2]$, for any $z \in \overline{D}_3$. The function $z^{1/\alpha}$ is regular in the complex plane, cut along the negative imaginary semiaxis, selecting the branch with $1^{1/\alpha} = 1$. Represent the integral on the right-hand sides of (3.90) and (3.91) in the form

$$\begin{aligned}\frac{1}{2\pi i}&\Big(\int_\delta^a + \int_a^d + \int_d^1\Big)\Big(\frac{1}{z+\widehat{g}(0,\tau;\mu)} - \frac{1}{z+\widehat{g}(t,\tau;\mu)-ib(\tau;\mu)t\tau}\Big)\frac{d\tau}{\tau}\\ &:= \frac{1}{2\pi i}\Big(\int_\delta^a + \int_a^d + \int_d^1\Big)\widehat{K}(z,\tau;t)\,d\tau := I_1(z) + I_2(z) + I_3(z),\end{aligned}$$

where $a = a(r, t; \mu)$ and $d = d(r; \mu)$ are defined by the relations

$$\operatorname{Re}\widehat{g}(t, a; \mu) = \hat{r} \quad \text{and} \quad \widehat{g}(0, d; \mu) = \hat{r},$$

respectively. Recall that $\hat{r} := r + \hat{c}_\mu$.

Let $z_1 := z_1(\mu)$ be the point satisfying $-z_1 \in \gamma_t$ and $\operatorname{Re} z_1 = r$. By $\hat{z}_1$ denote $z_1 + \hat{c}_\mu$. Denote also by $l_{r,\varepsilon}$ the part of the curve C_r cut out by the circles $c(-\hat{r}, \varepsilon)$ and $c(-\hat{z}_1, \varepsilon)$, where $c(\zeta, \varepsilon) := \{z \in \mathbb{C} : |z - \zeta| < \varepsilon\}$, $\zeta \in \mathbb{C}$. In the following we shall consider the curves $C_{r,\varepsilon} := C_r \setminus l_{r,\varepsilon}$, where the parameter $\varepsilon > 0$ is sufficiently small (see Figure 2).

It is not difficult to see that, for $\tau \in [\delta, 1]$ and $|t| \le t_0$,

$$\left| \int_{C_{r,\varepsilon}} \widehat{K}(z, \tau; t)\, dz \right| \le c(\delta, \mu) < \infty,$$

where $c(\delta, \mu)$ is a positive constant which does not depend on ε.

Since, for $\delta \le \tau < a$,

$$\widehat{g}(0, \tau; \mu) \le \operatorname{Re}\widehat{g}(t, \tau; \mu) < \hat{r}$$

and, for $d < \tau \le 1$,

$$\widehat{g}(0, \tau; \mu) > \hat{r},$$

we see with the help of the dominated convergence theorem and (3.88), that

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_{C_{r,\varepsilon}} I_1(z)\, dz &= \lim_{\varepsilon \to 0} \int_\delta^a \frac{1}{2\pi i} \int_{C_{r,\varepsilon}} \widehat{K}(z, \tau; t)\, dz\, d\tau \\ &= \int_\delta^a \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{C_{r,\varepsilon}} \widehat{K}(z, \tau; t)\, dz\, d\tau = 0 \end{aligned} \tag{3.93}$$

and

$$\lim_{\varepsilon \to 0} \int_{C_{r,\varepsilon}} I_3(z)\, dz = \int_d^1 \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{C_{r,\varepsilon}} \widehat{K}(z, \tau; t)\, dz\, d\tau = 0. \tag{3.94}$$

For $a < \tau < d$ we have the inequalities

$$\widehat{g}(0, \tau; \mu) < \hat{r} \quad \text{and} \quad \operatorname{Re}\widehat{g}(t, \tau; \mu) > \hat{r},$$

therefore

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_{C_{r,\varepsilon}} I_2(z)\, dz &= \int_a^d \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{C_{r,\varepsilon}} \widehat{K}(z, \tau; t)\, dz\, d\tau \\ &= \int_a^d \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{C_{r,\varepsilon}} \frac{1}{z + \widehat{g}(0, \tau; \mu)}\, dz\, \frac{d\tau}{\tau} \\ &= \int_a^d \frac{d\tau}{\tau} = \log \frac{d}{a}. \end{aligned} \tag{3.95}$$

Using (3.89)–(3.91) and (3.93)–(3.95), we see that, for $r \in (r_1, c_\mu/4)$,

$$f(t) = -\log\frac{\varphi(-z_1,t;\mu)}{\varphi(-r,t;\mu)} - \log\frac{d}{a}. \tag{3.96}$$

Using Lemma 3.14, (3.83) and (3.92), we deduce

$$\frac{\varphi(-z_1,t;\mu)}{\varphi(-r,t;\mu)} = \Big(1 + i\frac{\operatorname{Im}\rho(t,\alpha;\mu)}{\operatorname{Re}\rho(t,\alpha;\mu)}\Big)^{1/\alpha} \quad \text{and} \quad \frac{d}{a} = \Big(\frac{\operatorname{Re}\rho(t,\alpha;\mu)}{\rho(0,\alpha;\mu)}\Big)^{1/\alpha}.$$

Therefore we get from (3.96) the relation (3.86) for $t \in [-t_0, t_0]$. Since $\rho(t,\alpha;\mu)$ is a regular function in $\mathbb{C}$ and, as it is easy to see, $f(t)$ is regular in the domain $\{t \in \mathbb{C} : \operatorname{Re} t^2 > 0\}$, (3.86) holds for all $t \in \mathbb{R}$. The lemma is proved. □

LEMMA 3.19. *Let X have a stable distribution G with exponent $0 < \alpha < 2$. For $1 < \alpha < 2$, we assume $\mathbf{E}X = 0$. For $\alpha = 1$, we assume that, for some $a \in \mathbb{R}$, $G(x + a)$ has a Cauchy law. Then the sequence $\{\mu_n\}$ has a weak limit μ ($c_\mu = 1$), such that the function $\widehat{g}(t,u;\mu) - ib(u;\mu)tu$ has the form* (3.64)–(3.66). *Moreover, every such function is generated by a random variable X with distribution G from the class described above.*

One can prove this lemma with the help of elementary calculations. Therefore we omit the proof of this lemma.

LEMMA 3.20. *Assume that conditions* (1.2) *and* (1.8) *hold and that $\{\mu_n\}$ does not converge to $\mu \equiv 0$ in the vague topology. Then there exists $\alpha \in (0,2)$ such that*

$$\int_{(x,\infty)} \frac{1+u^2}{u^2}\, d\mu_n \to c_1(\alpha)\alpha^{-1}x^{-\alpha}, \qquad x > 0,$$

$$\int_{(-\infty,x)} \frac{1+u^2}{u^2}\, d\mu_n \to c_2(\alpha)\alpha^{-1}|x|^{-\alpha}, \qquad x < 0,$$

where the constants $c_j(\alpha)$ satisfy $c_j(\alpha) \ge 0$, $j = 1, 2$, and (2.6).

PROOF. In view of Lemma 3.1, $\{\mu_n\}_{n=1}^\infty$ contains a vaguely convergent subsequence to some limit measure μ such that $0 < \mu(\mathbb{R}) \le 1$. Let us show that this measure μ is not concentrated at zero. Assume, to the contrary, that μ is concentrated at zero. By Lemma 3.17,

$$\begin{aligned}\widehat{g}(t,\tau;\mu) - \hat{c}_\mu - ib(\tau;\mu)t\tau &= c_\mu\tau^2(1 + t^2/2)\\ &:= \tau^2\rho(t,2;\mu), \qquad \tau > 0, t \in \mathbb{R}.\end{aligned}$$

Then, by Lemma 3.18,

$$f(t) = -\tfrac{1}{2}\log(1 + t^2/2), \qquad t \in \mathbb{R}. \tag{3.97}$$

Return to relation (1.9). This relation, by (3.97), has the form

$$(3.98)\qquad -\frac{1}{2}\log(1+t^2/2)=\int_0^\infty\Big(\lim_{n\to\infty}\mathbf{E}e^{it\tau S_n/V_n}-1\Big)e^{-\tau^2}\,\frac{d\tau}{\tau},\qquad t\in\mathbb{R}.$$

By the uniqueness theorem for Laplace transform, we deduce from (3.98) the formula $\lim_{n\to\infty}\mathbf{E}e^{itS_n/V_n}=e^{-t^2/2}, t\in\mathbb{R}$. Then we see from Lemma 2.6 that X is in the domain of attraction of a normal law and $\mathbf{E}X=0$. Therefore $\mathbf{E}X^2\mathbf{I}\{|X|\le x\}$ is a slowly varying function at $+\infty$ [see Ibragimov and Linnik (1971), page 83]. With the help of well-known calculations [see Ibragimov and Linnik (1971), pages 79–83] this implies that $\{\mu_n\}$ has a weak limit μ such that $\mu(\{0\})=1$, which contradicts (1.8).

Thus, μ is not concentrated at zero. Using Lemma 3.18, we note that there exists $\alpha\in(0,2)$ such that

$$(3.99)\qquad -\frac{1}{\alpha}\log\frac{\rho(t,\alpha;\mu)}{\rho(0,\alpha;\mu)}=\int_0^\infty\big(g(t\tau)-1\big)e^{-\tau^2}\,\frac{d\tau}{\tau},\qquad t\in\mathbb{R},$$

where $\rho(t,\alpha;\mu)$ is defined in Lemma 3.14 and $g(t\tau):=\mathbf{E}e^{it\tau Z}=\lim_{n\to\infty}\mathbf{E}e^{it\tau S_n/V_n}$. Assume that $X, X_j,\ j\in\mathbb{N}$, are i.i.d. stable random variables with exponent α, satisfying the assumptions of Lemma 3.19. Let $g_1(t\tau):=\lim_{n\to\infty}\mathbf{E}e^{it\tau S_n/V_n}$, where $S_n,\ V_n$ are defined with the help of these stable random variables. By Lemma 2.4, $\lim_{n\to\infty}\mathfrak{L}(S_n/V_n)$ has no mass at the points $+1$ and -1. In addition, by Lemmas 2.4, 3.18 and 3.19, (3.99) holds with $g(t\tau)$ replaced by $g_1(t\tau)$. Using again the uniqueness theorem for Laplace transform, we deduce that $g(t)=g_1(t), t\in\mathbb{R}$, and we obtain that $\mathbf{P}(|Z|=1)=0$. Then, by Lemma 3.16, we conclude that the family $\{\mu_n\}$ is tight.

Now let us prove that the sequence $\{\mu_n\}$ has a *weak* limit. In view of (1.8), $\{\mu_n\}_{n=1}^\infty$ contains a weakly convergent subsequence to some probability measure μ which is not concentrated at zero. By Lemma 3.14, the function $\widehat{g}(t,u;\mu)-\hat{c}_\mu-ib(u;\mu)tu$, which corresponds to this measure, has the form (3.64)–(3.66). Note that in this case $c_\mu=1$. Let a subsequence $\{\mu_{n'}\}\subset\{\mu_n\}$ converge weakly to another measure μ_1. By Lemma 3.14, the function $\widehat{g}(t,u;\mu_1)-ib(u;\mu_1)tu$, which corresponds to this measure, is of the form (3.64)–(3.66) with a parameter $\alpha_1, 0<\alpha_1<2$. Let us show that $\alpha_1=\alpha$. By Lemma 3.18, we have the relation

$$(3.100)\qquad \left(\frac{\rho(t,\alpha;\mu)}{\rho(0,\alpha;\mu)}\right)^{1/\alpha}=\left(\frac{\rho(t,\alpha_1;\mu_1)}{\rho(0,\alpha_1;\mu_1)}\right)^{1/\alpha_1},\qquad t\in\mathbb{R}.$$

The power of the function $\rho_1(t)=\rho(t,\alpha;\mu)/\rho(0,\alpha;\mu)$ is defined by $(\rho_1(t))^{1/\alpha}=\exp\{\frac{1}{\alpha}\log\rho_1(t)\}$, where we take for $\log\rho_1(t)$ that branch of the logarithm for which $\log\rho_1(0)=0$ and which is continuous. We noted earlier that the function $\rho(t,\alpha;\mu),\ t\in\mathbb{R}$, admits an analytic continuation in $\mathbb{C}$ as an entire function for any $0<\alpha<2$. Let us assume for definiteness that $c_1(\alpha)\neq 0$. Since, as it is easy to see from (3.64)–(3.66), $\rho(iy,\alpha;\mu)\to-\infty$ as $y\to-\infty$, the function $\rho(z,\alpha;\mu)$

has a finite number of zeros on the semiaxis $z = iy$, $y < 0$. Denote these zeros by $iy_1, iy_2, \dots, iy_m$, where $y_1 \ge y_2 \ge \cdots \ge y_m$. Write $P(z) := (z - iy_1)\cdots(z - iy_m)$. Then $(\rho(z,\alpha;\mu)/P(z))^{1/\alpha}$ is a regular function on the semiaxis $z = iy$, $y < 0$. We conclude by (3.100) that $\rho^{1/\alpha_1}(z,\alpha_1;\mu_1)/P^{1/\alpha}(z)$ is also a regular function on the same semiaxis as well and that the following equality holds

$$\left(\frac{\rho(iy,\alpha;\mu)}{\rho(0,\alpha;\mu)P(iy)}\right)^{1/\alpha} = \left(\frac{\rho(iy,\alpha_1;\mu_1)}{\rho(0,\alpha_1;\mu_1)}\right)^{1/\alpha_1} \frac{1}{(P(iy))^{1/\alpha}}, \qquad y \le 0.$$

Recalling Lemma 3.15, we see that this equality cannot hold for sufficiently large $|y|$ if $\alpha \ne \alpha_1$. This proves $\alpha = \alpha_1$. Since the parameter $c_3(\alpha)$ is defined by the relation (3.62), we see that $\rho(0,\alpha;\mu) = \rho(0,\alpha;\mu_1)$ and we have from (3.100) $\rho(t,\alpha;\mu) = \rho(t,\alpha;\mu_1)$, $t \in \mathbb{R}$. This relation implies $\mu = \mu_1$. Hence $\{\mu_n\}$ converges weakly to the measure μ and, by Lemma 3.2, we arrive at the assertion of the lemma. $\square$

PROOF OF LEMMA 2.7. Note that if $\{\mu_n\}$ has vague limit 0, then

$$\mu_n\big((-\infty,-x)\cup(x,\infty)\big) \to 1, \qquad n \to \infty,$$

for every $x > 0$. Now we obtain Lemma 2.7 as obvious consequence of Lemma 2.2, Lemma 2.3 and Lemma 3.20, Lemma 2.1, Lemma 2.4. $\square$

PROOF OF LEMMA 2.8. Let us first outline the proof. Assumptions (1.2), (1.11) and Lemma 3.17 allow us to construct a triangular array X_{nk} of random variables which are independent in each row such that the sums $\sum_k X_{nk}$ converge weakly to $N(0,1)$ distribution and, on the other hand, converge weakly to a random variable Z in (1.2). Then, by the result of Giné, Götze and Mason (1997), we arrive at the assertion of the lemma.

In order to prove Lemma 2.8 we need the following well-known result [see Petrov (1975), page 95].

LEMMA 3.21. *Let $\{X_{nk}\}$ denote a triangular array of random variables which are independent in each row, and let $F_{nk}(x)$ denote the distribution function of X_{nk}. The condition of asymptotic negligibility, that is* $\max_{1\le k\le k_n}\mathbf{P}(|X_{nk}| \ge \varepsilon) \to 0$ *for every fixed* $\varepsilon > 0$, *holds and there will exist a sequence of constants* $\{b_n\}$ *such that the distribution of the sums* $\sum_k X_{nk} - b_n$ *converges weakly to the normal* (0, 1) *distribution, if and only if the following conditions are satisfied*:

$$\sum_k \mathbf{P}(|X_{nk}| \ge \varepsilon) \to 0 \tag{3.101}$$

for every $\varepsilon > 0$ *and*

$$\sum_k \left\{\int_{|x|<\tau} x^2\,dF_{nk}(x) - \left(\int_{|x|<\tau} x\,dF_{nk}(x)\right)^2\right\} \to 1 \tag{3.102}$$

for some $\tau > 0$. If these conditions are satisfied, we may write

$$b_n = \sum_k \int_{|x|<H} x\, dF_{nk}(x) + o(1), \tag{3.103}$$

where H is an arbitrary positive number. All admissible constants b_n satisfy this equation.

Using the diagonal procedure we conclude from assumption (1.11) that there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ of the positive integers such that $n_k \to \infty$ as $k \to \infty$ and

$$n_k \mathbf{P}\big(|X| > 2^{-k} B_{n_k}\big) \le \frac{1}{k}. \tag{3.104}$$

Denote $n' = n_k$. Let $X_{j,n'}$, $\mathcal{S}_{n'}$ and $\mathcal{V}_{n'}$ be defined as in (3.72) and (3.73), respectively, with $\varepsilon = 1$. Since S_n / V_n converges weakly to a random variable Z, it is not difficult to see that for the subsequence $\{n'\}$

$$G(x) = \lim_{n'\to\infty} \mathbf{P}\bigg(\frac{S_{n'}}{V_{n'}} < x\bigg) = \lim_{n'\to\infty} \mathbf{P}\bigg(\frac{\mathcal{S}_{n'}}{\mathcal{V}_{n'}} < x\bigg), \qquad x \in C(G), \tag{3.105}$$

where $C(G)$ is the continuity set of the distribution function $G(x) = \mathbf{P}(Z < x)$. By (3.104) and (1.5), we conclude that

$$n' \mathbf{E} \frac{X^2}{B_{n'}^2} \mathbf{I}\{|X| \le B_{n'}\} \to 1, \qquad n' \to \infty. \tag{3.106}$$

Since, for sufficiently large n',

$$\begin{aligned} \mathbf{E}\big(\mathcal{V}_{n'}^2 - \mathbf{E}\mathcal{V}_{n'}^2\big)^2 &= n' B_{n'}^{-4} \mathbf{E}\big(X_{1,n'}^2 - \mathbf{E}X_{1,n'}^2\big)^2 \le n' B_{n'}^{-4} \mathbf{E}X_{1,n'}^4 \\ &= n' B_{n'}^{-4} \mathbf{E}X_{1,n'}^4 \mathbf{I}\{|X_{1,n'}| \le 2^{-k} B_{n'}\} \\ &\quad + n' B_{n'}^{-4} \mathbf{E}X_{1,n'}^4 \mathbf{I}\{2^{-k} B_{n'} < |X_{1,n'}| \le B_{n'}\} \\ &\le 2^{-2k} n' B_{n'}^{-2} \mathbf{E}X_{1,n'}^2 + n' \mathbf{P}\big(|X_{1,n'}| > 2^{-k} B_{n'}\big) \le \frac{c}{k}, \end{aligned} \tag{3.107}$$

we deduce that $\mathcal{V}_{n'}^2 \to 1$ as $n' \to \infty$, in probability. Therefore we have

$$\lim_{n'\to\infty} \mathbf{P}\bigg(\frac{\mathcal{S}_{n'}}{\mathcal{V}_{n'}} < x\bigg) = \lim_{n'\to\infty} \mathbf{P}(\mathcal{S}_{n'} < x), \qquad x \in C(G). \tag{3.108}$$

We now apply Lemma 3.21 to the triangular array of random variables

$$\bigg\{ \frac{X_j}{B_{n'}} \mathbf{I}\{|X_j| \le B_{n'}\} \bigg\}_{j=1}^{n'}.$$

Since $B_{n'} \to \infty$ as $n' \to \infty$, the condition of asymptotic negligibility holds for these random variables. Condition (3.101) follows from (3.104). In view of (3.104), $\{\mu_{n'}\}$ converges weakly to the probability measure μ such that $\mu(\{0\}) = 1$. Applying Lemma 3.17 to the family $\{\mu_{n'}\}$, we obtain (3.82) for some subsequence $\{n'_1\} \subset \{n'\}$. Without loss of generality we may assume that $\{n'_1\} = \{n'\}$. From (1.6) and (3.104) it follows that (3.82) can be rewritten in the form

$$\lim_{n'\to\infty} \frac{n'}{B_{n'}} \mathbf{E} X \mathbf{I}\{|X| \le B_{n'}\} = 0. \tag{3.109}$$

It remains to note that (3.102) follows from (3.106) and (3.109), and, by (3.109), we have in (3.103) $b_{n'} = o(1)$. In view of Lemma 3.21, we conclude that $\mathcal{S}_{n'}$ converges to $N(0, 1)$ in distribution. Hence, (1.2), (3.105) and (3.108) together imply that the sequence S_n/V_n converges to $N(0, 1)$ in distribution. It remains to note that, in view of Lemma 2.6, X is in the domain of attraction of a normal law and $\mathbf{E}X = 0$. □

APPENDIX

PROOF OF LEMMA 2.3. We need the following result which is due to Darling (1952). To formulate it introduce the notation $X_n^* := \max\{|X_1|, \dots, |X_n|\}$.

LEMMA A.1. *Assume that* $X \ge 0$. *Then* $\mathbf{E}|S_n/X_n^* - 1| \to 0$ *as* $n \to \infty$, *if* $\mathbf{P}(X > x)$, $x > 0$, *is a slowly varying function at* $+\infty$.

Since

$$\left| \frac{|S_n|}{X_n^*} - 1 \right| \le \frac{|X_1| + \cdots + |X_n|}{X_n^*} - 1,$$

we see, by Lemma A.1, that $|S_n|/X_n^* \to 1$ as $n \to \infty$ a.s. Applying Lemma A.1 to $X_1^2, \dots, X_n^2$, it also follows that $V_n/X_n^* \to 1$ as $n \to \infty$ a.s. The assertion of Lemma 2.3 immediately follows from these relations. □

PROOF OF LEMMA 2.4. In the first step we shall prove assertions (i) and (iv) for $0 < \alpha < 1$. Since X is in the domain of attraction of a stable law, it is well known [see Feller (1971), pages 574–581], that (2.1) holds with $b_n = a_n$, where a_n is the lower bound of all $x > 0$ for which $nx^{-2}(1 + \int_{-x+0}^{x} y^2\, dF(y)) \le 1$. Now we shall show that $(S_n/b_n, (V_n/b_n)^2)$, $n \in \mathbb{N}$, has a limiting joint distribution. Write, for $t \in \mathbb{R}$, $s \in \mathbb{R}$,

$$\begin{aligned}
\mathbf{E}e^{itS_n/b_n + isV_n^2/b_n^2} &= \big(\mathbf{E}e^{itX/b_n + isX^2/b_n^2}\big)^n \\
&= \left(1 + \int_{-\infty}^{\infty} \frac{e^{itx+isx^2} - 1}{x^2} x^2\, dF(b_n x)\right)^n \qquad \text{(A.1)}\\
&:= \big(1 + I_n(t, s)\big)^n.
\end{aligned}$$

It is sufficient to prove that $\mathbf{E}e^{itS_n/b_n+isV_n^2/b_n^2}$ has a limit as $n \to \infty$ for any fixed $(t,s) \in \mathbb{R}^2$. Repeating the arguments of Feller [(1971), pages 579–581], we obtain

$$nI_n(t,s) \to \int_{-\infty}^{\infty} (e^{itx+isx^2} - 1)\frac{K(x)}{|x|^{1+\alpha}}\,dx, \qquad n \to \infty, \tag{A.2}$$

for any fixed $(t,s) \in \mathbb{R}^2$, where $K(x) = c_4$ if $x < 0$, $K(x) = c_5$ if $x > 0$, and the constants c_4, c_5 such that $c_4, c_5 \geq 0, c_4 + c_5 > 0$. Using (A.1) and (A.2), we obtain

$$\begin{aligned}&\lim_{n\to\infty} \mathbf{E}\exp\{itS_n/b_n + isV_n^2/b_n^2\}\\ &\quad = \exp\left\{\int_{-\infty}^{\infty} (\exp\{itx + isx^2\} - 1)\frac{K(x)}{|x|^{1+\alpha}}\,dx\right\}, \qquad (t,s) \in \mathbb{R}^2.\end{aligned} \tag{A.3}$$

Thus we conclude from (A.3) and the continuity theorem [Feller (1971), page 508] that $(S_n/b_n, (V_n/b_n)^2), n \in \mathbb{N}$, has a limiting distribution. Hence it follows that $S_n/V_n, n \in \mathbb{N}$, has a limiting distribution since $V_n/b_n, n \in \mathbb{N}$, has a limiting distribution concentrated on the positive semiaxis. Assertion (i) of the lemma is proved.

We have

$$\frac{1}{\pi}\int_0^{\infty} \lim_{n\to\infty} \mathbf{E}e^{itS_n/V_n}e^{-st}\,dt = \int_0^{\infty} e^{-s^2t^2/2}\mathcal{D}(t)\,dt, \qquad s > 0, \tag{A.4}$$

where

$$\mathcal{D}(t) = (1-\alpha)(2\pi^{-3})^{1/2}\frac{c_5 D_{\alpha-2}(-it) + c_4 D_{\alpha-2}(it)}{c_5 D_{\alpha}(-it) + c_4 D_{\alpha}(it)} \tag{A.5}$$

[see LMRS (1973), equation (4.14), page 796]. In the last equation $D_\nu(z), z \in \mathbb{C}$, denotes the parabolic cylinder function [see Magnus, Oberhettinger and Soni (1966)], which is an entire function of z for each real ν. Using the following asymptotic expansions for $D_\nu(z)$ [Magnus, Oberhettinger and Soni (1966), page 331]

$$D_\nu(z) = z^\nu \exp\{-z^2/4\}\big(1 + o(1)\big) \qquad \text{as } z \to \infty, |\arg z| < 3\pi/4, \tag{A.6}$$

we obtain

$$\mathcal{D}(t) = \big(1 + o(1)\big)(1-\alpha)(2\pi^{-3})^{1/2}t^{-2}, \qquad t \to +\infty. \tag{A.7}$$

The formulas (A.4), (A.5) and (A.7) hold for $1 < \alpha < 2$ as well. On the other hand, using the identity

$$\frac{t}{\sqrt{\pi}}\int_0^{\infty} e^{-(s^2/(4u))-ut^2}\,\frac{du}{\sqrt{u}} = e^{-st}, \qquad s > 0,\ t > 0,$$

and the integral representation

$$e^{-a^2/4}D_{-2}(-ia) = \int_0^{\infty} e^{iat-t^2/2}t\,dt, \qquad a \in \mathbb{C}$$

[Magnus, Oberhettinger and Soni (1966), page 328], we see that, for $s > 0$,

$$
\begin{aligned}
&\frac{1}{\pi}\int_0^\infty e^{-t^2/2+iat}e^{-st}\,dt \\
&\quad = \left(\frac{2}{\pi^3}\right)^{1/2}\int_0^\infty e^{-s^2t^2/2}e^{-a^2t^2/(4(t^2+1))}D_{-2}\left(-\frac{iat}{\sqrt{t^2+1}}\right)\frac{dt}{1+t^2},
\end{aligned}
\tag{A.8}
$$

where $a \in \mathbb{R}$. Note that the relation $\operatorname{Im} D_{-2}(-ia) = \sqrt{\pi/2}\,ae^{-a^2/4} \neq 0$ holds for $a \in \mathbb{R} \setminus \{0\}$. Thus, comparing (A.4), (A.7) and (A.8), and using the Laplace uniqueness theorem, we conclude that the limiting distribution of S_n/V_n is not Gaussian. By assertions (i)–(iv) [LMRS (1973), page 801] the limiting distribution has no mass at the points $+1$ and -1. Assertion (iv), with $0 < \alpha < 1$, of the lemma is proved.

Now we shall prove assertions (ii) and (iv) for $1 < \alpha < 2$ of the lemma. In the case $1 < \alpha < 2$ we write

$$
\begin{aligned}
&\mathbf{E}\exp\{it(S_n - \mathbf{E}S_n)/b_n + isV_n^2/b_n^2\} \\
&\qquad = \bigl(1 + I_{1n}(t,s) + I_{2n}(t,s) - I_{3n}(t,s)\bigr)^n
\end{aligned}
\tag{A.9}
$$

for $t \in \mathbb{R}$, $s \in \mathbb{R}$, where

$$
\begin{aligned}
I_{1n}(t,s) &:= \int_{-\infty}^\infty \frac{e^{itx+isx^2} - 1 - itx}{x^2}x^2\,dF(b_nx + \mathbf{E}X), \\
I_{2n}(t,s) &:= \int_{-\infty}^\infty e^{it(x-\mathbf{E}X/b_n)}\frac{e^{isx^2} - 1}{x^2}x^2\,dF(b_nx), \\
I_{3n}(t,s) &:= \int_{-\infty}^\infty e^{itx}\frac{e^{isx^2} - 1}{x^2}x^2\,dF(b_nx + \mathbf{E}X).
\end{aligned}
$$

We define the constants b_n in the same way as in the proof of assertion (i) of the lemma above. It is well known that in our case $b_n \to \infty$ as $n \to \infty$ and $b_n \le n^\gamma$, $\gamma < 1$, for sufficiently large n. Repeating again the arguments of Feller [(1971), pages 579–581], we obtain

$$
\begin{aligned}
nI_{1n}(t,s) &\to \int_{-\infty}^\infty \bigl(e^{itx+isx^2} - 1 - itx\bigr)\frac{K(x)}{|x|^{1+\alpha}}\,dx, \\
nI_{jn}(t,s) &\to \int_{-\infty}^\infty e^{itx}\bigl(e^{isx^2} - 1\bigr)\frac{K(x)}{|x|^{1+\alpha}}\,dx, \qquad j = 2, 3,
\end{aligned}
\tag{A.10}
$$

as $n \to \infty$, for any fixed $(t,s) \in \mathbb{R}^2$, where $K(x) = c_6$ if $x < 0$, $K(x) = c_7$ if $x > 0$, and the constants c_6, c_7 such that $c_6, c_7 \ge 0, c_6 + c_7 > 0$. From (A.9) and (A.10) it follows that $\mathbf{E}e^{it(S_n-\mathbf{E}S_n)/b_n+isV_n^2/b_n^2}$ has a limit as $n \to \infty$ for any fixed $(t,s) \in \mathbb{R}^2$. Since $V_n/b_n, n \in \mathbb{N}$, has a limiting distribution concentrated on the positive semiaxis, it follows that $(S_n - \mathbf{E}S_n)/V_n, n \in \mathbb{N}$, has a limiting distribution as well. Hence $S_n/V_n, n \in \mathbb{N}$, has a limiting distribution if and only if $\mathbf{E}X = 0$. In the same way as in the proof of assertion (i) we deduce assertion (iv) of the lemma in the case $\alpha > 1$ that the limiting distribution of $S_n/V_n, n \in \mathbb{N}$, is not Gaussian and has no mass at the points $+1$ and -1.

Let us prove assertions (iii) and (iv) for $\alpha = 1$ of the lemma. Assume that X is in the domain of attraction of Cauchy's law. Let us prove that S_n/V_n, $n \in \mathbb{N}$, converges weakly if and only if $\lim_{n\to\infty} q_n$ exists and is finite, where $q_n := n\mathbf{E}\sin(X/b_n)$. We define the constants b_n in the same way as in the proof of assertion (i) of the lemma above. In the first step we note, repeating the arguments of Feller [(1971), pages 579–581] together with those above, that

$$\begin{aligned}(A.11)\qquad &\lim_{n\to\infty} \mathbf{E}\exp\{it(S_n/b_n - q_n) + isV_n^2/b_n^2\}\\ &\quad = \exp\left\{\int_{-\infty}^{\infty} \left(\exp\{itx + isx^2\} - 1 - it\sin x\right)\frac{K(x)}{x^2}\,dx\right\}\end{aligned}$$

for any fixed $(t, s) \in \mathbb{R}^2$, where $K(x) = c_8$ for $x \in \mathbb{R}$ with the positive constant c_8. Hence

$$(A.12)\qquad \big(S_n/b_n - q_n, (V_n/b_n)^2\big), n \in \mathbb{N}, \text{ has a limiting distribution,}$$

given by the joint distribution of some random variables, say (S, V^2), such that

$$\mathbf{E}e^{itS+isV^2} = \lim_{n\to\infty} \mathbf{E}e^{it(S_n/b_n - q_n)+isV_n^2/b_n^2}, \qquad t \in \mathbb{R}, s \in \mathbb{R}.$$

Since $V_n/b_n, n \in \mathbb{N}$, converges weakly to a distribution concentrated on the positive semiaxis, it follows that $(S_n - b_n q_n)/V_n, n \in \mathbb{N}$, has a limiting distribution as well. By (A.11), it follows, for $a \in \mathbb{R}$,

$$\begin{aligned}(A.13)\qquad &\mathbf{E}\exp\{it(S+a) + isV^2\}\\ &\quad = \exp\left\{c_8 \int_{-\infty}^{\infty} \left(e^{isx^2}\cos(sx) - 1\right)\frac{dx}{x^2} + iat\right\}, \qquad t \in \mathbb{R}, s \in \mathbb{R}.\end{aligned}$$

Hence, repeating the arguments of LMRS [(1973), pages 794–798], we obtain an analog of (A.4), namely

$$(A.14)\qquad \frac{1}{\pi}\int_0^{\infty} \mathbf{E}e^{it(S+a)/V} e^{-st}\,dt = \int_0^{\infty} e^{-s^2t^2/2}\mathcal{D}_1(t, a)\,dt, \qquad s > 0,$$

with

$$(A.15)\qquad \mathcal{D}_1(t, a) = \left(\frac{2}{\pi^3}\right)^{1/2}\left(1 + te^{t^2/2}\left(\int_0^t e^{-u^2/2}\,du - ia/(2\pi)^{1/2}\right)\right)^{-1}.$$

Assume that $\lim_{n\to\infty} q_n$ exists and is finite. By (A.12), $(S_n/b_n, (V_n/b_n)^2)$, $n \in \mathbb{N}$, has a limiting distribution and thus $S_n/V_n, n \in \mathbb{N}$, has a limiting distribution as well.

On the other hand, assume that $S_n/V_n, n \in \mathbb{N}$, has a limiting distribution. We shall show that $\lim_{n\to\infty} q_n$ exists and is finite. By (A.12), we see that $\limsup_{n\to\infty} |q_n| < \infty$ and denote

$$\widehat{q}_1 := \liminf_{n\to\infty} q_n \quad \text{and} \quad \widehat{q}_2 := \limsup_{n\to\infty} q_n.$$

Let $\{n_1\}$ and $\{n_2\}$ denote subsequences of positive integers such that $\lim_{n_1\to\infty} q_{n_1} = \widehat{q}_1$ and $\lim_{n_2\to\infty} q_{n_2} = \widehat{q}_2$. In view of (A.12), it follows from (A.13) that

$$\lim_{n_j\to\infty} \mathbf{E}\exp\{itS_{n_j}/b_{n_j} + isV_{n_j}/b_{n_j}\}$$
$$= \exp\left\{c_8 \int_{-\infty}^{\infty} (\exp\{isx^2\}\cos(sx) - 1)\frac{dx}{x^2} + i\widehat{q}_j t\right\}, \qquad j = 1, 2.$$

By (A.14), we arrive at the relations

$$\text{(A.16)} \qquad \begin{aligned} &\frac{1}{\pi}\int_0^{\infty} \lim_{n_j\to\infty} \mathbf{E}e^{itS_{n_j}/V_{n_j}} e^{-st}\,dt \\ &\quad = \int_0^{\infty} e^{-s^2t^2/2}\mathcal{D}_1(t, \widehat{q}_j)\,dt, \qquad s > 0, j = 1, 2. \end{aligned}$$

Since the left-hand sides of (A.16) for $j = 1$ and $j = 2$ coincide, we see that the right-hand sides of (A.16) for $j = 1$ and $j = 2$ coincide as well. Using the uniqueness of Laplace transform, it follows $\widehat{q}_1 = \widehat{q}_2$. This proves assertion (iii) of the lemma.

Finally note that (A.4) holds with $\mathcal{D}(t)$ replaced by $\mathcal{D}_1(t, q)$ for a parameter $q := \lim_{n\to\infty} q_n$. It is easy to see by (A.15) that $|\mathcal{D}_1(t, q)| \le c\exp\{-t^2/2\}$ for sufficiently large $t > 0$. Comparing (A.4) and (A.8), we obtain that the limiting distribution of S_n/V_n is not Gaussian. By assertions (i)–(iv) [LMRS (1973), page 801], the limiting distribution has no mass at the points $+1$ and -1. Thus Lemma 2.4 is proved completely. □

PROOF OF LEMMA 2.5. The first assertion of the lemma follows immediately from Lemma 2.4, and from assertions (i)–(iv), LMRS [(1973), page 801]. The relation $D_{\alpha_1} \ne D_{\alpha_2}$ if $0 < \alpha_1 < \alpha_2 < 2$ follows from the relations (A.4), (A.7) and (A.4) with $\mathcal{D}(t)$ replaced by $\mathcal{D}_1(t, 0)$. □

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INSTITUTE FOR LOW TEMPERATURE PHYSICS
AND ENGINEERING
NATIONAL ACADEMY OF SCIENCES OF UKRAINE
47 LENIN AVENUE
61164 KHARKOV
UKRAINE
E-MAIL: chistyakov@ilt.kharkov.ua

FAKULTÄT FÜR MATHEMATIK
UNIVERSITÄT BIELEFELD
POSTFACH 100131
33501 BIELEFELD 1
GERMANY
E-MAIL: goetze@mathematik.uni-bielefeld.de