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RATE OF ESCAPE OF RANDOM WALKS ON WREATH PRODUCTS AND RELATED GROUPS[1]

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This article examines the rate of escape for a random walk on $G \wr \mathbb{Z}$ and proves laws of the iterated logarithm for both the inner and outer radius of escape. The class of G for which these results hold includes finite, G as well as groups of the form $H \wr \mathbb{Z}$, so the construction can be iterated. Laws of the iterated logarithm are also found for random walk on Baumslag–Solitar groups and a discrete version of the Sol geometry.

1. Introduction. One question about random walks that has been classically studied is determining the rate at which a random walk tends to infinity. For a random walk X_n on a finitely generated group G, R_n is said to be an outer radius for X_n if $|X_n| > R_n$ finitely often with probability 1 ($|g|$ denotes the length of g with respect to the generating set). Likewise, a function r_n is said to be an inner radius if $|X_n| < r_n$ finitely often with probability 1. For simple random walk on $\mathbb{Z}$, the law of the iterated logarithm implies that $R_n = \sqrt{(2+\varepsilon)n \log\log n}$ is an outer radius for all $\varepsilon > 0$. The names inner and outer radius to describe such functions are not universally used: the terminology herein follows Grigor'yan [9], but the names upper class and upper upper Lévy class are also used in place of outer radius, and lower lower Lévy class is used in place of inner radius (see, e.g., [16]).

For the case of an inner radius for simple random walk on $\mathbb{Z}^d$, Dvoretzky and Erdős [6] showed that for $d \geq 3$, an increasing function r_n is an inner radius if and only if $\sum (r(2^n)2^{-n/2})^{d-2}$ converges. Hebisch and Saloff-Coste [10] generalized this result to the case of finitely supported, symmetric random walks on groups G of polynomial growth.

For the case of random walks on groups with exponential growth, the main attention has focused on whether or not the walk escapes at a linear rate. A large body of work by Avez [2], Kaimanovich and Vershik [11], Derriennic [5] and Varopoulos [20] established the equivalence of a random walk having a linear rate of escape with the existence of nonconstant bounded harmonic functions as well as with the positivity of the entropy associated with the random walk. Vershik [22] has written a detailed survey that covers this subject.

The focus here is on determining some rates of escape for random walks on groups of exponential growth for examples in which the rate of escape is sublinear.

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Because precise asymptotics for transition probabilities are not known, I am unable to completely characterize inner and outer radii as thoroughly as on $\mathbb{Z}^d$, but for some walks I was able to find inner and outer radii r_n and R_n that are optimal up to constant factors.

The bulk of these examples are random walks on wreath products of the form $G \wr \mathbb{Z}$. Wreath products are a classical construction of semidirect products and a useful tool in group theory. In fact, in the theory of amenable groups, Vershik [21] discovered some interesting geometric properties of Følner sets on $\mathbb{Z} \wr \mathbb{Z}$. Another one of the first key uses of such an example was by Kaimanovich and Vershik [11], who showed that the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ is an example of a group with exponential growth on which a finitely supported, symmetric random walk has zero speed. Previously it had been known that positive speed implied exponential growth [20], but whether or not the converse was true was unknown. The same example was also used by Varopoulos [18, 19] as the first example of a group for which the return probability of random walk was shown to decay like $\exp(-n^{1/3})$.

Since then, random walks on wreath products have been studied in a number of different contexts. Schoolfield [17] studied wreath products of finite groups as a generalization of card shuffling models and used them as a set of baseline examples that are useful because of various comparison techniques. Lyons, Pemantle and Peres [13] showed that speed of an inward biased random walk on the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ can be positive even though the unbiased walk has zero speed. A slight modification of their example was given by Revelle [15], where the exact speed of escape is computed.

The decay of the return probability for symmetric random walks on groups $G \wr \mathbb{Z}$ was studied by Pittet and Saloff-Coste [14], who showed that by iterating the construction it is possible to get the return probability to decay faster than $\exp(-n^{\alpha})$ for any fixed $\alpha < 1$. This behavior is somewhat exotic, because for random walk on polycyclic groups of exponential growth, Alexopoulos [1] showed that the return probability decays like $\exp(-n^{1/3})$.

For some random walks on these iterated wreath products, Ershler [7, 8] independently studied $E|X_n|$, the expected distance from the origin after n steps. She gave examples of walks on groups of the form $G \wr \mathbb{Z}$ for which the drift $E|X_n|$ grows like n^{β} for any β of the form $1 - 2^{-k}$. She also examined some examples of the form $G \wr \mathbb{Z}^2$, in which case $E|X_n|$ can grow like $n/[\ln(\ln(\cdots \ln(n) \cdots))]$. These examples answered an open problem asked by Vershik as to whether or not the drift $E|X_n|$ could have intermediate growth that is between $\sqrt{n}$ and linear.

I likewise studied the rate of escape of random walks on groups of the form $G \wr \mathbb{Z}$, but instead of considering the expected drift, I prove a law of the iterated logarithm for both the inner and outer radii of some random walks on these groups. One consequence of these bounds is that the rate of escape is sublinear, so they give examples of groups with exponential growth with zero speed despite having return probabilities $\exp(-n^{\alpha})$ for α arbitrarily close to 1. At the same time, there are amenable groups (e.g., $\mathbb{Z}_2 \wr \mathbb{Z}^3$) on which the walker has positive speed despite having return probability $\exp(-n^{\alpha})$ for $\alpha < 1$ ($\alpha = 3/5$ in the case of $\mathbb{Z}_2 \wr \mathbb{Z}^3$). The rate of escape and the return probability thus behave quite differently, as it is possible to change the qualitative behavior of either one while holding the other fixed.

The structure of this article is as follows. The fundamental technique used in all of the examples is that the groups considered admit a natural projection onto $\mathbb{Z}$. Section 2 presents some notation that is repeatedly used to describe this projection. Section 3 gives a general connection between the escape rate of a random walk on G and that of a random walk on $G \wr \mathbb{Z}$. If the random walk on G has a rate of escape on the order of n^{α}, then the rate of escape on $G \wr \mathbb{Z}$ is on the order of $n^{(1+\alpha)/2}$ times iterated logarithm correction factors. In particular, when G is infinite and the random walk on G escapes at sublinear rate, the random walk considered on $G \wr \mathbb{Z}$ has an intermediate escape rate that is faster than that of classical random walks on $\mathbb{Z}^d$. Finally, Section 4 is devoted to some specific examples of groups with exponential growth. On $\mathbb{Z}^d \wr \mathbb{Z}$, the fact that $\mathbb{Z}^d$ is Abelian makes it possible to consider some more general random walks, all of which have intermediate escape rate of $n^{3/4}$ times iterated logarithm correction factors. For the case of $G \wr \mathbb{Z}$, where G is finite, as well as solvable Baumslag–Solitar groups, the outer radius is shown to be of order $\sqrt{n \log\log n}$ and the inner radius is shown to be of order $\sqrt{n/\log\log n}$. These examples are groups with exponential growth yet classical behavior of the outer radius. The inner radius on these groups differs from that on $\mathbb{Z}^d$, so that behavior is new. The final example that is considered is a discrete version of the Sol geometry, which gives another example of a group with exponential growth but the same outer radius as in the classical case of $\mathbb{Z}^d$.

2. Notation.

For a given group G and a finite generating set S, define a metric on G by letting $|g|$ be the minimal length of g in terms of the generators S. Changing the generating set S is a quasi-isometry, which means that it changes lengths by at most constant factors. Since interest here is only in the order of $|X_n|$ and constants are not being carefully calculated, the choice of the generating set S is suppressed.

Elements of the wreath product $G \wr \mathbb{Z}$ are pairs of the form (f, y), where $y \in \mathbb{Z}$ and $f \in \sum_{\mathbb{Z}} G$. Recall that elements of the direct sum $\sum_{\mathbb{Z}} G$ are functions from $\mathbb{Z}$ to G which are equal to the identity for all but finitely many values. The law is given by $(f, y)(g, z) = (f s(y) g, y + z)$, where $s(y)$ denotes right shift by y, that is, $s(y) g_i = g_{i-y}$. In this way, $\mathbb{Z}$ acts by coordinate shift on $\sum_{\mathbb{Z}} G$ and $G \wr \mathbb{Z} = \sum_{\mathbb{Z}} G \rtimes \mathbb{Z}$ is the resulting semidirect product.

For a random walk on $G \wr \mathbb{Z}$, the following notation is introduced: Let $X_n = (f_n, Y_n)$ denote the position of the walker, where $Y_n \in \mathbb{Z}$ and $f_n(i)$ denotes the coordinate functions in G. These groups and, in particular, the special case of $\mathbb{Z}_2 \wr \mathbb{Z}$, are often referred to as lamplighter groups, a name that was popularized by Kaimanovich and Vershik. The name comes from the following way to visualize the group $\mathbb{Z}_2 \wr \mathbb{Z}$: imagine an infinite row of lamps with a lamplighter walking up and down the row lighting them. The Y_n represents the position of the lamplighter, and the coordinate functions $f_n(i)$ that have values in $\mathbb{Z}_2$ denote whether the lamp at site i is on or off. For a more general group $G \wr \mathbb{Z}$, we imagine that each lamp can take on many different colors, each corresponding to an element in the group G.

Let $\theta^*_{i,n} = \#\{0 \le k \le n : Y_k = i\}$ denote the number of visits to the ith lamp by the lamplighter during the interval $[0, n]$. Because the walks that are considered here are slightly different at the endpoints than at other sites, it will be useful to consider the modified function $\theta_{i,n}$ given by $\theta_{i,n} = \theta^*_{i,n} - (1/2)\delta_{i,0} - (1/2)\delta_{i,Y_n}$. When dealing with a fixed time n, the time subscript will often be suppressed and θ_i will be written. Up to constants, the distance to the identity $|X_n|$ is the amount of time to turn off all of the lights ($\sum |f_n(i)|$) plus the necessary travel time for the lamplighter. Because of the classical LIL, the amount of time that the lamplighter needs to travel is at most on the order of $\sqrt{n \log\log n}$. As will be seen later, in most of the given examples $\sum |f_n(i)|$ grows faster and is the dominant term in the distance to the origin.

Many of the techniques used here require studying the range visited by Y_n, so let $M_n = \max\{i : \theta_{i,n} > 0\}$ and $m_n = \min\{i : \theta_{i,n} > 0\}$. Set $V_n = \#\{i : \theta_{i,n} > 0\}$ to be the number of sites visited by time n.

3. General bounds. To be able to say anything meaningful, some assumptions need to be made about random walks on the group G. In particular, assume that the rate of escape of the walk is reasonably well understood.

DEFINITION 1. *Let G be a finitely generated group with μ a symmetric probability measure on the group and let the random walk ξ_n generated by the pair (G, μ) be the Markov process given by $\xi_0 = e$ and $P(\xi_{n+1} = \xi_n g) = \mu(g)$. Then ξ_n has tight degree of escape α if there exist $\gamma, \delta > 0$ such that $P(|\xi_n| n^{-\alpha} > \gamma) \ge \delta$ and there exists a $\beta > 0$ such that $P(|\xi_n| n^{-\alpha} > x) \le c_1 \exp(-c_2 x^{\beta})$.*

Note that there are really quite a lot of such walks—for example, the bounds in [10] show that groups of polynomial growth have $\alpha = 1/2$ for any nontrivial, finitely supported symmetric generating set. Furthermore, any pair (G, μ) that generates a walk with linear rate of escape has $\alpha = 1$. All irreducible walks on nonamenable groups, for example, will thus have $\alpha = 1$. The case of $\alpha = 0$ corresponds to nontrivial, finite groups G: the bound on the tail implies that the random walk is positive recurrent and thus G must be finite, and it is easy to see that nontrivial, finite groups G have $\alpha = 0$. Examples of groups with exponential growth and with $\alpha = 1/2$ are given in Section 4.2. The main result is the following construction of groups with different values of α.

THEOREM 1. *Suppose (G, ν) generates a random walk with tight degree of escape $\alpha \in [0, 1]$ and that μ is a probability measure on $\mathbb{Z}$ satisfying $\sum x\mu(x) = 0$ and $\sum x^2\mu(x) < \infty$. Let $\tilde{\nu}$ be the probability measure on $G \wr \mathbb{Z}$ given by $\tilde{\nu}(g_0, 0) = \nu(g)$, where g_0 denotes the function that is g at 0 and the identity elsewhere. Let $\tilde{\mu}$ be the probability measure on $G \wr \mathbb{Z}$ given by $\tilde{\mu}(\mathbf{e}, x) = \mu(x)$, where $\mathbf{e}$ denotes the identity function. Then $(G \wr \mathbb{Z}, \tilde{\nu} * \tilde{\mu} * \tilde{\nu})$ generates a random walk with tight degree of escape $\alpha' = (\alpha + 1)/2$. Moreover, the random walk on $G \wr \mathbb{Z}$ satisfies*

$$0 < \limsup \frac{|X_n|}{n^{\alpha'}(\log\log n)^{1-\alpha'}} < \infty, \tag{3.1}$$

$$0 < \liminf \frac{|X_n|(\log\log n)^{1-\alpha'}}{n^{\alpha'}} < \infty. \tag{3.2}$$

REMARKS. In terms of the lamplighter model, the measure $\tilde{\nu} * \tilde{\mu} * \tilde{\nu}$ corresponds to the lamplighter adjusting the current lamp, moving and then adjusting the new lamp. The result of this is that the ith lamp is adjusted $2\theta_{i,n}$ times in the first n steps and, conditioned on the local times, the states of the lamps are independent.

For the case of a k-fold iteration of wreath products of the form $\mathbb{Z} \wr \mathbb{Z} \wr \cdots \wr \mathbb{Z}$, Theorem 1 says that for a suitable generating set (and up to $\log\log n$ factors), $|X_n|$ grows like n^α for $\alpha = 1 - 2^{-k-1}$. The drift on this example was independently considered by Ershler [7], who proved that $E|X_n|$ has this growth rate.

The bounds for the lim inf given in (3.2) demonstrate a new behavior that does not occur in the case of random walks on groups with polynomial growth. For random walks on groups of polynomial growth and $f(n)$ decreasing, $\liminf |X_n| f(n)$ is either zero or infinity (see [10]), and so there is no law of the iterated logarithm for the inner radius on those groups.

PROOF OF THEOREM 1. The case of $\alpha = 0$ is a special one for which a slightly stronger statement is proved in Theorem 3. For now, I thus restrict myself to the case of $\alpha > 0$.

Let S be a generating set for $G \wr \mathbb{Z}$ that has elements of the form $(\mathbf{e}, \pm 1)$ and $(g_0, 0)$, where g is in a generating set for G. Then $\sum |f_n(i)| \le |X_n| \le \sum |f_n(i)| + 2V_n$, where V_n is the number of sites visited by the lamplighter. Because $\limsup V_n/\sqrt{n \log\log n} < \infty$ a.s. and $\alpha > 0$ currently is assumed the contribution of V_n is of a lower order than the claimed bound of $\sum |f_n(i)|$. Thus it is only necessary to prove the bounds (3.1) and (3.2) for $\sum |f_n(i)|$.

First, the upper bounds for (3.1) and (3.2) are proved. Because it is not possible to escape faster than at a linear rate, the upper bounds are trivial when $\alpha = 1$.

Assuming that $\alpha \in (0, 1)$, let $Z_i = |f_n(i)|\theta_{i,n}^{-\alpha}$. By Hölder's inequality,

$$\sum_{i\in\mathbb{Z}} |f_n(i)| = \sum_{i:\theta_i>0} \theta_i^\alpha Z_i \tag{3.3}$$

$$\leq \left(\sum_i \theta_i\right)^\alpha \left(\sum_{i:\theta_i>0} Z_i^{1/(1-\alpha)}\right)^{1-\alpha} \tag{3.4}$$

$$= n^\alpha \left(\sum_{i:\theta_i>0} Z_i^{1/(1-\alpha)}\right)^{1-\alpha}. \tag{3.5}$$

To control this sum, the following large deviation estimate is needed:

LEMMA 1. *Let X_i be a sequence of nonnegative random variables that satisfy the uniform bound $P(X_i > x|X_j, j \neq i) \leq c_1 \exp(-c_2 x^\beta)$ for $c_1, c_2 > 0$ and $\beta \in (0, 1]$. Then for sufficiently large $\lambda = \lambda(c_1, c_2, \beta)$,*

$$P\left(\sum_{i=1}^n X_i > \lambda n\right) \leq c_3 \exp[-c_4(\lambda n)^\beta].$$

PROOF. The assumptions on the X_i imply that we can find an i.i.d. sequence Y_i such that $\sum_{i=1}^n X_i$ is stochastically dominated by $\sum_{i=1}^n Y_i$ and such that $P(Y_i > x) = c_1 \exp(-c_2 x^\beta) \wedge 1$. For $\beta = 1$ the result is then standard, and for $0 < \beta < 1$ it is a weaker version of statement (i) of Lemma 6.3 in [4]. □

Because $Z_i = |f_n(i)|\theta_i^{-\alpha}$ and $E[\varphi(Z_i)|\theta, Z_j, j \neq i] = E[\varphi(Z_i)|\theta]$ for any φ, the assumptions on ν show that for some $\beta > 0$,

$$\begin{aligned} P\big[Z_i^{1/(1-\alpha)} > x\big|\theta, Z_j, j \neq i\big] &= P\big[|f_n(i)|\theta_i^{-\alpha} > x^{1-\alpha}\big] \\ &\leq c_1 \exp\big[-c_2 x^{\beta(1-\alpha)}\big]. \end{aligned}$$

For large enough λ, Lemma 1 then shows that

$$P\left[\sum_{i:\theta_i>0} Z_i^{1/(1-\alpha)} > \lambda V_n \middle| V_n\right] \leq c_3 \exp\big[-c_4(\lambda V_n)^{\beta(1-\alpha)}\big]. \tag{3.6}$$

Because $V_n < n^{1/4}$ finitely often, this bound combined with the Borel–Cantelli lemma implies that $\sum Z_i^{1/(1-\alpha)} > \lambda V_n$ finitely often.

However, bounds for V_n are well known. In particular, there are constants c and c' such that $V_n < c\sqrt{n \log\log n}$ for all but finitely many n and $V_n < c'\sqrt{n/\log\log n}$ for infinitely many n (see, e.g., [16], Theorem 5.7). Combining everything yields $\sum |f_n(i)| \leq n^\alpha (c\lambda\sqrt{n \log\log n}\,)^{1-\alpha}$ for all but finitely many n, which is the upper bound in (3.1). For the upper bound in (3.2), $\sum |f_n(i)| \leq n^\alpha (c'\lambda\sqrt{n/\log\log n}\,)^{1-\alpha}$ infinitely often, which yields the claim.

The lower bounds for (3.1) and (3.2) again rely on an exponential Chebyshev argument. Since the lower bounds are no longer trivial for $\alpha = 1$, an argument that holds for all $\alpha \in (0, 1]$ is needed.

LEMMA 2. *Let* ξ_i, $i = 1, \dots, k$, *be a finite family of independent, nonnegative random variables with* $P(\xi_i \geq 1) \geq \delta$. *If* $\lambda_i \geq 0$ *is a sequence such that* $\max \lambda_i = 1$, *then*

$$P\left(\sum_{i=1}^{k} \lambda_i \xi_i < \frac{\delta}{2} \sum_{i=1}^{k} \lambda_i\right) \leq \exp\left(-\frac{\delta}{12} \sum_{i=1}^{k} \lambda_i\right).$$

PROOF. By Chebyshev's inequality,

$$\begin{aligned}
P\left(\sum_{i=1}^{k} \lambda_i \xi_i < \frac{\delta}{2} \sum_{i=1}^{k} \lambda_i\right) &\leq \exp\left(\frac{\delta}{2} \sum_{i=1}^{k} \lambda_i\right) E \exp\left(-\sum_{i=1}^{k} \lambda_i \xi_i\right) \\
&\leq \exp\left(\frac{\delta}{2} \sum_{i=1}^{k} \lambda_i\right) \prod_{i=1}^{k} [1 - \delta + \delta \exp(-\lambda_i)] \\
&= \exp\left(\frac{\delta}{2} \sum_{i=1}^{k} \lambda_i + \sum_{i=1}^{k} \log[1 - \delta + \delta \exp(-\lambda_i)]\right) \\
&\leq \exp \delta \left[\frac{1}{2} \sum_{i=1}^{k} \lambda_i + \sum_{i=1}^{k} [-1 + \exp(-\lambda_i)]\right] \\
&\leq \exp \delta \left[\frac{1}{2} \sum_{i=1}^{k} \lambda_i + \sum_{i=1}^{k} \frac{7}{12}(-\lambda_i)\right] \\
&= \exp\left[-\frac{\delta}{12} \sum_{i=1}^{k} \lambda_i\right],
\end{aligned}$$

where the last inequality comes from the fact that $\max_i \lambda_i = 1$. $\square$

The assumptions on ν imply that there exist γ and δ such that when $\theta_i > 0$, $P(|f_n(i)|\theta_i^{-\alpha} > \gamma) \geq \delta$. Letting $\lambda_i = \theta_i^\alpha / (\max \theta_i^\alpha)$ and $\xi_i = |f_n(i)|/(\gamma \theta_i^\alpha)$, Lemma 2 implies that for a fixed history of visits θ,

$$\begin{aligned}
P\left[\sum_i |f_n(i)| < \frac{\delta\gamma}{2} \sum_i \theta_i^\alpha \middle| \theta\right] &= P\left[\sum_{i : \theta_{i,n} > 0} \frac{|f_n(i)|}{\theta_i^\alpha \gamma} \frac{\theta_i^\alpha}{\max_i \theta_i^\alpha} < \frac{\delta}{2} \sum_i \frac{\theta_i^\alpha}{\max_i \theta_i^\alpha} \middle| \theta\right] \\
&\leq \exp\left(-\frac{\delta}{12 \max_i \theta_i^\alpha} \sum_i \theta_i^\alpha\right).
\end{aligned}$$

To control this quantity, note that because $\alpha \le 1$,

$$\sum_i \left(\frac{\theta_{i,n}}{\max_i \theta_{i,n}}\right)^{\alpha} \ge \sum_i \frac{\theta_{i,n}}{\max_i \theta_{i,n}} = \frac{n}{\max_i \theta_{i,n}},$$

and so

$$\sum_i \theta_{i,n}^{\alpha} \ge n\Bigl(\max_i \theta_{i,n}\Bigr)^{\alpha-1}. \tag{3.7}$$

Using this inequality, it can be seen that

$$P\left[\sum_i |f_n(i)| < \frac{\delta\gamma}{2}\sum_i \theta_i^{\alpha}\right] \le P\Bigl(\max_i \theta_i \ge n^{2/3}\Bigr) + \exp\left(\frac{-\delta n^{1/3}}{12}\right). \tag{3.8}$$

In [3], it was shown that $P(\max_i \theta_{i,n} > xn^{1/2})$ decays like $\exp(-x^2)$, so all of the expressions on the right decay like $\exp(-n^{1/3})$ or faster. By the Borel–Cantelli lemma, $\sum_i |f_n(i)| < (\delta\gamma/2)\sum_i \theta_i^{\alpha}$ finitely often. Using (3.7) a second time implies that $\sum |f_n(i)| < n(\delta\gamma/2)\max_i \theta_{i,n}^{\alpha-1}$ finitely often.

Kesten [12] showed that for appropriate c_1 and c_2, $\max_i \theta_i < \sqrt{c_1 n \log\log n}$ for all but finitely many n and $\max_i \theta_i < \sqrt{c_2 n/\log\log n}$ infinitely often. Combining these last two facts gives the lower bounds for (3.2) and (3.1).

The last piece of the theorem that is needed is that the walk $(G \wr \mathbb{Z}, \tilde{\nu} * \tilde{\mu} * \tilde{\nu})$ is itself a random walk with tight degree of escape α'. For the tail bound

$$P\bigl(|X_n| n^{-\alpha'} > x\bigr) \le c_1 \exp\bigl(-c_2 x^{\beta'}\bigr),$$

the same technique is used that was used in computing the upper bounds. The case of $\alpha = 1$ again needs to be treated separately, but if $\alpha = 1$, then $\alpha' = 1$. Because the walk X_n is finitely supported, $P(|X_n| > xn) = 0$ for sufficiently large x, so the bound is trivial. When $\alpha \in (0,1)$, then because $|X_n| \le \sum |f_n(i)| + 2V_n$, the fact that the tails of V_n have Gaussian decay gives

$$\begin{aligned} P\bigl[|X_n| n^{-\alpha'} > x\bigr] &\le P\left[\sum_i |f_n(i)| n^{-\alpha'} > x/2\right] + P\bigl[V_n n^{-\alpha'} > x/4\bigr] \\ &\le P\left[\sum_i |f_n(i)| n^{-\alpha'} > x/2\right] + c_1 \exp(-x^2 n^{\alpha}). \end{aligned}$$

Using (3.5) to estimate the first part of this expression gives

$$\begin{aligned} P\left[\sum_i |f_n(i)| n^{-\alpha'} > x/2\right] &\le P\left[\left(\sum_i Z_i^{1/(1-\alpha)}\right)^{1-\alpha} n^{\alpha} n^{-\alpha'} > x/2\right] \\ &= P\left[\sum_i Z_i^{1/(1-\alpha)} > (x/2)^{1/(1-\alpha)} n^{1/2}\right]. \end{aligned}$$

Breaking this last event up into two cases—one where $V_n < n^{1/2}(x/2)^{\varepsilon}$ and the other where $V_n > n^{1/2}(x/2)^{\varepsilon}$ for $\varepsilon = \beta/[2+(1-\alpha)\beta]$—and bounding crudely, this is at most

$$P\left[\sum_i Z_i^{1/(1-\alpha)} > (x/2)^{-\varepsilon+1/(1-\alpha)} V_n\right] + P\left[V_n > n^{1/2}(x/2)^{\varepsilon}\right].$$

By using (3.6) to estimate the first quantity and standard estimates for the number of visited sites for the second quality, both of these quantities are bounded by expressions of the form $c_1 \exp[-c_2 x^{\beta'}]$, where

$$\beta' = \frac{2\beta}{2+(1-\alpha)\beta}.$$

Because $\beta > 0$, this implies that $\beta' > 0$, as required.

For the other bound, note that (3.8) implies that

$$\begin{aligned} &P\left[\sum_i |f_n(i)| < \delta\gamma n^{\alpha'}\right] \\ &\quad \le P\left[\sum_i |f_n(i)| < \frac{\delta\gamma}{2}\sum_i \theta_i^{\alpha}\right] + P\left[\delta\gamma n^{\alpha'} > \frac{\delta\gamma}{2}\sum_i \theta_i^{\alpha}\right] \qquad (3.9)\\ &\quad \le c_1 \exp(-c_2 n^{1/3}) + P\Big(\max_i \theta_{i,n} > \sqrt{n}\Big). \end{aligned}$$

Csáki and Révész [3] showed that the distribution of $\max_i \theta_{i,n}^*/\sqrt{n}$ is comparable to $\max_x \zeta_x$, where ζ_x is the local time at x for Brownian motion at time 1. Therefore, $P(\max_i \theta_{i,n} > \sqrt{n})$ is bounded away from 1. Because (3.9) is arbitrarily close to $P(\max_i \theta_{i,n} > \sqrt{n})$, then for large n there exist δ', γ' such that $P(|X_n| n^{-\alpha'} > \gamma') > \delta'$. The desired bound thus holds for sufficiently large n, but extending it to all n is trivial because for any n, $P(|X_n| = n) > 0$. $\square$

4. Examples.

4.1. $\mathbb{Z}^d \wr \mathbb{Z}$. The primary drawback of Theorem 1 is that the proofs require a relatively special class of generators. For the case of $\mathbb{Z}^d \wr \mathbb{Z}$, however, the fact that $\mathbb{Z}^d$ is Abelian makes it possible to obtain results for more generators.

THEOREM 2. *Let π denote the natural projection from $\mathbb{Z}^d \wr \mathbb{Z}$ onto $\mathbb{Z}$ and let μ be a finitely supported probability measure on $\mathbb{Z}^d \wr \mathbb{Z}$. For a group element $g = (f, x)$, let f_i denote the ith coordinate function of f. If μ is centered in the sense that $\sum_g \pi(g)\mu(g) = 0$ and for all i, j, $\sum_{g \in \pi^{-1}(j)} f_i(g)\mu(g) = 0$, then*

$$\limsup \frac{|X_n|}{n^{3/4}(\log\log n)^{1/4}} < \infty, \qquad (4.10)$$

$$\liminf \frac{|X_n|(\log\log n)^{1/4}}{n^{3/4}} < \infty. \qquad (4.11)$$

If $\sum_g \pi(g)\mu(g) \neq 0$ *or* $\sum_{i,g} f_i(g)\mu(g) \neq 0$, *then* $\lim |X_n|/n > 0$.

These are the same upper bounds given in Theorem 1 for the convolution measures, but unfortunately the arguments used for the matching lower bounds do not extend to this situation.

PROOF OF THEOREM 2. The second part is easier to prove. If $\sum_g \pi(g) \times \mu(g) \neq 0$, then the projection onto $\mathbb{Z}$ does not have mean zero, so $\lim |\pi(X_n)|/n > 0$. However, there exists a constant $c > 0$ such that $|X_n| \geq c|\pi(X_n)|$, so $\lim |X_n|/n > 0$. For the other uncentered case, $f_i(X_j) - f_i(X_{j-1}) = f_{i-Y_{j-1}} \times (X_{j-1}^{-1}X_j)$, so

$$E\sum_i f_i(X_j) - f_i(X_{j-1}) = \sum_{i,g} f_i(g)\mu(g).$$

If $\sum_{i,g} f_i(g)\mu(g) \neq 0$, then rewriting $\sum_i f_i(X_n)$ as $\sum_{j=1}^n \sum_i f_i(X_j) - f_i(X_{j-1})$ and using the strong law of large numbers shows that $\sum_i f_i(X_n)/n \to \sum_{i,g} f_i(g)\mu(g) \neq 0$. Because $|X_n| \geq c|\sum_i f_i(X_n)|$ for an appropriate constant c, this limit proves the claim.

The idea behind the proof for the case when μ is centered is similar to that of Theorem 1 and so it is only sketched here. Let $\theta_{i,n}^j$ denote the number of times up to time n that $\pi(X_k) = i$ and $\pi(X_{k+1}) = i + j$. Let $\xi_{i,t}^j$ be a family of independent random walks with time t on $\sum_{\mathbb{Z}} \mathbb{Z}^d$ generated by the measure ν_j, where $\nu_j(f) = \mu(f, j)/\mu[\pi^{-1}(j)]$. By assumption, $\xi_{i,t}^j$ has zero mean in each of its coordinate functions. Somewhat as before,

$$|X_n| \leq c'\left(|\pi(X_n)| + \sum_{i,j}\left|\xi^j_{i,2\theta^j_{i,n}}\right|\right)$$

and the right-hand side is approximated by $\sum_{i,j}(\theta_{i,n}^j)^{1/2}$. From this point on, the analysis is the same as in the proof of the upper bounds in Theorem 1. □

4.2. *Lamplighter groups.* As remarked earlier, any finitely supported, symmetric random walk on a group of polynomial growth has tight degree of escape $\alpha = 1/2$. These are hardly the only examples. For the case of $\alpha = 0$, which as explained earlier is equivalent to G being a nontrivial, finite group, the rate of escape of any finitely supported random walk can be studied.

THEOREM 3. *Let G be a nontrivial, finite group and let η be a finitely supported probability measure that has support that generates $G \wr \mathbb{Z}$. Let π denote the natural projection onto $\mathbb{Z}$. If the measure $\pi(\eta)$ has mean zero, then $(G \wr \mathbb{Z}, \eta)$ has tight degree of escape $\alpha = 1/2$, and the random walk X_n generated by η*

satisfies the bounds

$$0 < \liminf \frac{|X_n|\sqrt{\log\log n}}{\sqrt{n}} < \infty, \tag{4.12}$$

$$0 < \limsup \frac{|X_n|}{\sqrt{n \log\log n}} < \infty. \tag{4.13}$$

If $\pi(\eta)$ *does not have mean zero*, *then the rate of escape is linear.*

As a side note, walks that are not translation invariant on these groups can have fairly surprising behavior. Lyons, Pemantle and Peres [13] found that an inward-biased random walk on $\mathbb{Z}_2 \wr \mathbb{Z}$ can have linear rate of escape, and for a similar example on $\mathbb{Z}_k \wr \mathbb{Z}$, Revelle [15] evaluated $\lim |X_n|/n$. Because both of these examples are biased toward a specific point, neither one is translation invariant and so are not covered by Theorem 3.

PROOF OF THEOREM 3. Let $Y_n = \pi(X_n)$ be the walk performed by the lamplighter on $\mathbb{Z}$, let $M_n = \max\{Y_k : k \le n\}$ and let $m_n = \min\{Y_k : k \le n\}$. Let $c_1 = \max\{|(f, y)| : f_z = 0, z \ne 0, |y| \le 1\}$, that is to say, c_1 denotes the maximum length of words that correspond to the lamplighter adjusting the current lamp and then possibly moving to an adjacent site. Let $c_2 = \max\{|k| : \pi(\eta)(k) > 0\}$ be the maximum step size that the lamplighter can take. Then, at any given time, the distance to the origin in the random walk can be estimated by

$$|Y_n|/c_2 \le |X_n| \le c_1(2M_n - 2m_n + 2), \tag{4.14}$$

where since $Y_n \in \mathbb{Z}$, $|Y_n|$ denotes the standard absolute value, but $|X_n|$ is the length of X_n in the group. If $\pi(\eta)$ does not have mean zero, then $|Y_n|$ grows linearly, so the claim follows from the lower bound of (4.14).

When Y_n is a finitely supported, mean zero random walk on $\mathbb{Z}$, $P(M_n > x) \le a_1 \exp(-a_2 x^2/n)$ for suitable constants a_1, a_2, and a similar bound holds for $P(m_n < -x)$. Therefore, b_1, b_2 can be found such that $P(|X_n| n^{-1/2} > x) \le b_1 \exp(-b_2 x^2)$, which is the upper bound needed to show that X_n has tight degree of escape $1/2$. The lower bound for tight degree of escape is simply the fact that Y_n is a nontrivial random walk on $\mathbb{Z}$, so there is a δ such that $P(|X_n| n^{-1/2} > \gamma) \ge P(|Y_n| > \gamma c_2 \sqrt{n}) \ge \delta > 0$.

To prove (4.13), use is made of the estimate from (4.14). The result follows from the fact that laws of the iterated logarithms hold for both $|Y_n|$ and $M_n - m_n$.

The upper bound of (4.12) follows from the upper bound in (4.14) and the fact that for any finitely supported mean zero random walk on $\mathbb{Z}$, there exists c_3 such that $M_n - m_n < c_3 \sqrt{n/\log\log n}$ infinitely often. The difficult part of Theorem 3 is the lower bound of (4.12). The idea is that there are some lamps near the end points of the lamplighter's path that are on. To prove this, first consider the case of a convolution measure, and then extend to arbitrary measures.

Suppose that η is of the form $\nu * \mu$, where μ is a finitely supported probability measure such that $\pi(\mu)$ has mean zero and ν is a probability measure supported on elements of the form $(g_0, 0)$. Suppose moreover that ν is not a point mass, so there exists an $\varepsilon > 0$ such that $\nu(g_0, 0) \geq \varepsilon$ for at least two distinct $g \in G$. Let $F_n^- = \min\{i : f_n(i) \neq e\}$ and $F_n^+ = \max\{i : f_n(i) \neq e\}$. In the lamplighter description, these are flags that mark the most extreme lamps that are on at time n. By the assumptions on ν, any visited lamp is on with probability at least ε, even conditioning on the states of all of the other lamps. Because the lamplighter can only take steps of size c_2, we thus have $P(M_n - F_n^+ > k) < \varepsilon^{k/c_2}$. Likewise $P(F_n^- - m_n > k) < \varepsilon^{k/c_2}$. Taking $k = n^{1/3}$ and applying the Borel–Cantelli lemma yields that $M_n - F_n^+ > n^{1/3}$ finitely often and likewise $F_n^- - m_n > n^{1/3}$ finitely often. To return to the identity, the lamplighter needs to visit both F_n^+ and F_n^-, so $|X_n| \geq (F_n^+ - F_n^-)/c_2$. Since $F_n^+ - F_n^-$ is comparable to $M_n - m_n$, and because there exists c_4 such that $M_n - m_n > c_4\sqrt{n/\log\log n}$ for all but finitely many n, this implies the lower bound of (4.12).

For an arbitrary measure η, define a new measure ν to represent changes that happen only at the current lamp and define another measure μ to describe other types of steps in the group, and then use a stopping time to compare η with $\nu * \mu$. More precisely, since the support of η generates $G \wr \mathbb{Z}$, there exists an $l > 0$ such that $\eta^{(*l)}(g_0, 0) > 0$ for some $g \neq e \in G$. Because $|X_n|$ and $|X_{l\lfloor n/l \rfloor}|$ differ by at most a constant, assume without loss of generality that $\eta(g_0, 0) > 0$ for some $g \neq e \in G$.

Let $p = \sum_{g \in G} \eta(g_0, 0)$. Define $\tilde{\eta}$ to be the probability measure supported on elements of the form $(g_0, 0)$ given by $\tilde{\eta}(g_0, 0) = \eta(g_0, 0)/p$. Let ν be the probability measure given by

$$\nu = \sum_{i=0}^{\infty} p^i (1-p) \tilde{\eta}^{(*i)},$$

where $\eta^{(*0)}$ denotes a point mass at the identity, so $\nu(e, 0) \geq 1 - p$. For (f, x) not of the form $(g_0, 0)$, let μ be the measure given by $\mu(f, x) = \eta(f, x)/(1 - p)$. Let $T_n = \inf\{j > T_{n-1} : \mu(X_{j-1}^{-1} X_j) > 0\}$ denote the nth time that the lamplighter does something other than adjust the current lamp. Let Z_n be the random walk generated by $\nu * \mu$. Because $\nu(e, 0) \geq 1 - p$ and $\nu(g_0, 0) > 0$ for some $g \neq e$ by assumption, the lower bound of (4.12) is obtained for Z_n by the convolution case. However, $Z_n \overset{\mathcal{D}}{=} X_{T_n}$, $|X_j|$ and $|X_{T_n}|$ differ by at most c_1 for $j \in [T_n, T_{n+1})$, and $\lim T_n/n = 1/(1 - p)$ by the strong law of large numbers, which proves that the lower bound of (4.12) holds for X_n as well. □

4.3. *Baumslag–Solitar groups.* Baumslag–Solitar groups are another source of examples of groups of exponential growth for which random walks also have an inner radius of order $(n/\log\log n)^{1/2}$ and outer radius of order $(n \log\log n)^{1/2}$.

The main significance of this example is that, unlike lamplighter groups, the Baumslag–Solitar groups are finitely presented. For an integer $q \geq 2$, the elements of the corresponding Baumslag–Solitar group are affine mappings of the form $x \mapsto q^a x + b$, where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}[1/q]$, that is to say, b is a q-ary rational. Multiplication consists of composition of maps and random walk consists of right multiplication.

THEOREM 4. *Let ν be a finitely supported, symmetric probability measure supported on elements of the form $x \mapsto x+z$, where $z \in \mathbb{Z}$, and $\nu^{(*2)}(x \mapsto x+b) > 0$ for $b = 0, 1, \dots, q-1$. Let μ be a finitely supported, symmetric probability measure on elements of the form $x \mapsto q^a x$. Then for the random walk X_n generated by $\nu * \mu * \nu$,*

$$0 < \limsup \frac{|X_n|}{\sqrt{n \log\log n}} < \infty, \tag{4.15}$$

$$0 < \liminf \frac{|X_n|\sqrt{\log\log n}}{\sqrt{n}} < \infty. \tag{4.16}$$

PROOF. The analysis of the rate of escape on this group is somewhat similar to that of a wreath product with $\mathbb{Z}$, partly because there is a natural projection onto $\mathbb{Z}$. More precisely, the walk at any time can be viewed as being at the map $x \mapsto q^{Y_n} x + b_n$, where $Y_n \in \mathbb{Z}$ and $b_n = \pm \sum \varepsilon_i q^i$ is a q-ary rational (i.e., the ε_i range over $0, 1, \dots, q-1$). For the generating measure $\nu * \mu * \nu$ under consideration, it turns out that the transitions are given by

$$P\big[Y_{n+1} = Y_n + y, b_{n+1} = b_n + q^{Y_n}(q^y x + z)\big] = \nu(x)\mu(y)\nu(z). \tag{4.17}$$

By breaking down the changes in b_n into different components of the form aq^i, b_n can be expressed by

$$b_n = \sum_i Z^i_{2\theta_{i,n}} q^i, \tag{4.18}$$

where Z^i_t are a family of independent random walks on $\mathbb{Z}$ generated by ν and $\theta_{i,n}$ are the local times of Y_n with the usual modification at 0 and Y_n. More precisely, let $\theta_{i,n} = \#\{0 \leq k \leq n : Y_k = i\} - (1/2)(\delta_{i,0} + \delta_{i,Y_n})$. To compute the ε_i terms in the case when $b_n \geq 0$, the relationship

$$\sum Z^i_{2\theta_{i,n}} q^i = \sum \varepsilon_i q^i$$

gives the recursive solution

$$q^i \varepsilon_i \equiv \left[q^i Z^i_{2\theta_{i,n}} + \sum_{k<i} (Z^k_{2\theta_{k,n}} - \varepsilon_k) q^k \right] \pmod{q^{i+1}}. \tag{4.19}$$

A similar expression holds when b_n is negative.

Looking at (4.17) might help to visualize this group as being similar to the lamplighter group. Again Y_n can be thought of as a lamplighter and b_n can be viewed as a configuration of lamps, each with brightness ε_i. The q^{Y_n} term in the increments of b_n corresponds to the lamplighter changing lamps only near his or her current position. The fact that b_n is a q-ary rational means that the lamps now interact: adding q to the current lamp maintains its state, but adds 1 to the next lamp to the right. The key idea for computing the inner radius is that, after the lamplighter Y_n has visited a given site i, the configuration ε_i is somewhat random. Now ε_i is nonzero over most of the range of sites visited by the lamplighter, which is at least $cn^{1/2}\log\log n^{-1/2}$ for all but finitely many n.

More formally, let π denote the natural projection from $\mathbb{Z}$ to $\mathbb{Z}/q\mathbb{Z}$ and consider the random walk on $\mathbb{Z}/q\mathbb{Z}$, the distribution of which, at time n, is given by $\pi[\nu^{(*2n)}]$. By hypothesis, $\pi[\nu^{(*2n)}](b) > 0$ for all $n > 0$ and $b \in \mathbb{Z}/q\mathbb{Z}$. Since the walk converges to uniform, a p can be found such that $\pi[\nu^{(*2n)}](b) \geq p > 0$ for all $n > 0$ and $b \in \mathbb{Z}/q\mathbb{Z}$. The significance of this bound is that it implies, along with the expansion (4.19), that if $\theta_{i,n} > 0$ and $i \neq 0, Y_n$, we have $E(\delta_{\varepsilon_i,1}|\theta, \varepsilon_k, k < i) \geq p$. Using this, it is easy to show that for any interval $[a, b]$, we have $P\{\varepsilon_i = 0, i \in [a, b]\} < (1 - p)^r$ where $r = \#\{i \in [a, b] : \theta_i > 0, i \neq 0, Y_n\}$.

As before, let $m_n = \min\{Y_k : k < n\}$ and $M_n = \max\{Y_k : k < n\}$. Since μ is finitely supported, there exists a largest step size $s = \max\{x : \mu(x) > 0\}$ for the lamplighter Y_n. From above, $P\{\varepsilon_i = 0, i \in [m_n, m_n(1-\delta) + \delta M_n]\} < (1 - p)^r$, where $r = \delta(M_n - m_n)s^{-1} - 2$. Likewise, $P\{\varepsilon_i = 0, i \in [\delta m_n + (1 - \delta)M_n, M_n]\} < (1 - p)^r$. Let $\mathcal{A}$ denote the event

$$\mathcal{A} = \{\text{there is no pair}(i, j) : \varepsilon_i = \varepsilon_j = 1, j - i > (1 - 2\delta)(M_n - m_n)\}.$$

Then the upper bound

$$\begin{aligned} P(\mathcal{A}) &= P(\mathcal{A} \cap \{M_n - m_n > n^{1/3}\}) + P(\mathcal{A} \cap \{M_n - m_n < n^{1/3}\}) \\ &\leq 2(1 - p)^{\delta n^{1/3} s^{-1} - 2} + P(M_n - m_n < n^{1/3}) \\ &\leq c_1 \exp(-c_2 n^{1/3}) \end{aligned}$$

holds for suitable c_1 and c_2. By the Borel–Cantelli lemma, $\mathcal{A}$ occurs finitely often with probability 1, which shows that the distance to the identity is at least comparable to $M_n - m_n$. Because $M_n - m_n$ has an inner radius of $n^{1/2}/(\log\log n)^{1/2}$, this proves the lower bound of (4.16).

For the upper bound, (4.17) and the fact that ν is finitely supported imply that $b_n \leq cnq^{M_n}$. This implies that $\varepsilon_i = 0$ outside of the interval $[m_n, M_n + \log_q cn]$. To return to the identity just requires Y_n visiting the set $\{i : \varepsilon_i \neq 0\}$ and then returning to 0, so $|X_n| \leq c_3(M_n - m_n) + c_4 \log n$. Since $\log n$ is of a lower order, this yields the upper bound of (4.16).

For the lower bound of (4.15), note that because $|Y_n| > c\sqrt{n \log\log n}$ infinitely often, the outer radius is known to be at least of that order. For the upper bounds of (4.15), as before we have $|X_n| \leq c_3(M_n - m_n) + c_4 \log n$ and the result follows from the fact that $(n \log\log n)^{1/2}$ is an outer radius for $M_n - m_n$. $\square$

4.4. *Sol.* Another example of a walk with tight rate of escape is the following discrete version of the Sol geometry. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and consider the group $G = \mathbb{Z}^2 \rtimes \mathbb{Z}$, where the action of $\mathbb{Z}$ on $\mathbb{Z}^2$ is given by A, that is to say, the group law is $[(x, y), n][(z, w), m] = [(x, y) + A^n(z, w), n + m]$. This group is the example of this section that can be embedded discretely into a Lie group. It is also of interest to geometric group theorists because it is isomorphic to a cocompact subgroup of the Sol geometry, which is one of the eight simply connected three-dimensional geometries. The Sol geometry as a group can be viewed as $\mathbb{R}^2 \rtimes \mathbb{R}$, with the group law given by $[(x, y), t][(x', y'), t'] = [(x, y) + A_t(x', y'), t + t']$, where $A_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$. The name Sol comes from the fact that it is in many ways the simplest solvable group with exponential growth.

Again consider the random walk generated by a convolution measure of the form $\nu * \mu * \nu$. This time, take $\mu[(0, 0), \pm 1] = 1/2$ and $\nu[(\pm 1, 0), 0] = 1/3 = \nu(e)$. At time n, let the random walk X_n be denoted by $X_n = [(W_n, V_n), Y_n]$. As usual, let π be the projection onto $\mathbb{Z}$ given by $\pi[(x, y), n] = n$. Then $\pi(X_n) = Y_n$ is performing a simple random walk on $\mathbb{Z}$. Let $\theta^*_{i,n} = \#\{0 \le k \le n : Y_k = i\}$ be the local time of Y_n and let $\theta_{i,n} = \theta^*_{i,n} - 1/2(\delta_{0,i} + \delta_{i,Y_n})$ be the local time modified at 0 and Y_n as before.

THEOREM 5. *The random walk on Sol generated by the probability measure* $\nu * \mu * \nu$ *described above has tight degree of escape* $\alpha = 1/2$. *Moreover, with probability* 1, $\limsup |X_n|/\sqrt{n \log\log n} \in (0, \infty)$ *and* $\liminf |X_n|\sqrt{\log\log n}/\sqrt{n} < \infty$.

Before proving this theorem, we need a way to estimate the lengths of words. Let λ_1 and λ_2 denote the eigenvalues of A and pick an orthonormal eigenbasis v_1, v_2. Let α_1, α_2 be such that $(1, 0) = \alpha_1 v_1 + \alpha_2 v_2$. By again decomposing the walk into the increments that occur when $Y_n = i$ and taking advantage of the fact that $A^i(1, 0) = \alpha_1 v_1 \lambda_1^i + \alpha_2 v_2 \lambda_2^i$, $[(W_n, V_n), Y_n]$ can be expressed in a form that is more useful for analyzing the distance to the origin. More precisely,

$$(W_n, V_n) = \sum_{i \in \mathbb{Z}} Z^i_{2\theta_{i,n}} (\alpha_1 v_1 \lambda_1^i + \alpha_2 v_2 \lambda_2^i),$$

where Z^i_n is a family of independent, simple random walks with holding probability $1/3$ and time index n. That is to say, $P(Z^i_{n+1} = Z^i_n \pm 1) = P(Z^i_{n+1} = Z^i_n) = 1/3$. Moreover, all the Z^i_t's are required to be independent of Y_n. This decomposition just comes from letting $Z^i_{2\theta_{i,n}}$ count the number of times $A^i(1, 0)$ is added to the vector (W_n, V_n).

The idea for estimating the distance to the origin is that the length of this vector in terms of the group generators is, as in the previous examples, comparable to the number of i's such that $\theta_{i,n} > 0$. To do this, the relationships given by the characteristic polynomial $\lambda^2 - 3\lambda + 1$ are used to reduce the coefficients in the above summation. This essentially gives an upper bound for a solution to the word problem on this group.

LEMMA 3. *Let $a = \{a_m, \dots, a_M\}$ be a finite sequence of integers and let $f(a)$ denote the Laurent polynomial given by $f(a)(x) = \sum a_i x^i$. Let $K = \max\{|a_i|\}$. Then there exists a sequence of integers $b = \{b_{m-1}, \dots, b_{M+1}\}$ such that $0 \le b_i \le 2$ for $i \in [m, M]$ and $|b_{m-1}|, |b_{M+1}| \le K + 1$ that satisfies $f(b)(\lambda) = f(a)(\lambda)$ for both eigenvalues of A, where $f(b)(x) = \sum b_i x^i$ is the Laurent polynomial with coefficients b_i.*

Apply Lemma 3 with the sequence $a_i = Z^i_{2\theta_{i,n}}$. The sequence b_i corresponds to a relatively short way to write the group element X_n, and the relationship $f(a)(\lambda) = f(b)(\lambda)$ implies that the two corresponding decompositions of X_n are indeed the same element.

Note that this is in many ways similar to the analysis for the Baumslag–Solitar group. In that group, each element is expressed as a pair of a q-ary rational and an integer. Here group elements are expressed as a pair of a ternary (except at the endpoints) string and an integer. This can be thought of as a pair that comprises a "rational" number in "base λ" and an integer. In the Baumslag–Solitar group, there was an error factor at M_n that corresponded to bits carrying to the right: here an additional error factor is needed at m_n to compensate for the fact that one of the eigenvalues is less than 1, which causes some terms to "carry" to the left.

PROOF OF LEMMA 3. This lemma is proved this by constructing a sequence b that satisfies the required conditions. Let b^0 be the sequence such that

$$f(b^0)(x) = f(a)(x) - K(x - 3 + x^{-1})(x^M + \cdots + x^m).$$

Because $(-x^2 + 3x - 1)(x^{l-2} + \cdots + x + 1) = -x^l + 2x^{l-1} + x^{l-2} + \cdots + x^3 + x^2 + 2x - 1$, the effect of this is that b^0 is a sequence with $b^0_i \ge 0$ for $i \in [m, M]$ and b^0_i is 0 for $i > M + 1$ and $i < m - 1$. For a sequence $b^k = \{b^k_i\}_{i \in \mathbb{Z}}$, let b^{k+1} be the sequence obtained in the following way: Let $j = \min\{l \in [m, M] : b^k_l \ge 3\}$ and let b^{k+1}_i be the sequence such that $f(b^{k+1})(x) = f(b^k)(x) + x^{j-1}(x^2 - 3x + 1)$. If there is no $l \in [m, M]$ such that $b^k_l \ge 3$, then the process terminates. Moreover, because at any given step $\sum |b^k_i|$ decreases, the process eventually terminates. Let b be the final sequence. Since $f(b^{k+1})(\lambda) = f(b^k)(\lambda)$ at each step, $f(b)(\lambda) = f(a)(\lambda)$. The fact that b_i is zero outside of $[m - 1, M + 1]$ and bounded by $0 \le b_i \le 2$ on $[m, M]$ is clear from the construction. What remains to be shown is that $|b_{M+1}|$ and $|b_{m-1}| \le K + 1$.

Consider the polynomial $g(x) = [f(b)(x) - f(a)(x)]x^{1-m}$. By construction, g is divisible by $x^2 - 3x + 1$. Denote g by $\alpha_l x^l + \cdots + \alpha_1 x + \alpha_0$, where $l = M - m + 2$. Because $0 \le b_i \le 2$ for $i \in [m, M]$, then $|\alpha_j| \le K + 2$ for all $j \in (0, l)$. To show that $|\alpha_l|$ and $|\alpha_0| < K + 2$, it suffices to show that there is no polynomial g such that $x^2 - 3x + 1$ divides g, with $|\alpha_j| \le K + 2$ for $j \in (0, l)$ but with $|\alpha_l|$ or $|\alpha_0|$ larger than $K + 1$.

To do this, assume there is such a g and argue by contradiction. Because the roots of $x^2 - 3x + 1$ are reciprocals, replacing x by $1/x$ and multiplying through by x^l allows the assumption that $|\alpha_l| \geq |\alpha_0|$. Moreover, replacing g by $-g$ does not change divisibility by $x^2 - 3x + 1$, so $\alpha_l > 0$ may be assumed.

Suppose that $\alpha_l \geq K + 2$. Then let

$$h = \beta_l x^l + \cdots + \beta_0$$

be a new polynomial obtained by

$$h = (K+2)(-x^2 + 3x - 1)(x^{l-2} + \cdots + x + 1) + g.$$

This means that $\beta_j = \alpha_j + (K+2)$ for $j \in (1, l-1)$ and $\beta_j = \alpha_j + 2(K+2)$ for $j = 1, l-1$. As a result, $\beta_j \geq 0$ for $j > 0$ and $\beta_1, \beta_{l-1} \geq K + 2$. Moreover, $\beta_0 = \alpha_0 - (K+2)$ and $\beta_l = \alpha_l - (K+2)$, so because $\alpha_l \geq |\alpha_0|$, then $-\beta_0 = K + 2 - \alpha_0 \leq K + 2 + \alpha_l = 2(K+2) + \beta_l$. Because $\beta_1, \beta_{l-1} \geq K + 2$ and $\beta_j \geq 0$ for $j > 0$, this implies that $-\beta_0 \leq \sum_{j>0} \beta_j$. However, one of the roots, call it λ_1, of $x^2 - 3x + 1$ is greater than 1. Because $\beta_j \geq 0$ for $j > 0$, $h(\lambda_1) \geq \sum_j \beta_j > 0$, which means that $x^2 - 3x + 1$ cannot divide h, which is a contradiction. Hence $\alpha_l \leq K + 1$ as claimed. $\square$

This lemma yields an upper bound for the word problem on Sol that is sufficient for the purposes here. Each element b_i reveals how many times $A^i(1, 0)$ needs to be subtracted to return to the identity. Thus

$$|Y_n| \leq |X_n| \leq 2(M_n - m_n + 2) + 2(K_n + 2), \tag{4.20}$$

where $K_n = \max |Z^i_{2\theta_{i,n}}|$ is obtained. This representation thus easily yields the upper bounds on the probability of being far from the origin, from which it is almost immediate that the group has tight degree of escape $\alpha = 1/2$.

PROOF OF THEOREM 5. To control the upper bound in (4.20), use the decomposition

$$\{K_n > n^{1/3}\} \subset \{\max \theta_i > \sqrt{n \log n}\} \cup \{\max \theta_i \leq \sqrt{n \log n}, K_n > n^{1/3}\}.$$

Since $\max_i \theta_{i,n} > \sqrt{n \log n}$ finitely often and the second event has probability bounded by $n \exp[-a_1 n^{2/3} + a_2 (n \log n)^{1/2}]$, the event $\{K_n > n^{1/3}\}$ occurs finitely often. The K_n term is thus of lower order and can be neglected. The theorem then follows from classical results about $|Y_n|$ and $M_n - m_n$ (see, e.g., [16]). $\square$

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REFERENCES

[1] ALEXOPOULOS, G. (1992). A lower estimate for central probabilities on polycyclic groups. *Canad. J. Math.* **44** 897–910.

[2] AVEZ, A. (1974). Théorème de Choquet–Deny pour les groupes à croissance non exponentielle. *C. R. Acad. Sci. Paris Ser. I Math.* **279A** 25–28.

[3] CSÁKI, E. and RÉVÉSZ, P. (1983). Strong invariance for local times. *Probab. Theory Related Fields* **62** 263–278.

[4] DEMBO, A., GANTERT, N., PERES, Y. and ZEITOUNI, O. (2002). Large deviations for random walks on Galton–Watson trees: Averaging and uncertainty. *Probab. Theory Related Fields* **122** 241–288.

[5] DERRIENNIC, Y. (1980). Quelques applications du théorème ergodique sousadditif. *Astérisque* **74** 183–201.

[6] DVORETZKY, A. and ERDŐS, P. (1951). Some problems on random walks in space. *Proc. Second Berkeley Symp. Math. Statist. Probab.* 353–367. Univ. California Press.

[7] ERSHLER (DYUBINA), A. (2001). On the asymptotics of the rate of departure to infinity. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov* (*POMI*) **283** 251–257 (in Russian).

[8] ERSHLER (DYUBINA), A. (2001). Asymptotics of drift and entropy for a random walk on groups. *Uspekhi Mat. Nauk* **56** 179–180. [Translation (2001) *Russian Math. Surveys* **56** 580–581.]

[9] GRIGOR'YAN, A. (1999). Escape rate of Brownian motion on Riemannian manifolds. *Appl. Anal.* **71** 63–89.

[10] HEBISCH, W. and SALOFF-COSTE, L. (1993). Gaussian estimates for Markov chains and random walks on groups. *Ann. Probab.* **21** 673–709.

[11] KAIMANOVICH, V. and VERSHIK, A. (1983). Random walks on discrete groups: Boundary and entropy. *Ann. Probab.* **11** 457–490.

[12] KESTEN, H. (1965). An iterated logarithm law for the local time. *Duke Math. J.* **32** 447–456.

[13] LYONS, R., PEMANTLE, R. and PERES, Y. (1996). Random walks on the lamplighter group. *Ann. Probab.* **24** 1993–2006.

[14] PITTET, C. and SALOFF-COSTE, L. (2002). On random walk on wreath products. *Ann. Probab.* **30** 948–977.

[15] REVELLE, D. (2001). Biased random walk on lamplighter groups and graphs. *J. Theoret. Probab.* **14** 379–391.

[16] RÉVÉSZ, P. (1990). *Random Walk in Random and Non-random Environments*. World Scientific, Singapore.

[17] SCHOOLFIELD, C. H. (2003). Random walks on wreath products of groups. *J. Theoret. Probab.* To appear.

[18] VAROPOULOS, N. (1983). Random walks on soluble groups. *Bull. Sci. Math.* **107** 337–344.

[19] VAROPOULOS, N. (1983). A potential theoretic property of soluble groups. *Bull. Sci. Math.* **108** 263–273.

[20] VAROPOULOS, N. (1985). Long range estimates for Markov chains. *Bull. Sci. Math.* **109** 225–252.

[21] VERSHIK, A. (1973). Amenability and approximation of infinite groups. [Translation (1982) *Selecta Math. Soviet.* **2** 311–330.]

[22] VERSHIK, A. (2000). Dynamic theory of growth in groups: Entropy, boundaries, examples. *Russian Math. Surveys* **55** 667–733.

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