A NOTE ON BOUNDS FOR THE ODDS THEOREM OF OPTIMAL STOPPING

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The odds theorem gives a unified answer to a class of stopping problems on sequences of independent indicator functions. The success probability of the optimal rule is known to be larger than Re^{-R}, where R defined in the theorem satisfies $R \geq 1$ in the more interesting case. The following findings strengthen this result by showing that $1/e$ is then a lower bound. Knowing that this is the best possible uniform lower bound motivates this addendum.

Let $I_1, I_2, \ldots, I_n$ be independent indicator functions on a probability space $(\Omega, \mathcal{F}, P)$. Consider the problem of stopping on the last success up to time n, that is, on the last indicator $I_k = 1$ with $k \leq n$. (If there is no such k or if we stop too early, then we lose by definition.)

The optimal rule is given by the odds theorem; see Bruss (2000), Theorem 1. Let $p_k = \mathrm{E}(I_k)$, $q_k = 1 - p_k$ and $r_k = p_k/(1 - p_k)$. The r_k's are called *odds*. It is optimal to add up the odds backwards, $r_n + r_{n-1} + \cdots$, until this sum becomes more than or equal to 1, at index s say (we put $s = 1$ if all odds add up to less than 1), and to stop at the first index $k \geq s$ with $I_k = 1$. The optimal success probability is obtained at the same time [see Bruss (2000), Section 2.1] by

$$(1) \qquad V = V(p_n, p_{n-1}, \ldots, p_1) = \prod_{j=s}^{n} q_j \sum_{k=s}^{n} r_k.$$

It is the elegance of simplicity which makes the odds theorem attractive. It can be readily applied to natural stopping problems such as, for example, the secretary problem [Bruss (2000)], the group-interview problem [Hsiau and Yang (2000)] and the "last-peak" problem [Tamaki (2001)], but also to many other simple problems of games, betting or investment. Hence easy bounds for V in (1) are of interest.

Theorem 2 of Bruss (2000) shows that $V > Re^{-R}$, where $R = r_n + r_{n-1} + \cdots + r_s$. This makes $1/e$ what we called before a "typical" lower bound. However, as we shall see now, this can be refined to yield the best possible uniform lower bound.

THEOREM.

$$If \sum_{k=1}^{n} r_k \geq 1 \qquad then \ V > \frac{1}{e}.$$

PROOF.   Since we prove a statement about the value of the optimal strategy in-
dependently of its form, we can renumerate the p_k's, and hence the q_k's and r_k's, in
their natural order. To keep the same notation, we simply rename $p_1 := p_n$, $p_2 :=
p_{n-1}, \ldots, p_n := p_1$ and, correspondingly, $r_1 := r_n, r_2 := r_{n-1}, \ldots, r_n := r_1$. Let
now, in this new notation, $R_k = r_1 + r_2 + \cdots + r_k$ and let $t = \inf\{k : R_k \geq 1\}$. We
then must show that

$$(2) \qquad V := R_t \prod_{k=1}^{t} (1 - p_k) > \frac{1}{e}.$$

By definition of t, $t = 1$ implies $r_1 \geq 1$, that is, $p_1 \geq 1/2$, and hence $V \geq 1/2 >
1/e$. Therefore, the statement is true for $t = 1$. Let thus $t \geq 2$. Use $1/(1 - p_j) =
1 + r_j$ and rewrite V in the form

$$(3) \qquad V = R_t \prod_{j=1}^{t-1} \left(\frac{1}{(1 - p_j)} \right)^{-1} \left(\frac{1}{(1 - p_t)} \right)^{-1} = \frac{R_t}{(1 + r_t) \prod_{j=1}^{t-1} (1 + r_j)}.$$

Maximizing $\prod_{j=1}^{t-1}(1 + u_j)$ with respect to $u_1, u_2, \ldots, u_{t-1}$ subject to the
constraints $u_j \geq 0$ and $u_1 + u_2 + \cdots + u_{t-1} = R_{t-1}$ shows that this maximum
is obtained by $u_1 = u_2 = \cdots = u_{t-1} = R_{t-1}/(t - 1)$. Recall that $t \geq 2$. Therefore,
from (3),

$$(4) \qquad V \geq \frac{R_t}{(1 + r_t)(1 + (R_{t-1}/(t - 1)))^{t-1}} \geq \frac{R_{t-1} + r_t}{(1 + r_t)e^{R_{t-1}}},$$

where we used $(1 + a/k)^k \uparrow e^a$ as $k \to \infty$ and $R_t = R_{t-1} + r_t$ in the second
inequality. Now let

$$(5) \qquad g(x, y) = \frac{x + y}{(1 + y)e^x} \qquad \text{for } 0 \leq x < 1, \ 1 - x \leq y.$$

On the specified domain of g we have $\partial g/\partial y \geq 0$ so that g increases in y for each
$0 \leq x < 1$. Therefore, with (5),

$$(6) \qquad g(x, y) \geq g(x, 1 - x) = \frac{1}{(2 - x)e^x}.$$

Since $0 \leq R_{t-1} < 1 \leq R_t$ by definition of t we have that R_{t-1} and r_t satisfy
the domain specification of g for x and y, respectively. Thus, using (4) and (6),
$V \geq 1/((2 - R_{t-1})e^{r_t})$. Therefore, to prove the uniform lower bound $1/e$ it suffices
to show that $g(x, y) \geq 1/e$ on the whole domain of g. This is now evident from (6)
because

$$\log\big(g(x, y)\big) \geq -x - \log(2 - x)$$
$$= -x - \log\big(1 + (1 - x)\big) > -x - (1 - x) = -1,$$

where $0 \leq x < 1$ implies the strict inequality. Hence the proof. $\square$

A better bound depending on t is, of course, obvious from the middle expression of (4), but it is the uniform bound which attracts our interest. As we know already from the asymptotic value of some special best-choice problems [see, e.g., Samuels (1992) for a review], this bound $1/e$ is the best possible. There is another special case: If at least one p_j is equal to one (implying that the sum of all odds is equal to infinity), then the lower bound $1/e$ follows also from the prophet inequality of Hill and Krengel (1992).

It is noteworthy that, as we have proved, this lower bound extends to the general setting of the odds theorem.

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