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GENERALIZED COVARIATIONS, LOCAL TIME AND STRATONOVICH ITÔ'S FORMULA FOR FRACTIONAL BROWNIAN MOTION WITH HURST INDEX $H \geq \frac{1}{4}$

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Given a locally bounded real function g, we examine the existence of a 4-covariation $[g(B^H), B^H, B^H, B^H]$, where B^H is a fractional Brownian motion with a Hurst index $H \geq \frac{1}{4}$. We provide two essential applications. First, we relate the 4-covariation to one expression involving the derivative of local time, in the case $H = \frac{1}{4}$, generalizing an identity of Bouleau–Yor type, well known for the classical Brownian motion. A second application is an Itô formula of Stratonovich type for $f(B^H)$. The main difficulty comes from the fact B^H has only a finite 4-variation.

1. Introduction. The present paper is devoted to generalized covariation processes and an Itô formula related to the fractional Brownian motion. The classical Itô formula and classical covariations constitute the core of stochastic calculus with respect to semimartingales. Fractional Brownian motion, which, in general, is not a semimartingale, has been studied intensively in stochastic analysis, and it is considered in many applications in hydrology, telecommunications, economics and finance. Finance is the most recent one in spite of the fact that, according to [34], the general assumption of no arbitrage opportunity is violated. Interesting remarks have recently been made in [7] and [40].

Recall that a mean-zero Gaussian process $X = B^H$ is a fractional Brownian motion with Hurst index $H \in]0, 1[$ if its covariance function is given by

$$K_H(s,t) = \frac{1}{2}\big(|s|^{2H} + |t|^{2H} - |s-t|^{2H}\big), \qquad (s,t) \in \mathbb{R}^2. \tag{1.1}$$

An easy consequence of this property is that

$$\mathrm{E}(B_t^H - B_s^H)^2 = (t-s)^{2H}. \tag{1.2}$$

Before concentrating on this self-similar Gaussian process, we would like to make some general observations.

Calculus with respect to integrands that are not semimartingales is now 20 years old. A large number of papers has been produced, and it is impossible to list them here; however, we are still not close to having a truly efficient approach for applications.

There are essentially three techniques for studying non-semimartingale integrators:

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- Pathwise and related techniques;
- Dirichlet forms;
- anticipating techniques (Malliavin calculus, Skorohod integration and so on).

Pathwise-type integrals are often defined using discretization as the limit of Riemann sums: an interesting survey on the subject is the book by Dudley and Norvaisa [13]. They emphasize the large historical literature in the deterministic case. The first contribution in the stochastic framework was provided by Föllmer [18] in 1981; through this significant and simply written contribution, the author wished to discuss integration with respect to a Dirichlet process X, that is to say, a local martingale plus a zero quadratic variation (or sometimes zero energy) process. This approach has been continued and performed by Bertoin [4].

Since 1991, Russo and Vallois [35] have developed a regularization procedure, whose philosophy is similar to discretization. They introduced forward (generalizing Itô), backward and symmetric (generalizing Stratonovich) stochastic integrals and a generalized quadratic variation. Their techniques are of a pathwise nature, but they are not truly pathwise. They make large use of ucp (uniform convergence in probability) related topology. More recently, several papers have followed that strategy; see, for instance, [16], [36]–[38] and [41]. One advantage of the regularization technique is that it allows us to generalize directly the classical Itô integral. Our forward integral of an adapted square-integrable process with respect to the classical Brownian motion is exactly Itô's integral; the integral via discretization is a sort of Riemann integral and it does allow us to define easily, for instance, a totally discontinuous function as the indicator of rational numbers on [0, 1]. However, the theorems contained in this paper can be translated without any difficulty into the language of discretization.

The terminology "Dirichlet processes" is inspired by the theory of Dirichlet forms. Tools from that theory have been developed to understand such processes as integrators; see, for instance, [27] and [28]. Dirichlet processes belong to the class of finite quadratic variation processes.

Even though Dirichlet processes generalize semimartingales, fractional Brownian motion is a finite quadratic variation process (even Dirichlet) if and only if the Hurst index is greater than or equal to $\frac{1}{2}$. When $H = \frac{1}{2}$, one obtains the classical standard Brownian motion. If $H > \frac{1}{2}$, it is even a zero quadratic variation process. Moreover, fractional Brownian motion is a semimartingale if and only if it is a classical Brownian motion.

The regularization, or discretization technique, for those and related processes has been performed by [15], [17], [22], [39], [43] and [44] in the case of zero quadratic variation, so $H > \frac{1}{2}$. Young's [42] integral can often be used under this circumstance. This integral coincides with the forward (but also with the backward or symmetric) integral since the covariation between the integrand and integrator is always 0.

As we will explain later, when the integrator has paths with finite p-variation for $p > 2$, there is no hope to make use of forward and backward integrals and the reference integral will be for us the symmetric integral which is a generalization of the Stratonovich integral.

The following step was done by Lyons and co-authors, see [25] and [26], who considered, through an absolutely pathwise approach based on the Lévy stochastic area, integrators having p-variation for any $p > 1$, provided one could construct a canonical geometric rough path associated with the process. This construction was done in [8] when the integrator is a fractional Brownian motion with Hurst parameter $H > \frac{1}{4}$; in that case, paths are almost surely of finite p-variation for $p > 4$.

Using Russo–Vallois regularization techniques, Errami and Russo [16] have considered a stochastic calculus and some ordinary SDEs with respect to integrators with finite p-variation when $p \leq 3$. This applies directly to the fractional Brownian motion case for $H \geq \frac{1}{3}$. A significant object introduced in [16] was the concept of n-covariation $[Y_1, \dots, Y_n]$ of n processes $Y_1, \dots, Y_n$.

Since fractional Brownian motion is a Gaussian process, it was natural to use the Skorohod–Malliavin approach, which, as we said, constitutes a powerful tool for the analysis of integrators that are not semimartingales. Using this approach, integration with respect to fractional Brownian motion, was attacked by Decreusefonds and Ustunel [11] and it was studied intensively, see [1], [2] and [6] even when the integrator is a more general Gaussian process. Malliavin–Skorohod techniques allow to treat integration with respect to processes, in several situations where the variation is larger than 2. In particular, [1] includes the case of a fractional Brownian motion B^H such that $H > \frac{1}{4}$. The key tool there is the Skorohod integral, which can be related to the symmetric-Stratonovich integral, up to a trace term of some Malliavin derivative of the integrand. In the case of fractional Brownian motion, [1] discussed an Itô formula for the Stratonovich integral when the Hurst index H is strictly greater than $\frac{1}{4}$.

Other significant and interesting references about stochastic calculus with fractional Brownian motion, especially for $H > \frac{1}{2}$, are [12], [14], [24], [29] and [30]. Some activity is also going on with stochastic PDEs driven by fractional sheets; see [21].

Our paper follows “almost pathwise calculus techniques” developed by Russo and Vallois, and it reaches the $H = \frac{1}{4}$ barrier, developing very detailed Gaussian calculations. As we said, one motivation of this paper, was to prove an Itô–Stratonovich formula for the fractional Brownian motion $X = B^H$ for $H \geq \frac{1}{4}$. Such a process has a finite 4-variation in the sense of [16] and a finite pathwise p-variation for $p > 4$, if one refers, for instance, to [14] and [25]. We even prove that the cubic variation in the sense of [16] is 0 even when the Hurst index is strictly bigger than $\frac{1}{6}$; see Proposition 2.3.

If one wants to remain in the framework of “pathwise” calculus, Itô’s formula has to be of Stratonovich type. In fact, if $H < \frac{1}{2}$, such a formula cannot make use of the forward integral $\int_0^{\cdot} g(B^H)\,d^- B^H$ considered, for instance, in [36] because that integral, as well as the bracket $[g(B^H), B^H]$, is not defined since an explosion occurs in the regularization. For instance, as pointed out in [1], the forward integral $\int_0^T B_s^H\,d^- B_s^H$ does not exist. The use of the Stratonovich-symmetric integral is natural and it provides cancellation of the term involving the second derivative.

Our Itô formula is of the following type:

$$f(B_t^H) = f(B_0^H) + \int_0^t f'(B_u^H)\,d^\circ B_u^H.$$

As we said, when $H > \frac{1}{4}$, the previous formula has already been treated in [1] using Malliavin calculus techniques.

The natural way to prove an Itô formula for an integrator having a finite 4-variation is to write a fourth-order Taylor expansion,

$$\begin{aligned} f(X_{t+\varepsilon}) &= f(X_t) + f'(X_t)(X_{t+\varepsilon} - X_t) + \frac{f''(X_t)}{2}(X_{t+\varepsilon} - X_t)^2 \\ &\quad + \frac{f^{(3)}(X_t)}{6}(X_{t+\varepsilon} - X_t)^3 + \frac{f^{(4)}(X_t)}{24}(X_{t+\varepsilon} - X_t)^4, \end{aligned}$$

plus a remainder term, which can be neglected. The second- and third-order terms can be essentially controlled because one will prove the existence of suitable covariations, and the fourth-order term provides a finite contribution because X has a finite fourth variation. If $H = \frac{1}{4}$, the third-order term can be expressed in terms of a 4-covariation term $[f^{(3)}(X), X, X, X]$; it compensates then for the fourth-order term.

From our point of view, the main achievement of this paper is the proof of the existence of the 4-covariation $[g(B^H), B^H, B^H, B^H]$ for $H \geq \frac{1}{4}$, g being locally bounded; see Theorem 3.7. Moreover, we prove that it is Hölder continuous with parameter strictly smaller than $\frac{1}{4}$. The local boundedness assumption on g can, of course, be relaxed, making a more careful analysis on the density of fractional Brownian motion at each instant. For the moment, we have not investigated this generality.

This result provides, as an application, the Itô–Stratonovich formula for $f(B^H)$, f being of class C^4; see Theorem 4.1.

A second application is a generalized Bouleau–Yor formula for fractional Brownian motion. Fractional Brownian motion B^H has a local time $(l_t^H(a))$ which has a continuous version in (a, t), for any $0 < H < 1$, as the density of the occupation measure; see, for instance, [3] and [20]. In particular, one has

$$\int_0^t g(B_s^H)\,ds = \int_{\mathbb{R}} g(a) l_t^H(a)\,da.$$

First, we mention the result for the classical Brownian motion $B = B^{1/2}$. A direct consequence of [5], [19] and [38] is the following: for a locally bounded function f, we have the equality

$$[f(B), B]_t = -\int_{\mathbb{R}} f(a) l_t^{1/2}(da),$$

where the right-hand side is well defined, since $(l_t^{1/2}(a))_{a\in\mathbb{R}}$ is a semimartingale. We will refer to the previous equality as the *Bouleau–Yor identity*.

Our generalization of the Bouleau–Yor identity is the following:

$$[f(B^{1/4}), B^{1/4}, B^{1/4}, B^{1/4}]_t = -3\int_{\mathbb{R}} f(a)(l_t^{1/4})'(a)\,da.$$

This is done in Corollary 3.8. We recall also that, for $H > \frac{1}{3}$, a Tanaka-type formula has been obtained in [9] involving the Skorohod integral.

The technique used here is a "pedestrian" but accurate exploitation of the Gaussian feature of fractional Brownian motion. Other recent papers where similar techniques have been used are, for instance, [23] and [31]. Some of the computations are made using a Maple procedure.

A natural question is the following: is $H = \frac{1}{4}$ an absolute barrier for the validity of the Bouleau–Yor identity and for the Itô–Stratonovich pathwise formula?

Concerning the extended Bouleau–Yor identity, this is certainly not the case. Similar methods with more technicalities allow one to establish the $2n$-covariation $[g(B^H), B^H, \dots, B^H]$ and its relation with the local time of B^H when $H = 1/2n, n \geq 3$. We have decided not to develop these details because of the heavy technicalities.

As far as the "pathwise" Itô formula is concerned, it is a different story. It is, of course, immediate to see that, for any $0 < H < 1$, if $B = B^H$, one has $B_t^2 = 2\int_0^t B_s\,d^\circ B_s$. On the other hand, proceeding by an obvious Taylor expansion, one would expect

$$B_t^3 = 3\int_0^t B_s^2\,d^\circ B_s - \tfrac{1}{2}[B, B, B]_t, \tag{1.3}$$

provided that $[B, B, B]_t$ exists; Remark 2.4 says that for $H < \frac{1}{6}$ this quantity does not exist and for $H > \frac{1}{6}$ it is 0. Therefore, an Itô formula of the type (1.3) is valid for $H > \frac{1}{6}$ and not valid for $H < \frac{1}{6}$. The study of a pathwise Itô formula for $H \in]\frac{1}{4}, \frac{1}{6}[$ is under investigation.

The paper is organized as follows: we recall some basic definitions and results in Section 2. In Section 3, we state the theorems, we make some basic remarks and we prove part of the results. Section 4 is devoted to the proof of the Itô formula, and Section 5 contains the technical proofs.

2. Notation and recalls of preliminary results.

We start by recalling some definitions and results established in previous papers (see [36]–[39]). In the following, X and Y will be continuous processes. The space of continuous processes will be a metrizable Fréchet space $\mathcal{C}$ if it is endowed with the topology of the *uniform convergence in probability on each compact interval* (ucp). The space of random variables is also a metrizable Fréchet space, denoted by $L^0(\Omega)$, and it is equipped with the topology of the convergence in probability.

We define the *forward integral*

$$\int_0^t Y_u\, d^- X_u := \lim_{\varepsilon \downarrow 0} \text{ucp} \frac{1}{\varepsilon} \int_0^t Y_u (X_{u+\varepsilon} - X_u)\, du \tag{2.1}$$

and the *covariation*

$$[X, Y]_t := \lim_{\varepsilon \downarrow 0} \text{ucp} \frac{1}{\varepsilon} \int_0^t (X_{u+\varepsilon} - X_u)(Y_{u+\varepsilon} - Y_u)\, du. \tag{2.2}$$

The *symmetric-Stratonovich integral* is defined as

$$\int_0^t Y_u\, d^\circ X_u := \lim_{\varepsilon \downarrow 0} \text{ucp} \frac{1}{2\varepsilon} \int_0^t Y_u (X_{u+\varepsilon} - X_{(u-\varepsilon)\vee 0})\, du, \tag{2.3}$$

and the following fundamental equality is valid

$$\int_0^t Y_u\, d^\circ X_u = \int_0^t Y_u\, d^- X_u + \tfrac{1}{2}[X, Y]_t, \tag{2.4}$$

provided that the right-hand side is well defined. However, as we will see in the next section, the left-hand side may exist even if the covariation $[X, Y]$ does not exist. On the other hand, the symmetric-Stratonovich integral can also be written as

$$\int_0^t Y_u\, d^\circ X_u = \lim_{\varepsilon \downarrow 0} \text{ucp} \int_0^t (Y_{u+\varepsilon} + Y_u) \frac{X_{u+\varepsilon} - X_u}{2\varepsilon}\, du. \tag{2.5}$$

Previous definitions will be relaxed later.

If X is such that $[X, X]$ exists, X is called a *finite quadratic variation process*. If $[X, X] = 0$, then X will be called a *zero quadratic variation process*. In particular, a *Dirichlet process* (the sum of a local martingale and a zero quadratic variation process) is a finite quadratic variation process. If X is a finite quadratic variation process and if $f \in C^2(\mathbb{R})$, then the following Itô formula holds:

$$f(X_t) = f(X_0) + \int_0^t f'(X_u)\, d^- X_u + \tfrac{1}{2}[f'(X), X]_t. \tag{2.6}$$

We recall that finite quadratic variation processes are stable by C^1-transformations. In particular, if $f, g \in C^1$ and the vector (X, Y) is such that all mutual covariations exist, then $[f(X), g(Y)]_t = \int_0^t f'(X_s) g'(X_s)\, d[X, Y]_s$. Hence, formulas (2.4) and (2.6) give

$$f(X_t) = f(X_0) + \int_0^t f'(X_u)\, d^\circ X_u. \tag{2.7}$$

REMARK 2.1. (i) If X is a continuous semimartingale and Y is a suitable previsible process, then $\int_0^{\cdot} Y_u \, d^- X_u$ is the classical Itô integral (for details, see [36]).

(ii) If X and Y are (continuous) semimartingales, then $\int_0^{\cdot} Y_u \, d^\circ X_u$ is the Fisk–Stratonovich integral and $[X, Y]$ is the ordinary square bracket.

(iii) If $X = B^H$, then its paths are a.s. Hölder continuous with parameter strictly less than H. Therefore, it is easy to see that, if $H > \frac{1}{2}$, then B^H is a zero quadratic variation process. When $H = \frac{1}{2}$, $B = B^{1/2}$ is the classical Brownian motion and so $[B^{1/2}, B^{1/2}]_t = t$. In particular, Itô formula (2.7) holds for $H \geq \frac{1}{2}$.

(iv) If $X = B$ is a classical Brownian motion, then formula (2.6) holds even for $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{R})$ (see [19] and [38]). On the other hand, if $(l_t(a))$ is the local time associated with B, then in [5] it was shown that

$$f(B_t) = f(B_0) + \int_0^t f'(B_u) \, dB_u - \frac{1}{2} \int_{\mathbb{R}} f'(a) l_t(da). \tag{2.8}$$

The integral involving local time on the right-hand side of (2.8) was defined directly by Bouleau and Yor for a general semimartingale. However, in the case of Brownian motion, Corollary 1.13 in [5] states that, for fixed $t > 0$, $(l_t(a))_{a \in \mathbb{R}}$ is a classical semimartingale; indeed, that integral has a meaning as a deterministic Itô integral. Thus, for $g \in L^2_{\mathrm{loc}}(\mathbb{R})$, setting f such that $f' = g$ and using (2.6) and (2.8), we obtain the *Bouleau–Yor identity*:

$$\int_{\mathbb{R}} g(a) l_t(da) = -[g(B), B]_t. \tag{2.9}$$

Corollary 3.8 will generalize this result to the case of fractional Brownian motion $B^{1/4}$.

(v) An accurate study of "pathwise stochastic calculus" for finite quadratic variation processes has been done in [39]. One provides necessary and sufficient conditions on the covariance of a Gaussian process X so that X is a finite quadratic variation process and X has a deterministic quadratic variation.

Since the quadratic variation is not defined for B^H when $H < \frac{1}{2}$, we need to find a substitution tool. A concept of α-variation was already introduced in [39]. Here it will be called *strong α-variation* and is the following increasing continuous process:

$$[X]^{(\alpha)}_t := \lim_{\varepsilon \downarrow 0} \mathrm{ucp} \int_0^t \frac{|X_{u+\varepsilon} - X_u|^\alpha}{\varepsilon} \, du. \tag{2.10}$$

A real attempt to adapt the previous approach to integrators X which are not of finite quadratic variation has been done in [16]. For a positive integer n, in [16] one defines the *n-covariation* $[X^1, \dots, X^n]$ of a vector $(X^1, \dots, X^n)$ of real continuous processes in the following way:

$$[X_1, \dots, X_n]_t := \lim_{\varepsilon \downarrow 0} \mathrm{ucp} \int_0^t \frac{(X^1_{u+\varepsilon} - X^1_u) \cdots (X^n_{u+\varepsilon} - X^n_u)}{\varepsilon} \, du. \tag{2.11}$$

Clearly, if $n = 2$, the 2-covariation $[X_1, X_2]$ is the covariation previously defined. In particular, if all the processes X_i are equal to X, then the definition gives

$$\underbrace{[X, \dots, X]}_{n \text{ times}}(t) := \lim_{\varepsilon \downarrow 0} \text{ucp} \int_0^t \frac{(X_{u+\varepsilon} - X_u)^n}{\varepsilon} \, du, \tag{2.12}$$

which is called the *n-variation* of process X. Clearly, for even integer n, $[X]^{(n)} = \underbrace{[X, \dots, X]}_{n \text{ times}}$.

REMARK 2.2. (i) If the strong n-variation of X exists, then, for all $m > n$, $\underbrace{[X, \dots, X]}_{m \text{ times}} = 0$ (see [16], Remark 2.6.3, page 7).

(ii) If $\underbrace{[X, \dots, X]}_{n \text{ times}}$ and $[X]^{(n)}$ exist, then, for $g \in C(\mathbb{R})$,

$$\lim_{\varepsilon \downarrow 0} \text{ucp} \int_0^t g(X_u) \frac{(X_{u+\varepsilon} - X_u)^n}{\varepsilon} \, du = \int_0^t g(X_u) \, d[X, X, \dots, X]_u \tag{2.13}$$

(see [16], Remark 2.6.6, page 8, and Remark 2.1, page 5).

(iii) Let $f_1, \dots, f_n \in C^1(\mathbb{R})$ and let X be a strong n-variation continuous process. Then

$$[f_1(X), \dots, f_n(X)]_t = \int_0^t f_1'(X_u) \cdots f_n'(X_u) \, d \underbrace{[X, \dots, X]}_{n \text{ times}}(u).$$

(iv) In [16], Proposition 3.4, one writes an Itô-type formula for X a continuous strong 3-variation process and for $f \in C^3(\mathbb{R})$:

$$f(X_t) = f(X_0) + \int_0^t f'(X_u) \, d^\circ X_u - \frac{1}{12} \int_0^t f^{(3)}(X_u) \, d[X, X, X]_u. \tag{2.14}$$

In particular, the previous point implies that

$$f(X_t) = f(X_0) + \int_0^t f'(X_u) \, d^\circ X_u - \frac{1}{12} [f''(X), X, X]_t.$$

(v) Let us return to the process $X = B^H$. In [16], Proposition 3.1, it is proved that its strong 3-variation exists if $H \geq \frac{1}{3}$ but, even for the limiting case $H = \frac{1}{3}$, we have that the 3-covariation $[B^H, B^H, B^H] \equiv 0$.

(vi) In [39], Proposition 3.14, page 22, it is proved that the strong $\frac{1}{H}$-variation of B^H exists and equals $\rho_H t$, where $\rho_H = \mathrm{E}[|G|^{1/H}]$, with G a standard normal random variable. Consequently,

$$[B^H]_t^{(4)} = \begin{cases} 3t, & \text{if } H = \frac{1}{4}, \\ 0, & \text{if } H > \frac{1}{4}. \end{cases} \tag{2.15}$$

In Section 4, we will be able to write an Itô formula for the fractional Brownian motion with index $\frac{1}{4} \le H < \frac{1}{3}$. Let us stress that, in this case, B^H admits a (strong) 4-variation but not a strong 3-variation.

We end this section with the following remark: as it follows from the fifth part of the remark above, the 3-variation of a fractional Brownian motion B^H is 0 when $H \ge \frac{1}{3}$. This result can be extended to the case of lower Hurst index:

PROPOSITION 2.3. *Assume* $H > \frac{1}{6}$. *Then the* 3-*covariation* $[B^H, B^H, B^H]$ *exists and vanishes*.

PROOF. For simplicity, we fix $t = 1$. It suffices to prove that the limit, when ε goes to 0, of $\mathrm{E}[(\int_0^1 \frac{1}{\varepsilon}(B^H_{u+\varepsilon} - B^H_u)^3)^2]$ is 0. We will prove, in fact, that the limit, when $\varepsilon \downarrow 0$, of the following integral,

$$\mathcal{I}_\varepsilon := 2 \iint_{0<u<v<1} \mathrm{E}\biggl(\frac{(B^H_{u+\varepsilon} - B^H_u)^3 (B^H_{v+\varepsilon} - B^H_v)^3}{\varepsilon^2}\biggr)\, du\, dv,$$

equals 0.

For any centered Gaussian random vector (N, N'), we have

$$\mathrm{E}\bigl(N^3 (N')^3\bigr) = 6\, \mathrm{Cov}^3(N, N') + 9\, \mathrm{Cov}(N, N')\, \mathrm{Var}(N)\, \mathrm{Var}(N').$$

Indeed, it is enough to write $\mathrm{E}(N^3(N')^3) = \mathrm{E}[N^3\, \mathrm{E}((N')^3|N)]$ and to use linear regression (see also the proof of Lemma 3.7, page 15, in [39] for a similar computation).

Denote $(N, N') = (B^H_{u+\varepsilon} - B^H_u, B^H_{v+\varepsilon} - B^H_v)$ and $\eta_\varepsilon(u, v) = \mathrm{Cov}(N, N')$. Therefore, the previous integral $\mathcal{I}_\varepsilon$ can be written as

$$\begin{aligned}
\mathcal{I}_\varepsilon &= 12 \iint_{0<u<v<1} \frac{(\eta_\varepsilon(u, v))^3}{\varepsilon^2}\, du\, dv \\
&\quad + 9 \cdot 2^{4H+1} \varepsilon^{4H-2} \iint_{0<u<v<1} \eta_\varepsilon(u, v)\, du\, dv \\
&=: \mathcal{I}^1_\varepsilon + \mathcal{I}^2_\varepsilon.
\end{aligned}$$

Since

$$\eta_\varepsilon(u, v) = \tfrac{1}{2}\bigl(|v - u + \varepsilon|^{2H} + |v - u - \varepsilon|^{2H} - 2|v - u|^{2H}\bigr),$$

a direct computation shows that

$$\begin{aligned}
&\int_0^v \eta_\varepsilon(u, v)\, du \\
&\quad = \frac{1}{2(2H+1)} \begin{cases} (v+\varepsilon)^{2H+1} + (v-\varepsilon)^{2H+1} - 2v^{2H+1}, & \text{if } v \ge \varepsilon, \\ (v+\varepsilon)^{2H+1} - (\varepsilon - v)^{2H+1} - 2v^{2H+1}, & \text{if } 0 \le v \le \varepsilon, \end{cases}
\end{aligned}$$

and then

$$\iint_{0<u<v<1} \eta_\varepsilon(u,v)\,du\,dv$$
$$= \int_0^\varepsilon dv \int_0^v \eta_\varepsilon(u,v)\,du + \int_\varepsilon^1 dv \int_0^v \eta_\varepsilon(u,v)\,du$$
$$\sim \frac{1}{H}\varepsilon^2 - \frac{1}{2H(H+1)(2H+1)}\varepsilon^{2H+2}$$
$$\sim \frac{1}{H}\varepsilon^2 \qquad \text{as } \varepsilon \downarrow 0.$$

Hence, $\mathcal{I}_\varepsilon^2 \sim 9 \cdot 2^{4H+1}\frac{1}{H}\varepsilon^{4H}$, when $\varepsilon \downarrow 0$, for any $H > 0$, and $\lim_{\varepsilon\downarrow 0} \mathcal{I}_\varepsilon^2 = 0$ for any $H > 0$.

To compute $\mathcal{I}_\varepsilon^1$, we set $\zeta = v - u$. Then

$$\begin{aligned}\mathcal{I}_\varepsilon^1 &= \frac{3}{2\varepsilon^2}\int_0^1 \left((\zeta+\varepsilon)^{2H} + |\zeta-\varepsilon|^{2H} - 2\zeta^{2H}\right)^3 (1-\zeta)\,d\zeta \\ &= 3\varepsilon^{6H-1}\int_0^{1/\varepsilon} \left((\theta+1)^{2H} + |\theta-1|^{2H} - 2\theta^{2H}\right)^3 (1-\varepsilon\theta)\,d\theta \\ &=: 3\varepsilon^{6H-1}\mathcal{I}_\varepsilon^{11} - 3\varepsilon^{6H}\mathcal{I}_\varepsilon^{12}.\end{aligned}$$

Clearly,

$$\lim_{\varepsilon\downarrow 0}\mathcal{I}_\varepsilon^{11} = \int_0^\infty \left((\theta+1)^{2H} + |\theta-1|^{2H} - 2\theta^{2H}\right)^3 d\theta < \infty \qquad \text{if } H < \tfrac{5}{6}.$$

A similar calculation shows that the second term tends to a convergent integral under the same condition on H. This yields

$$\mathcal{I}_\varepsilon^2 \sim 3\varepsilon^{6H-1}\int_0^\infty \left((\theta+1)^{2H} + |\theta-1|^{2H} - 2\theta^{2H}\right)^3 d\theta \qquad \text{as } \varepsilon \downarrow 0$$

and gives the conclusion, since $H > \frac{1}{6}$. □

REMARK 2.4. From the previous proof, we can also deduce that

$$\lim_{\varepsilon\downarrow 0} \mathrm{E}\left[\left(\int_0^1 \frac{1}{\varepsilon}(B_{u+\varepsilon}^H - B_u^H)^3\right)^2\right]$$

is infinite for $H < \frac{1}{6}$; therefore, if $H < \frac{1}{6}$, then the 3-variation $[B^H, B^H, B^H]$ virtually does not exist.

3. Third-order-type integrals and 4-covariations. To understand the case of fractional Brownian motion for $H \geq \frac{1}{4}$, besides the family of integrals introduced until now, we need to introduce a new class of integrals.

Let again X, Y be continuous processes. We define the following *third-order integrals* for $t > 0$:

$$\begin{aligned}
&\int_0^t Y_u\, d^{-3}X_u := \lim_{\varepsilon\downarrow 0}\text{prob}\, \frac{1}{\varepsilon}\int_0^t Y_u(X_{u+\varepsilon}-X_u)^3\, du,\\
(3.1)\qquad &\int_0^t Y_u\, d^{+3}X_u := \lim_{\varepsilon\downarrow 0}\text{prob}\, \frac{1}{\varepsilon}\int_0^t Y_u(X_u-X_{(u-\varepsilon)\vee 0})^3\, du,\\
&\int_0^t Y_u\, d^{\circ 3}X_u := \lim_{\varepsilon\downarrow 0}\text{prob}\, \frac{1}{2\varepsilon}\int_0^t (Y_u+Y_{u+\varepsilon})(X_{u+\varepsilon}-X_u)^3\, du.
\end{aligned}$$

We will call them, respectively, (definite) *forward*, *backward* and *symmetric third-order integrals*. If the above $L^0(\Omega)$-valued function

$$t \mapsto \int_0^t Y_u\, d^{-3}X_u, \quad \text{respectively} \quad t \mapsto \int_0^t Y_u\, d^{+3}X_u, t \mapsto \int_0^t Y_u\, d^{\circ 3}X_u$$

exists for any $t > 0$ (and equals 0 for $t = 0$), and it admits a continuous version, then such a version will be called a *third-order forward* (respectively, *backward*, *symmetric*) *integral*, and it will be denoted again by

$$\left(\int_0^t Y_u\, d^{-3}X_u\right)_{t\geq 0}, \quad \text{respectively} \quad \left(\int_0^t Y_u\, d^{+3}X_u\right)_{t\geq 0}, \left(\int_0^t Y_u\, d^{\circ 3}X_u\right)_{t\geq 0}.$$

REMARK 3.1. If X is a strong 3-variation process, then $[X, X, X]$ will be a finite variation process and

$$(3.2)\qquad \int_0^t Y_u\, d^{-3}X_u = \int_0^t Y_u\, d^{+3}X_u = \int_0^t Y_u\, d[X, X, X]_u.$$

In particular, if $X = B^H$ is a fractional Brownian motion, with $H \geq \frac{1}{3}$, all the quantities in (3.2) are 0. If $H < \frac{1}{3}$, the strong 3-variation does not exist (see [16], Proposition 3). Recall that if $\frac{1}{6} < H < \frac{1}{3}$, the 3-covariation $[B^H, B^H, B^H]$ exists and vanishes (see Proposition 2.3); hence, $\int_0^t Y_u\, d[X, X, X]_u = 0$. We shall prove that if $\frac{1}{4} < H < \frac{1}{3}$ and if $Y = g(B^H)$ then the third-order integrals also vanish, so (3.2) is still true (see Theorem 3.4). If $H = \frac{1}{4}$ and $Y = g(B^H)$, the third-order integrals are not necessarily 0.

The following results relate third-order integrals with the notion of 4-covariation.

PROPOSITION 3.2. (i)

$$\int_0^t Y_u\, d^{\circ 3}X_u = \tfrac{1}{2}\left(\int_0^t Y_u\, d^{-3}X_u + \int_0^t Y_u\, d^{+3}X_u\right),$$

provided two of the three previous quantities exist.

(ii)

$$\int_0^t Y_u \, d^{+3} X_u - \int_0^t Y_u \, d^{-3} X_u = [Y, X, X, X]_t,$$

provided two of the three previous quantities exist.

COROLLARY 3.3. *Let X be a continuous process having a 4-variation and take $f \in C^1(\mathbb{R})$.*

(i) *If $\int_0^t f(X_u) \, d^{-3} X_u$ exists, then $\int_0^t f(X_u) \, d^{+3} X_u$ exists and*

$$\int_0^t f(X_u) \, d^{+3} X_u = \int_0^t f(X_u) \, d^{-3} X_u + \int_0^t f'(X_u) \, d[X, X, X, X]_u.$$

(ii) *If $\int_0^t f'(X_u) \, d^{-3} X_u$ exists and if furthermore $f \in C^2(\mathbb{R})$, then*

$$[f(X), X, X]_t = \int_0^t f'(X_u) \, d^{-3} X_u + \frac{1}{2} \int_0^t f''(X_u) \, d[X, X, X, X]_u.$$

PROOF. The first point follows immediately from Proposition 3.2 and Remark 2.2(iii). To prove the second part, a second-order Taylor expansion gives, for $s, \varepsilon > 0$,

$$f(X_{s+\varepsilon}) - f(X_s) = f'(X_s)(X_{s+\varepsilon} - X_s) + \frac{f''(X_s)}{2}(X_{s+\varepsilon} - X_s)^2 \\ + R(f, \varepsilon, s)(X_{s+\varepsilon} - X_s)^2,$$

where $R(f, \varepsilon, s)$ converges to 0, ucp in s, when ε goes to 0, by the uniform continuity of f and of paths of X on each compact interval. Multiplying the previous expression by $(X_{s+\varepsilon} - X_s)^2$, integrating from 0 to t, dividing by ε and using Remark 2.2(ii), we obtain the result. □

In spite of the now classical notion of the symmetric integral given in (2.5), we need to relax this definition. From now on, we will say that the symmetric integral of a process Y with respect to an integrator X exists if

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t Y_u (X_{u+\varepsilon} - X_{(u-\varepsilon) \vee 0}) \, du$$

exists *in probability* and the limiting $L^0(\Omega)$-valued function has a continuous version. We will still denote that process (unique up to indistinguishability) by $\int_0^t Y_u \, d^\circ X_u$.

Similarly, in this paper, the concept of 4-covariation will be understood in a weaker sense with respect to (2.11). We will say that the 4-covariation $[X^1, X^2, X^3, X^4]$ exists if

$$\lim_{\varepsilon \downarrow 0} \int_0^t \frac{(X^1_{u+\varepsilon} - X^1_u) \cdots (X^4_{u+\varepsilon} - X^4_u)}{\varepsilon} \, du$$

exists in probability and the limiting $L^0(\Omega)$-valued function has a continuous version.

Clearly, if $\int_0^t Y_u \, d^\circ X_u$ exists in the classical sense of Russo and Vallois, then it also exists in this relaxed meaning; similarly, if $[X^1, X^2, X^3, X^4]$ exists in the (2.11) sense, then it will exist in the relaxed sense. We note that when all the processes are equal, then a Dini-type lemma, as in [39], allows us to show that the two definitions of 4-covariations are equivalent. We note that Proposition 3.2 and Corollary 3.3 are still valid with these conventions.

From now on, we will concentrate on the case when $X = B^H$ is the fractional Brownian motion with Hurst index H.

In the statement of the fundamental result of this section, we use the following definition: we say that a real function g fulfills the *subexponential inequality* if

$$|g(x)| \le L e^{l|x|} \qquad \text{with } l, L \text{ positive constants.} \tag{3.3}$$

THEOREM 3.4. *Let $\frac{1}{4} \le H < \frac{1}{3}$, let $t > 0$ and let g be a real locally bounded function. The following properties hold*:

(a) *The third-order integrals $\int_0^t g(B_u^H) \, d^{\pm 3} B_u^H$ exist and vanish if $\frac{1}{4} < H < \frac{1}{3}$. Henceforth, we assume $H = \frac{1}{4}$.*

(b) *The third-order integrals $\int_0^t g(B_u^{1/4}) \, d^{\pm 3} B_u^{1/4}$ exist and are opposite, that is, for any $t > 0$,*

$$\int_0^t g(B_u^{1/4}) \, d^{+3} B_u^{1/4} = -\int_0^t g(B_u^{1/4}) \, d^{-3} B_u^{1/4}. \tag{3.4}$$

Moreover, the processes $(\int_0^t g(B_u^{1/4}) \, d^{\pm 3} B_u^{1/4})_{t \ge 0}$ are Hölder continuous with parameter strictly less than $\frac{1}{4}$.

(c) *If, furthermore, g fulfills the subexponential inequality* (3.3), *the expectation and the second moment of third-order integrals are given by*

$$\begin{aligned} &\mathrm{E}\left\{\int_0^t g(B_u^{1/4}) \, d^{-3} B_u^{1/4}\right\} \\ &\qquad = -\mathrm{E}\left\{\int_0^t g(B_u^{1/4}) \, d^{+3} B_u^{1/4}\right\} = -\frac{3}{2} \int_0^t \frac{du}{\sqrt{u}} \mathrm{E}[g(B_u^{1/4}) B_u^{1/4}] \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} &\mathrm{E}\left\{\left(\int_0^t g(B_u^{1/4}) \, d^{\pm 3} B_u^{1/4}\right)^2\right\} \\ &\qquad = \tfrac{9}{2} \iint_{0<u<v<t} du \, dv \, \mathrm{E}[g(B_u^{1/4}) g(B_v^{1/4}) \\ &\qquad\qquad \times (\lambda_{11}\lambda_{12}(B_u^{1/4})^2 \\ &\qquad\qquad\quad + (\lambda_{11}\lambda_{22} + \lambda_{12}^2) B_u^{1/4} B_v^{1/4} \\ &\qquad\qquad\quad + \lambda_{12}\lambda_{22}(B_v^{1/4})^2 - \lambda_{12})], \end{aligned} \tag{3.6}$$

where the right-hand sides of (3.5) *and* (3.6) *are absolute convergent integrals. Here*

$$
\begin{aligned}
\lambda_{11} &= \frac{\sqrt{v}}{\sqrt{uv - K_{1/4}(u, v)^2}}, \\
\lambda_{22} &= \frac{\sqrt{u}}{\sqrt{uv - K_{1/4}(u, v)^2}}, \qquad (3.7)\\
\lambda_{12} &= -\frac{K_{1/4}(u, v)}{\sqrt{uv - K_{1/4}(u, v)^2}}.
\end{aligned}
$$

(d) *If* $g \in C^1(\mathbb{R})$, *then the quantity in* (3.4) *is equal to* $\frac{1}{2}\int_0^t g'(B_u^{1/4})\, d[B^{1/4}]_u^{(4)}$.

The proof of Theorem 3.4 is postponed to the last section. Let us note that composing Borel functions and fractional Brownian motion is authorized.

REMARK 3.5. If g is a Lebesgue a.e. defined, locally bounded Borel function, then the composition $g(B_t^H)$, $t > 0$, is well defined, up to an a.s. equivalence, random variable. Precisely, if g_1, g_2 are two Lebesgue a.e. modifications of g, then $g_1(B_t^H) = g_2(B_t^H)$ a.s. (since B_t^H has a density function). Consequently, $\int_0^t g_1(B_u^H)\, d^{\pm 3} B_u^H$ exists if and only if $\int_0^t g_2(B_u^H)\, d^{\pm 3} B_u^H$ exists and they are equal.

The proof of the following result is easily obtained by a localization argument.

PROPOSITION 3.6. *The maps*

$$g \mapsto \int_0^t g(B_u^{1/4})\, d^{\pm 3} B_u^{1/4} \quad \text{and} \quad g \mapsto \int_0^t g(B_u^{1/4})\, d^{\circ 3} B_u^{1/4}$$

are continuous from $L^\infty_{\mathrm{loc}}(\mathbb{R})$ *to* $L^0(\Omega)$.

The next result states the existence of a significant fourth-order covariation related to the fractional Brownian motion B^H with Hurst index $H = \frac{1}{4}$. Its proof is obvious using parts (b) and (d) in Theorem 3.4, Proposition 3.6, Proposition 3.2(ii) and Remark 2.2(iii).

THEOREM 3.7. *Let* $g \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ *and fix* $t > 0$. *The process* $([g(B^{1/4}), B^{1/4}, B^{1/4}, B^{1/4}]_t)_{t \geq 0}$ *is well defined, has Hölder continuous paths of parameter strictly less than* $\frac{1}{4}$ *and is given by*

$$
\begin{aligned}
&[g(B^{1/4}), B^{1/4}, B^{1/4}, B^{1/4}]_t \\
&\quad = 2\int_0^t g(B_u^{1/4})\, d^{+3} B_u^{1/4} = -2\int_0^t g(B_u^{1/4})\, d^{-3} B_u^{1/4}. \qquad (3.8)
\end{aligned}
$$

One consequence of Theorem 3.7 concerns the local time of the fractional Brownian motion. Let $(l_t^H(a))$ be the local time as the occupation measure density (see [3] and [20]). It exists for any $0 < H < 1$; moreover, if $H < \frac{1}{3}$, it is absolutely continuous with respect to a. We denote by $(l_t^H)'(a)$ the corresponding derivative. The following result extends to the fractional Brownian motion with $H = \frac{1}{4}$ the Bouleau–Yor-type equality (2.9) discussed in Remark 2.1 for the case of classical Brownian motion.

COROLLARY 3.8. *Let $g \in L^\infty_{\mathrm{loc}}$. Then, for fixed $t > 0$,*

$$[g(B^{1/4}), B^{1/4}, B^{1/4}, B^{1/4}]_t = -3 \int g(a)(l_t^{1/4})'(a)\, da. \tag{3.9}$$

PROOF. Recall that $[g(B^{1/4}), B^{1/4}, B^{1/4}, B^{1/4}]_t = 3t$ and so $[g(B^{1/4}), B^{1/4}, B^{1/4}, B^{1/4}]_t = 3\int_0^t g'(B_s^{1/4})\, ds$, whenever $g \in C^1(\mathbb{R})$ with compact support. By the density occupation formula, the previous expression becomes $-3\int g'(a) \times l_t^{1/4}(a)\, da$. Integrating by parts, we obtain the right member of (3.9). This shows the equality for smooth g. To obtain the final statement, we regularize $g \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ by taking $g_n = g * \phi_n$, where (ϕ_n) is a sequence of mollifiers converging to the Dirac delta function, we apply the equality for g being smooth and we take the limit. For the limit of left members, we use the continuity of the considered 4-covariation. For the right members, we use the Lebesgue dominated convergence theorem: in fact, we recall that $a \to \lambda_t'(a)$ is integrable with compact support and on each compact the upper bound of $|g_n|$ is bounded by the upper bound of $|g|$. □

4. Itô formula. Let B^H be again a fractional Brownian motion with Hurst index H.

THEOREM 4.1. *Let $H \geq \frac{1}{4}$ and $f \in C^4(\mathbb{R})$. Then the symmetric integral $\int_0^t f'(B_u^H)\, d^\circ B_u^H$ exists and an Itô-type formula can be written as*

$$f(B_t^H) = f(B_0^H) + \int_0^t f'(B_u^H)\, d^\circ B_u^H. \tag{4.1}$$

REMARK 4.2. The most interesting case concerns the critical limiting case $H = \frac{1}{4}$. When $H > \frac{1}{4}$, the result was also established in [1] using other methods.

PROOF OF THEOREM 4.1. Theorem 4.1 will be a consequence of Theorem 3.4. Fix $t > 0$. In fact, we prove that, for any $f \in C^4(\mathbb{R})$,

$$f(B_t^H) = f(B_0^H) + \int_0^t f'(B_u^H)\, d^\circ B_u^H - \frac{1}{12}\int_0^t f^{(3)}(B_u^H)\, d^{\circ 3} B_u^H, \tag{4.2}$$

which implies the final result since $\int_0^t f^{(3)}(B_s^H)\, d^{\circ 3} B_s^H$ vanishes [see Theorem 3.4(a) and (b) and Proposition 3.2(i)].

We start with the Taylor formula: for $a, b \in \mathbb{R}$, we have

$$f(b) - f(a) = f'(a)(b-a) + f''(a)\frac{(b-a)^2}{2} + f^{(3)}(a)\frac{(b-a)^3}{6} + \frac{(b-a)^4}{6}\int_0^1 \lambda^3 f^{(4)}\big(\lambda a + (1-\lambda)b\big)\,d\lambda \tag{4.3}$$

and also

$$\begin{aligned} f(a) - f(b) &= f'(b)(a-b) + f''(b)\frac{(a-b)^2}{2} + f^{(3)}(b)\frac{(a-b)^3}{6} \\ &\quad + \frac{(a-b)^4}{6}\int_0^1 \lambda^3 f^{(4)}\big(\lambda b + (1-\lambda)a\big)\,d\lambda \\ &= -f'(b)(b-a) + f''(b)\frac{(b-a)^2}{2} - f^{(3)}(b)\frac{(b-a)^3}{6} \\ &\quad + \frac{(b-a)^4}{6}\int_0^1 (1-\lambda)^3\big(f^{(4)}\big(\lambda a + (1-\lambda)b\big)\big)\,d\lambda. \end{aligned}$$

Since

$$f''(b) = f''(a) + f^{(3)}(a)(b-a) + (b-a)^2\int_0^1 \lambda\big(f^{(4)}\big(\lambda a + (1-\lambda)b\big)\big)\,d\lambda$$

and

$$f^{(3)}(b) = f^{(3)}(a) + (b-a)\int_0^1 f^{(4)}\big(\lambda a + (1-\lambda)b\big)\,d\lambda,$$

we can write

$$f(a) - f(b) = -f'(b)(b-a) + f''(a)\frac{(b-a)^2}{2} + f^{(3)}(a)\frac{(b-a)^3}{3} + (b-a)^4\int_0^1 \left(\frac{\lambda^2}{2} - \frac{\lambda^3}{6}\right) f^{(4)}\big(\lambda a + (1-\lambda)b\big)\,d\lambda. \tag{4.4}$$

Taking the difference between (4.3) and (4.4) and dividing by 2, we get

$$f(b) - f(a) = \frac{f'(a) + f'(b)}{2}(b-a) - \frac{1}{12} f^{(3)}(a)(b-a)^3 + (b-a)^4\int_0^1 \left(\frac{\lambda^3}{6} - \frac{\lambda^2}{4}\right) f^{(4)}\big(\lambda a + (1-\lambda)b\big)\,d\lambda. \tag{4.5}$$

On the other hand, exchanging the roles of a and b, we get

$$\begin{aligned} &f(a) - f(b) \\ &\quad = -\frac{f'(a) + f'(b)}{2}(b-a) + \frac{1}{12} f^{(3)}(b)(b-a)^3 \\ &\quad\quad + (b-a)^4\int_0^1 \left(\frac{(1-\lambda)^3}{6} - \frac{(1-\lambda)^2}{4}\right) f^{(4)}\big(\lambda a + (1-\lambda)b\big)\,d\lambda. \end{aligned} \tag{4.6}$$

Taking this time the difference between (4.5) and (4.6) and dividing by 2, we obtain

$$\begin{aligned} f(b) - f(a) = {} & \frac{f'(a) + f'(b)}{2}(b-a) \\ & - \frac{f^{(3)}(a) + f^{(3)}(b)}{24}(b-a)^3 + (b-a)^4 J(a,b), \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} J(a,b) &= \int_0^1 \left(\frac{\lambda^3}{6} - \frac{\lambda^2}{4} + \frac{1}{24}\right) f^{(4)}\big(\lambda a + (1-\lambda)b\big)\, d\lambda \\ &= \int_0^1 \left(\frac{\lambda^3}{6} - \frac{\lambda^2}{4} + \frac{1}{24}\right)\big(f^{(4)}\big(\lambda a + (1-\lambda)b\big) - f^{(4)}(a)\big)\, d\lambda, \end{aligned}$$

since

$$\int_0^1 \left(\frac{\lambda^3}{6} - \frac{\lambda^2}{4} + \frac{1}{24}\right) d\lambda = 0.$$

Setting in (4.7) $a = B_u^H$ and $b = B_{u+\varepsilon}^H$, we get

$$\begin{aligned} f(B_{u+\varepsilon}^H) - f(B_u^H) = {} & \big(f'(B_u^H) + f'(B_{u+\varepsilon}^H)\big)\frac{B_{u+\varepsilon}^H - B_u^H}{2} \\ & - \frac{f^{(3)}(B_u^H) + f^{(3)}(B_{u+\varepsilon}^H)}{2}\frac{(B_{u+\varepsilon}^H - B_u^H)^3}{12} \\ & + J(B_u^H, B_{u+\varepsilon}^H)(B_{u+\varepsilon}^H - B_u^H)^4. \end{aligned} \tag{4.8}$$

Using the uniform continuity on each compact real interval I of $f^{(4)}$ and of B^H, we observe that $\sup_{u \in I} J(B_u^H, B_{u+\varepsilon}^H) \to 0$ in probability when $\varepsilon \downarrow 0$. Take $t > 0$, integrate (4.8) in u on $[0,t]$ and divide by ε:

$$\begin{aligned} & \frac{1}{\varepsilon}\int_0^t \big(f(B_{u+\varepsilon}^H) - f(B_u^H)\big)\, du \\ & \quad = \int_0^t \big(f'(B_{u+\varepsilon}^H) + f'(B_u^H)\big)\frac{B_{u+\varepsilon}^H - B_u^H}{2\varepsilon}\, du \\ & \qquad - \int_0^t \frac{f^{(3)}(B_u^H) + f^{(3)}(B_{u+\varepsilon}^H)}{2}\frac{(B_{u+\varepsilon}^H - B_u^H)^3}{12\varepsilon}\, du \\ & \qquad + \int_0^t J(B_u^H, B_{u+\varepsilon}^H)\frac{(B_{u+\varepsilon}^H - B_u^H)^4}{\varepsilon}\, du. \end{aligned}$$

By a simple change of variable, we can transform the left-hand side and obtain

$$
\begin{aligned}
&\frac{1}{\varepsilon}\int_t^{t+\varepsilon} f(B_u^H)\,du - \frac{1}{\varepsilon}\int_0^{\varepsilon} f(B_u^H)\,du \\
&\quad = \int_0^t \big(f'(B_{u+\varepsilon}^H) + f'(B_u^H)\big)\frac{B_{u+\varepsilon}^H - B_u^H}{2\varepsilon}\,du \\
&\qquad - \int_0^t \frac{f^{(3)}(B_u^H) + f^{(3)}(B_{u+\varepsilon}^H)}{2}\frac{(B_{u+\varepsilon}^H - B_u^H)^3}{12\varepsilon}\,du \\
&\qquad + \int_0^t J(B_u^H, B_{u+\varepsilon}^H)\frac{(B_{u+\varepsilon}^H - B_u^H)^4}{\varepsilon}\,du.
\end{aligned}
\tag{4.9}
$$

The left-hand side of (4.9) tends, as $\varepsilon \downarrow 0$, toward $f(B_t^H) - f(B_0^H)$. Since $\sup_{u\in[0,t]} J(B_u^H, B_{u+\varepsilon}^H)$ tends to 0, the last term on the right-hand side of (4.9) also tends to 0, by the existence of the strong 4-variation. The second term on the right-hand side converges to $\int_0^t f^{(3)}(B_u^H)\,d^{\circ 3}B_u^H$, which exists by Theorem 3.4. Therefore, the first term on the right-hand side of (4.9) is also forced to have a limit in probability. According to part (b) of Theorem 3.4, the symmetric third-order integral has a continuous version in t; therefore, the second term must have a continuous version and it will, of course, be the symmetric integral $\int_0^t f'(B_u^H)\,d^{\circ}B_u^H$. Equation (4.2) is proved. □

5. Proofs of existence and properties of third-order integrals. The main topic of this section is the proof of Theorem 3.4, which will be articulated from Step I to Step VI.

Recall that $\frac{1}{4} \le H < \frac{1}{3}$. We will consider only the third-order forward integral, since for the third-order backward integral the reasoning is similar. Hence, let us denote

$$
I_\varepsilon(g)(t) := \frac{1}{\varepsilon}\int_0^t g(B_u^H)(B_{u+\varepsilon}^H - B_u^H)^3\,du \tag{5.1}
$$

and recall that the forward third-order integral $\int_0^t g(B_u^H)\,d^{-3}B_u^H$ was defined as the limit in probability of $I_\varepsilon(g)(t)$. For simplicity, we will fix $t = 1$ and simply denote $I_\varepsilon(g) := I_\varepsilon(g)(1)$.

First, let us outline the proof of Theorem 3.4.

I. Computation of $\lim_{\varepsilon\downarrow 0} \mathrm{E}[I_\varepsilon(g)]$. The limit vanishes for $\frac{1}{4} < H < \frac{1}{3}$. If $H = \frac{1}{4}$ and assuming the existence stated in part (b), the computation also gives (3.5).

II. Computation of $\lim_{\varepsilon\downarrow 0} \mathrm{E}[I_\varepsilon(g)^2]$. We state Lemma 5.1, which allows us to give an equivalent of this second moment as $\varepsilon \downarrow 0$. Again, the limit vanishes for $\frac{1}{4} < H < \frac{1}{3}$; hence, we get part (a). Henceforth, we assume $H = \frac{1}{4}$. Equation (3.6) is obtained assuming again the existence stated in (b).

III. Integrals on the right-hand sides of (3.5) and (3.6) are absolute convergent and the proof of part (c) is complete.

IV. Proof of the existence of the forward third-order integral [as a first step in proving (b)]. First, we reduce the study to the case of a bounded function g and then we establish the existence under this hypothesis.

V. We prove the existence of a continuous version of the forward third-order integral and the Hölder regularity of its paths.

VI. End of part (b) proof. We verify (3.4) proving at the same time (d). We state and use Lemma 5.3.

The end of the section is devoted to the proofs of Lemmas 5.1 and 5.3, which are stated at Steps II and VI and used in the proof of parts (b) and (d) of Theorem 3.4.

STEP I (Computation of $\lim_{\varepsilon\downarrow 0} \mathrm{E}[I_\varepsilon(g)]$). To compute the expectation of $I_\varepsilon(g)$, we will use the linear regression for $B^H_{u+\varepsilon} - B^H_u$, which is a centered Gaussian random variable with variance ε^{2H}. It can be written as

$$B^H_{u+\varepsilon} - B^H_u = \frac{K_H(u, u+\varepsilon) - K_H(u,u)}{K_H(u,u)} B^H_u + Z_\varepsilon, \tag{5.2}$$

where Z_ε is a Gaussian mean-zero random variable, independent of B^H_u with variance $\varepsilon^{2H} - (1/4u^{2H})((u+\varepsilon)^{2H} - u^{2H} - \varepsilon^{2H})^2$. Therefore,

$$B^H_{u+\varepsilon} - B^H_u = \alpha_\varepsilon(u) B^H_u + \beta_\varepsilon(u) N, \tag{5.3}$$

where N is a standard normal random variable independent from B^H_u and where, for $u > 0$ fixed, as $\varepsilon \downarrow 0$,

$$\alpha_\varepsilon(u) := \frac{1}{2u^{2H}}\big((u+\varepsilon)^{2H} - u^{2H} - \varepsilon^{2H}\big) = \frac{1}{2}\left(\frac{\varepsilon}{u}\right)^{2H} \phi_0\left(\frac{\varepsilon}{u}\right) \tag{5.4}$$

and

$$\beta^2_\varepsilon(u) := \varepsilon^{2H} - \alpha^2_\varepsilon(u) u^{2H} = \varepsilon^{2H}\phi_1\left(\frac{\varepsilon}{u}\right), \tag{5.5}$$

where $x^{2H}\phi_0(x) := (1+x)^{2H} - 1 - x^{2H}$ and $\phi_1(x) := (1 - \frac{1}{4}x^{2H}\phi_0^2(x))_+$, with ϕ_0 being a continuous bounded function and ϕ_1 a bounded function with the property $\lim_{x\downarrow 0}\phi_0(x) = -1$, $\lim_{x\downarrow 0}\phi_1(x) = 1$. Since $2H < 1$, we can also write

$$\alpha_\varepsilon(u) = -\frac{\varepsilon^{2H}}{2u^{2H}}(1 - 2Hu^{2H-1}\varepsilon^{1-2H} + o(\varepsilon^{1-2H})) \qquad \text{as } \varepsilon \downarrow 0. \tag{5.6}$$

Moreover,

$$\beta^2_\varepsilon(u) = \varepsilon^{2H}\left(1 - \frac{\varepsilon^{2H}}{4u^{2H}}\right) + o(\varepsilon^{4H}) \qquad \text{as } \varepsilon \downarrow 0. \tag{5.7}$$

We can now compute the first moment of $I_\varepsilon(g)$. Replacing (5.3) in the expression of $I_\varepsilon(g)$ and from the independence of N and B_u^H, we obtain

$$\mathrm{E}[I_\varepsilon(g)] = \int_0^1 \frac{\alpha_\varepsilon^3(u)}{\varepsilon} \mathrm{E}\big[g(B_u^H)(B_u^H)^3\big]\,du + \int_0^1 \frac{3\alpha_\varepsilon(u)\beta_\varepsilon^2(u)}{\varepsilon} \mathrm{E}\big[g(B_u^H)B_u^H\big]\,du.$$

The Cauchy–Schwarz inequality and the hypothesis on g imply that, for $0 < u < 1$,

$$\begin{aligned}\mathrm{E}\big[|g(B_u^H)B_u^H|\big] &\le L\mathrm{E}\big[e^{l|B_u^H|}|B_u^H|\big] \\ &\le L'\mathrm{E}\big[e^{lB_u^H}|B_u^H|\big] \le \text{const}\sqrt{\mathrm{E}\big[(B_u^H)^2\big]} \le \text{const}\, u^H < \infty.\end{aligned}$$

In a similar way, it follows that

$$\mathrm{E}\big[|g(B_u^H)(B_u^H)^3|\big] \le \text{const}\sqrt{\mathrm{E}[(B_u^H)^6]} = \text{const}\, u^{3H}.$$

Hence, since $\frac{1}{4} \le H < \frac{1}{3}$, as $\varepsilon \downarrow 0$,

$$\frac{\alpha_\varepsilon^3(u)}{\varepsilon} u^{3H} = \frac{1}{8}\frac{\varepsilon^{6H-1}}{u^{3H}} \phi_0^3\Big(\frac{\varepsilon}{u}\Big) \qquad \text{with } \int_0^1 \frac{du}{u^{3H}} < \infty.$$

Since $\frac{1}{4} \le H < \frac{1}{3}$, letting ε go to 0, we get

$$\lim_{\varepsilon\downarrow 0} \mathrm{E}[I_\varepsilon(g)] = \int_0^1 \Big(\lim_{\varepsilon\downarrow 0} \frac{3\alpha_\varepsilon(u)\beta_\varepsilon^2(u)}{\varepsilon}\Big) \mathrm{E}\big[g(B_u^H)B_u^H\big]\,du$$

and (3.5) is obtained using (5.4) and (5.5). Indeed, since $\frac{1}{4} \le H < \frac{1}{3}$, we have

$$\frac{\alpha_\varepsilon(u)\beta_\varepsilon^2(u)}{\varepsilon} u^H = \frac{1}{2}\frac{\varepsilon^{4H-1}}{u^H}(\phi_0\phi_1)\Big(\frac{\varepsilon}{u}\Big) \qquad \text{with } \int_0^1 \frac{du}{u^H} < \infty.$$

Clearly,

$$\lim_{\varepsilon\downarrow 0} \mathrm{E}[I_\varepsilon(g)] = 0 \qquad \text{if } \tfrac{1}{4} < H < \tfrac{1}{3}. \tag{5.8}$$

If $H = \frac{1}{4}$, Lebesgue dominated convergence implies that

$$\lim_{\varepsilon\downarrow 0} \mathrm{E}[I_\varepsilon(g)] = -\frac{3}{2}\int_0^1 \frac{1}{\sqrt{u}} \mathrm{E}\big[g(B_u^H)B_u^H\big]\,du$$

and then (3.5) follows, assuming the existence in the first part of (b) of Theorem 3.4.

Let us also explain the opposite sign in (3.5) for the backward third-order integral. We need to consider [see (5.3)]

$$B_u^H - B_{u-\varepsilon}^H = \hat{\alpha}_\varepsilon(u)B_u^H + \hat{\beta}_\varepsilon(u)N \qquad \text{(assume that } u - \varepsilon > 0\text{)},$$

where [see (5.4) and (5.5)]

$$\hat{\alpha}_\varepsilon(u) = \frac{1}{2u^{2H}}\big(u^{2H} - (u-\varepsilon)^{2H} + \varepsilon^{2H}\big), \qquad \hat{\beta}_\varepsilon(u)^2 = \varepsilon^{2H} - \hat{\alpha}_\varepsilon(u)^2 u^{2H}.$$

Hence [see (5.6)],

$$\hat{\alpha}_\varepsilon(u) = \frac{\varepsilon^{2H}}{2u^{2H}}\big(1 + 2Hu^{2H-1}\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\big) \qquad \text{as } \varepsilon \downarrow 0,$$

while (5.7) is still true for $\hat{\beta}_\varepsilon(u)^2$. These relations give the opposite sign in (3.5) for the backward third-order integral. □

STEP II (Computation of $\lim_{\varepsilon\downarrow 0} \mathrm{E}[I_\varepsilon(g)^2]$). The computation of the second moment of $I_\varepsilon(g)$ is done using again the Gaussian feature of the process. We express the linear regression for the random vector $(B^H_{u+\varepsilon} - B^H_u, B^H_{v+\varepsilon} - B^H_v)$. We denote by $G = (G_1, G_2, G_3^\varepsilon, G_4^\varepsilon)$ the Gaussian mean-zero random vector $(B^H_u, B^H_v, B^H_{u+\varepsilon} - B^H_u, B^H_{v+\varepsilon} - B^H_v)$ and we use a similar idea as in Step I. For instance, (5.2) will be replaced by

$$\begin{pmatrix} G_3^\varepsilon \\ G_4^\varepsilon \end{pmatrix} = A_\varepsilon \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} + \begin{pmatrix} Z_1^\varepsilon \\ Z_2^\varepsilon \end{pmatrix}, \tag{5.9}$$

where the Gaussian mean-zero random vector $Z^\varepsilon = (Z_1^\varepsilon, Z_2^\varepsilon)$ is independent of (G_1, G_2). Clearly,

$$I_\varepsilon(g)^2 = 2\iint_{0<u<v<1} g(B^H_u) g(B^H_v) \frac{(B^H_{u+\varepsilon} - B^H_u)^3}{\varepsilon} \frac{(B^H_{v+\varepsilon} - B^H_v)^3}{\varepsilon}\, du\, dv.$$

Hence,

$$\begin{aligned} &\mathrm{E}[I_\varepsilon(g)^2] \\ &\quad = 2\mathrm{E}\biggl\{\iint_{0<u<v<1} g(G_1)g(G_2)\,\mathrm{E}\biggl(\frac{(G_3^\varepsilon)^3(G_4^\varepsilon)^3}{\varepsilon^2}\Big| G_1, G_2\biggr)\, du\, dv\biggr\}. \end{aligned} \tag{5.10}$$

Therefore, we need to compute the conditional expectation in (5.10). For that reason, we need the following lemma, which will be useful again in Step IV(2), where we prove the existence of the L^2-limit of I_ε. For random variables $\xi, \zeta, \phi_\varepsilon$, we will denote

$$\xi \overset{\text{(law)}}{=} \zeta + o(\varepsilon) \qquad \text{as } \varepsilon \downarrow 0,\ \text{if } \xi \overset{\text{(law)}}{=} \zeta + \varepsilon\phi_\varepsilon,\ \text{with } \mathrm{E}\Bigl[\sup_{0<\varepsilon<1} |\phi_\varepsilon|^p\Bigr] < \infty\ \forall p.$$

LEMMA 5.1. *Consider the Gaussian mean-zero random vector*

$$\begin{aligned} G &= \big(G_1(u), G_2(v), G_3^\varepsilon(u), G_4^\varepsilon(v)\big) \\ &:= (B^H_u, B^H_v, B^H_{u+\varepsilon} - B^H_u, B^H_{v+\varepsilon} - B^H_v) \end{aligned} \tag{5.11}$$

and denote

$$\begin{pmatrix}\lambda_{11} & \lambda_{12}\\ \lambda_{12} & \lambda_{22}\end{pmatrix} := \begin{pmatrix} u^{2H} & K_H(u,v)\\ K_H(v,u) & v^{2H}\end{pmatrix}^{-1} = \mathrm{Cov}^{-1}_{(G_1,G_2)}, \tag{5.12}$$

$$\begin{aligned} Q_1(u,v) &:= -\tfrac{1}{2}(\lambda_{11}G_1 + \lambda_{12}G_2),\\ Q_2(u,v) &:= -\tfrac{1}{2}(\lambda_{12}G_1 + \lambda_{22}G_2).\end{aligned} \tag{5.13}$$

(a) *For* $\frac{1}{4} \le H < \frac{1}{3}$, *as* $\varepsilon \downarrow 0$,

$$\mathrm{E}\biggl(\frac{(G_3^\varepsilon)^3(G_4^\varepsilon)^3}{\varepsilon^2}\Big|G_1, G_2\biggr) \overset{\text{(law)}}{=} \varepsilon^{8H-2}\biggl(9Q_1Q_2 - \frac{9}{4}\lambda_{12} + o(1)\biggr); \tag{5.14}$$

(a′) *for* $\frac{1}{4} \le H < \frac{1}{3}$, *as* $\varepsilon \downarrow 0$,

$$\begin{aligned} \mathrm{E}\biggl(\frac{(G_3^\varepsilon)^3}{\varepsilon}\Big|G_1, G_2\biggr) &\overset{\text{(law)}}{=} \varepsilon^{4H-1}\bigl(3Q_1 + o(1)\bigr),\\ \mathrm{E}\biggl(\frac{(G_4^\varepsilon)^3}{\varepsilon}\Big|G_1, G_2\biggr) &\overset{\text{(law)}}{=} \varepsilon^{4H-1}\bigl(3Q_2 + o(1)\bigr).\end{aligned} \tag{5.15}$$

(b) *Denote* $G_4^\delta(v) = B^H_{v+\delta} - B^H_v$ *and* $G_1, G_2, G_3^\varepsilon$ *as previously. Then, for* $H = \frac{1}{4}$, *as* $\varepsilon \downarrow 0, \delta \downarrow 0$,

$$\mathrm{E}\biggl(\frac{(G_3^\varepsilon)^3(G_4^\delta)^3}{\varepsilon\delta}\Big|G_1, G_2\biggr) \overset{\text{(law)}}{=} 9Q_1Q_2 - \frac{9}{4}\lambda_{12} + o(1). \tag{5.16}$$

(c) *Equivalents in* (5.14), (5.15) *and* (5.16) *are uniform on* $\{1 < u, 1 < v - u\}$.

(d) *For* $\kappa > 0$,

$$\begin{aligned} &\bigl(G_1(\kappa u), G_2(\kappa v), G_3^{\kappa\varepsilon}(\kappa u), G_4^{\kappa\varepsilon}(\kappa v)\bigr)\\ &\quad \overset{\text{(law)}}{=} \kappa^H\bigl(G_1(u), G_2(v), G_3^\varepsilon(u), G_4^\varepsilon(v)\bigr)\end{aligned} \tag{5.17}$$

and

$$\begin{aligned} &\bigl(G_1(\kappa u), G_2(\kappa v), Q_1(\kappa u, \kappa v)Q_2(\kappa u, \kappa v) - \tfrac{1}{4}\lambda_{12}(\kappa u, \kappa v)\bigr)\\ &\quad \overset{\text{(law)}}{=} \Bigl(\kappa^H G_1(u), \kappa^H G_2(v), \kappa^{-2H}\bigl(Q_1(u,v)Q_2(u,v) - \tfrac{1}{4}\lambda_{12}(u,v)\bigr)\Bigr).\end{aligned} \tag{5.18}$$

REMARK 5.2. The computation of limits when ε or (ε, δ) go to 0 requires asymptotic equivalent expressions of the conditional expectations [parts (a) and (b) of Lemma 5.1]. However, since we have to integrate on the domain $\{0 < u < v < 1\}$, we need to check that those are uniform on u, v [see part (c) of Lemma 5.1].

We postpone the proof of Lemma 5.1 and we finish the proof of (3.6). Let $0 < \rho < 1$. The second moment of $I_\varepsilon(g)$ can be written as

$$\begin{aligned}\mathrm{E}\left[\frac{1}{2}I_\varepsilon^2(g)\right] &= \iint_{0<u<\varepsilon^{1-\rho},u<v<1} \mathrm{E}\left\{g(G_1)g(G_2)\frac{(G_3^\varepsilon)^3(G_4^\varepsilon)^3}{\varepsilon^2}\right\}du\,dv \\ &\quad + \iint_{0<v-u<\varepsilon^{1-\rho},0<u,v<1} \mathrm{E}\left\{g(G_1)g(G_2)\frac{(G_3^\varepsilon)^3(G_4^\varepsilon)^3}{\varepsilon^2}\right\}du\,dv \\ &\quad + \iint_{\varepsilon^{1-\rho}<u<1,\varepsilon^{1-\rho}<v-u<1,v<1} \mathrm{E}\left\{g(G_1)g(G_2)\frac{(G_3^\varepsilon)^3(G_4^\varepsilon)^3}{\varepsilon^2}\right\}du\,dv.\end{aligned}$$

Using assumptions on g, we can bound the first term by

$$\mathrm{const}\iint_{0<u<\varepsilon^{1-\rho},u<v<1}\frac{\varepsilon^{3H}\varepsilon^{3H}}{\varepsilon^2}\,du\,dv = \mathrm{const}\;\varepsilon^{6H-2+1-\rho}.$$

In the rest of this step, we will use in a significant way part (d) of Lemma 5.1.

Choosing $0 < \rho < 6H - 1$, we can see that the first term converges to 0, as $\varepsilon \downarrow 0$. A similar reasoning implies that the second term also converges to 0. Let us denote $\varepsilon^{1-\rho} = \kappa$ and $\varepsilon^\rho = \tilde{\varepsilon}$ (hence $\varepsilon = \kappa\tilde{\varepsilon}$). In the third term, we make the change of variables $u = \kappa\tilde{u}$ and $v = \kappa\tilde{v}$. Hence, as $\varepsilon \downarrow 0$,

$$\begin{aligned}&\iint_{\kappa<u<1,\kappa<v-u<1,v<1} \mathrm{E}\left\{g(G_1(u))g(G_2(v))\frac{(G_3^\varepsilon(u))^3(G_4^\varepsilon(v))^3}{\varepsilon^2}\right\}du\,dv \\ &\quad = \iint_{1<\tilde{u}<1/\kappa,1<\tilde{v}-\tilde{u}<1/\kappa,\tilde{v}<1/\kappa} \\ &\qquad\quad \times \mathrm{E}\left\{g(G_1(\kappa\tilde{u}))g(G_2(\kappa\tilde{v}))\frac{(G_3^{\kappa\tilde{\varepsilon}}(\kappa\tilde{u}))^3(G_4^{\kappa\tilde{\varepsilon}}(\kappa\tilde{v}))^3}{\kappa^2\tilde{\varepsilon}^2}\right\}\kappa^2\,d\tilde{u}\,d\tilde{v} \\ &\quad \overset{(5.17)}{=} \iint_{1<\tilde{u}<1/\kappa,1<\tilde{v}-\tilde{u}<1/\kappa,\tilde{v}<1/\kappa} \\ &\qquad\quad \times \mathrm{E}\left\{g(\kappa^H G_1(\tilde{u}))g(\kappa^H G_2(\tilde{v}))\frac{\kappa^{6H}(G_3^{\tilde{\varepsilon}}(u))^3(G_4^{\tilde{\varepsilon}}(v))^3}{\tilde{\varepsilon}^2}\right\}d\tilde{u}\,d\tilde{v} \\ &\quad = \iint_{1<\tilde{u}<1/\kappa,1<\tilde{v}-\tilde{u}<1/\kappa,\tilde{v}<1/\kappa} \\ &\qquad\quad \times \mathrm{E}\Bigg\{g(\kappa^H G_1(\tilde{u}))g(\kappa^H G_2(\tilde{v}))\kappa^{6H} \\ &\qquad\qquad \times \mathrm{E}\left(\frac{(G_3^{\tilde{\varepsilon}}(u))^3(G_4^{\tilde{\varepsilon}}(v))^3}{\tilde{\varepsilon}^2}\Big|G_1(\tilde{u}),G_2(\tilde{v})\right)\Bigg\}d\tilde{u}\,d\tilde{v}\end{aligned}$$

$$\begin{aligned}
&\overset{(5.14)}{\sim} \iint_{1<\tilde{u}<1/\kappa,1<\tilde{v}-\tilde{u}<1/\kappa,\tilde{v}<1/\kappa} \\
&\qquad \times \mathrm{E}\biggl\{g(\kappa^H G_1(\tilde{u}))g(\kappa^H G_2(\tilde{v}))\kappa^{6H}\tilde{\varepsilon}^{8H-2} \\
&\qquad\qquad \times \biggl(9Q_1(\tilde{u},\tilde{v})Q_2(\tilde{u},\tilde{v}) - \frac{9}{4}\lambda_{12}(\tilde{u},\tilde{v})\biggr)\biggr\}\, d\tilde{u}\, d\tilde{v} \\
&= \iint_{\kappa<u<1,\kappa<v-u<1,v<1} \\
&\qquad \times \mathrm{E}\biggl\{g\biggl(\kappa^H G_1\Bigl(\frac{u}{\kappa}\Bigr)\biggr)g\biggl(\kappa^H G_2\Bigl(\frac{v}{\kappa}\Bigr)\biggr)(\kappa\tilde{\varepsilon})^{6H}\tilde{\varepsilon}^{2H-2} \\
&\qquad\qquad \times \biggl(9Q_1\Bigl(\frac{u}{\kappa},\frac{v}{\kappa}\Bigr)Q_2\Bigl(\frac{u}{\kappa},\frac{v}{\kappa}\Bigr) - \frac{9}{4}\lambda_{12}\Bigl(\frac{u}{\kappa},\frac{v}{\kappa}\Bigr)\biggr)\biggr\}\frac{du\, dv}{\kappa^2} \\
&\overset{(5.18)}{=} \iint_{\kappa<u<1,\kappa<v-u<1,v<1} \\
&\qquad \times \mathrm{E}\biggl\{g(G_1(u))g(G_2(v))(\kappa\tilde{\varepsilon})^{6H}\tilde{\varepsilon}^{2H-2}\kappa^{2H-2} \\
&\qquad\qquad \times \biggl(9Q_1(u,v)Q_2(u,v) - \frac{9}{4}\lambda_{12}(u,v) + o(1)\biggr)\biggr\}\, du\, dv \\
&= \varepsilon^{8H-2}\iint_{\kappa<u<1,\kappa<v-u<1,v<1} \\
&\qquad\qquad \times \mathrm{E}\biggl\{g(G_1(u))g(G_2(v)) \\
&\qquad\qquad\qquad \times \biggl(9Q_1(u,v)Q_2(u,v) - \frac{9}{4}\lambda_{12}(u,v) + o(1)\biggr)\biggr\}\, du\, dv,
\end{aligned}$$

where we have also used part (c) of Lemma 5.1 to replace the conditional expectation by the uniform equivalent asymptotics in (5.14) on $\{1 < \tilde{u}, 1 < \tilde{v} - \tilde{u}\}$. Therefore, as $\varepsilon \downarrow 0$,

$$\begin{aligned}
\mathrm{E}[I_\varepsilon(g)^2] \sim \varepsilon^{8H-2}\mathrm{E}\biggl\{\tfrac{9}{2}\iint du\, dv\, g(G_1)g(G_2) \\
\times \bigl((\lambda_{11}G_1 + \lambda_{12}G_2)(\lambda_{12}G_1 + \lambda_{22}G_2) - \lambda_{12}\bigr)\biggr\}.
\end{aligned}$$

Equation (3.6) follows from the above expression. Moreover,

$$\lim_{\varepsilon\downarrow 0}\mathrm{E}[I_\varepsilon(g)^2] = 0 \qquad \text{if } \tfrac{1}{4} < H < \tfrac{1}{3}, \tag{5.19}$$

which together with (5.8) gives (a) of Theorem 3.4. $\square$

STEP III [Absolute convergence of the integrals in (3.5) and (3.6)]. The absolute convergence of the integral on the right-hand side of (3.5) is already explained by the reasoning given in Step I. We need to justify, however, the absolute convergence of the integral on the right-hand side of (3.6), which means

$$J := \iint_{0<u<v<1} du\,dv\,\mathrm{E}\big|g(B_u^{1/4})g(B_v^{1/4})\big(\lambda_{11}\lambda_{12}(B_u^{1/4})^2 + (\lambda_{11}\lambda_{22}+\lambda_{12}^2)B_u^{1/4}B_v^{1/4} + \lambda_{12}\lambda_{22}(B_v^{1/4})^2 - \lambda_{12}\big)\big| < \infty.$$

We can write $J = J_1 + J_2 + J_3 + J_4$, where

$$J_i := \iint_{0<u<v<1} \mathrm{E}\big(|\mathcal{E}_i(u,v)|\big)\,du\,dv, \qquad i = 1,2,3,$$

$$J_4 := \iint_{0<u<v<1} \mathrm{E}\big(|g(B_u^{1/4})g(B_v^{1/4})\lambda_{12}|\big)\,du\,dv,$$

where

$$\begin{aligned}
\mathcal{E}_1(u,v) &= g(B_u^{1/4})g(B_v^{1/4})(\lambda_{11}\lambda_{12}+\lambda_{11}\lambda_{22}+\lambda_{12}^2+\lambda_{12}\lambda_{22})(B_u^{1/4})^2,\\
\mathcal{E}_2(u,v) &= g(B_u^{1/4})g(B_v^{1/4})(\lambda_{11}\lambda_{22}+\lambda_{12}^2)B_u^{1/4}(B_v^{1/4}-B_u^{1/4}),\\
\mathcal{E}_3(u,v) &= g(B_u^{1/4})g(B_v^{1/4})\lambda_{12}\lambda_{22}(B_v^{1/4}+B_u^{1/4})(B_v^{1/4}-B_u^{1/4}).
\end{aligned} \tag{5.20}$$

We set $v = u(1+\eta)$ so that

$$J_i = \iint_{0<u<1,0<\eta<1/u-1} \mathrm{E}(|\mathcal{E}_i(u,\eta)|)u\,du\,d\eta, \qquad i = 1,2,3,$$

$$J_4 = \iint_{0<u<1,0<\eta<1/u-1} \mathrm{E}\big(|g(B_u^{1/4})g\big(B_{u(1+\eta)}^{1/4}\big)\lambda_{12}(u,\eta)|\big)u\,du\,d\eta.$$

We introduce the following notation:

$$K_{1/4}\big(u,u(1+\eta)\big) = \sqrt{u}\hat{K}(\eta), \quad \text{with } \hat{K}(\eta) := \tfrac{1}{2}\big(1+\sqrt{1+\eta}-\sqrt{\eta}\big),$$

$$\sqrt{u\cdot u(1+\eta)} - K_{1/4}^2\big(u,u(1+\eta)\big) = u\hat{\Delta}(\eta), \quad \text{with } \hat{\Delta}(\eta) := \sqrt{1+\eta} - \hat{K}^2(\eta).$$

We note that

$$\hat{K}(\eta) \sim 1 \text{ as } \eta \downarrow 0 \quad \text{and} \quad \hat{K}(\eta) \sim \tfrac{1}{2} \quad \text{as } \eta \uparrow \infty,$$

$$\hat{\Delta}(\eta) \sim \sqrt{\eta} \text{ as } \eta \downarrow 0 \quad \text{or} \quad \text{as } \eta \uparrow \infty.$$

Using (3.7), we can write

$$\lambda_{11} = \frac{1}{\sqrt{u}} \frac{\sqrt{1+\eta}}{\hat{\Delta}(\eta)}, \qquad \lambda_{22} = \frac{1}{\sqrt{u}} \frac{1}{\hat{\Delta}(\eta)}, \qquad \lambda_{12} = -\frac{1}{\sqrt{u}} \frac{\hat{K}(\eta)}{\hat{\Delta}(\eta)}.$$

We can now prove that each J_i is a convergent double integral. To illustrate this fact, we prove the convergence of J_2, the computation being similar for the other integrals J_i. We recall that

$$\begin{aligned} J_2 &= \iint_{0<u<1, 0<\eta<1/u-1} \mathrm{E}\big(|\lambda_{11}\lambda_{22} + \lambda_{12}^2| \\ &\qquad \times \big|g(B_u^{1/4}) g\big(B_{u(1+\eta)}^{1/4}\big) B_u^{1/4}\big(B_{u(1+\eta)}^{1/4} - B_u^{1/4}\big)\big|\big) u\, du\, d\eta \\ &= \iint_{0<u<1, 0<\eta<1/u-1} \frac{\sqrt{1+\eta} + \hat{K}^2(\eta)}{\hat{\Delta}^2(\eta)} \\ &\qquad \times \mathrm{E}\big(\big|g(B_u^{1/4}) g\big(B_{u(1+\eta)}^{1/4}\big) B_u^{1/4}\big(B_{u(1+\eta)}^{1/4} - B_u^{1/4}\big)\big|\big)\, du\, d\eta. \end{aligned}$$

By the Cauchy–Schwarz inequality and taking in to account the assumption on g, we can write

$$\mathrm{E}\big|g(B_u^{1/4}) g\big(B_{u(1+\eta)}^{1/4}\big) B_u^{1/4}\big(B_{u(1+\eta)}^{1/4} - B_u^{1/4}\big)\big| \le \mathrm{const}\, u^{1/2} \eta^{1/4}.$$

On the other hand,

$$\frac{\sqrt{1+\eta} + \hat{K}^2(\eta)}{\hat{\Delta}^2(\eta)} \sim \frac{2}{\eta} \qquad \text{as } \eta \downarrow 0$$

and

$$\frac{\sqrt{1+\eta} + \hat{K}^2(\eta)}{\hat{\Delta}^2(\eta)} \sim \frac{1}{\sqrt{\eta}} \qquad \text{as } \eta \uparrow \infty.$$

Hence, we now need to study respectively the integrals

$$\iint_{0<u<1, 0<\eta<1} \frac{u^{1/2}}{\eta^{3/4}}\, du\, d\eta < \infty,$$

$$\begin{aligned} \iint_{0<u<1, 1<\eta<1/u-1} \frac{u^{1/2}}{\eta^{1/4}}\, du\, d\eta &= \int_1^\infty \frac{d\eta}{\eta^{1/4}} \int_0^{1/(\eta+1)} u^{1/2}\, du \\ &= \frac{2}{3} \int_1^\infty \frac{d\eta}{\eta^{1/4}(\eta+1)^{3/2}} < \infty. \end{aligned}$$

This concludes the proof of part (c) of Theorem 3.4. □

STEP IV (Proof of the forward third-order integral existence).

(1) (Reduction to the case of a bounded function g). Suppose, for a moment, that we know the result when g is bounded. Since the paths of $B^{1/4}$ are continuous,

we prove by localization that the result is true when g is only locally bounded. Let $\alpha > 0$. We will show that $\{I_\varepsilon(g) : \varepsilon > 0\}$ is Cauchy with respect to the convergence in probability, that is,

$$\lim_{\varepsilon\downarrow 0,\delta\downarrow 0} \mathrm{P}(|I_\varepsilon(g) - I_\delta(g)| \geq \alpha) = 0.$$

Let $M > 0$, $\Omega_M = \{|B_u^{1/4}| \leq M,\ \forall u \in [0, t+1]\}$. On Ω_M, we have $I_\varepsilon(g) = I_\varepsilon(g_M)$ and $I_\delta(g) = I_\delta(g_M)$, where g_M is a function with compact support, which coincides on g on the compact interval $[-M, M]$.

Therefore, $\mathrm{P}(\{|I_\varepsilon(g) - I_\delta(g)| \geq \alpha\} \cap \Omega_M^c) \leq \mathrm{P}(\Omega_M^c)$. We choose M large enough, so that $\mathrm{P}(\Omega_M^c)$ is uniformly small with respect to ε and δ. Then

$$\begin{aligned}\mathrm{P}\big(\{|I_\varepsilon(g) - I_\delta(g)| \geq \alpha\} \cap \Omega_M\big) &= \mathrm{P}\big(\{|I_\varepsilon(g_M) - I_\delta(g_M)| \geq \alpha\} \cap \Omega_M\big) \\ &\leq \mathrm{P}\big(|I_\varepsilon(g_M) - I_\delta(g_M)| \geq \alpha\big).\end{aligned}$$

Since g_M has compact support, $I_\varepsilon(g_M)$ converges in probability.

(2) (Proof of the existence when g is a bounded function). Thus, it remains to prove that the sequence $\{I_\varepsilon(g) : \varepsilon > 0\}$ converges in probability, when g is bounded. For this purpose, we even show that, in this case, the sequence is even Cauchy in $L^2(\Omega)$.

We will prove the Cauchy criterion for $\{I_\varepsilon(g) : \varepsilon > 0\}$:

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0,\delta\downarrow 0} \mathrm{E}\big(|I_\varepsilon(g) - I_\delta(g)|^2\big) \\ &\qquad = \lim_{\varepsilon\downarrow 0,\delta\downarrow 0} \mathrm{E}[I_\varepsilon(g)^2] + \mathrm{E}[I_\delta(g)^2] - 2\mathrm{E}[I_\varepsilon(g) I_\delta(g)] = 0.\end{aligned}$$

The first two terms converge to the same limit given in (3.6) as $\varepsilon \downarrow 0$ and $\delta \downarrow 0$. It remains to show that $\lim_{\varepsilon\downarrow 0,\delta\downarrow 0} \mathrm{E}[I_\varepsilon(g) I_\delta(g)]$ equals the right-hand side of (3.6), and then the Cauchy criterion will be fulfilled. A simple change of variable gives

$$\begin{aligned}&I_\varepsilon(g) I_\delta(g) \\ &\quad = \iint_{0<u<v<1} g(B_u^{1/4}) g(B_v^{1/4}) \frac{(B_{u+\varepsilon}^{1/4} - B_u^{1/4})^3}{\varepsilon} \frac{(B_{v+\delta}^{1/4} - B_v^{1/4})^3}{\delta}\, du\, dv \\ &\qquad + \iint_{0<u<v<1} g(B_u^{1/4}) g(B_v^{1/4}) \frac{(B_{u+\delta}^{1/4} - B_u^{1/4})^3}{\delta} \frac{(B_{v+\varepsilon}^{1/4} - B_v^{1/4})^3}{\varepsilon}\, du\, dv.\end{aligned}$$

Taking the expectation of the expression above gives

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0,\delta\downarrow 0} \mathrm{E}[I_\varepsilon(g) I_\delta(g)] \\ &\qquad = 2 \lim_{\varepsilon\downarrow 0,\delta\downarrow 0} \mathrm{E}\biggl\{\iint_{0<u<v<1} g(G_1) g(G_2) \mathrm{E}\biggl(\frac{(G_3^\varepsilon)^3 (G_4^\delta)}{\varepsilon\delta}\Big| G_1, G_2\biggr)\, du\, dv\biggr\}\end{aligned}$$

so that the result will be a consequence of (5.16). □

STEP V (Proof of the existence of a Hölder-continuous version). It is enough to show the existence of a continuous version for $t \in [0, T]$ for any $T > 0$.

Suppose, for a moment, that for every g bounded we can show the existence of a (Hölder) continuous version for $(\int_0^t g(B_u^{1/4})\, d^{-3} B_u^{1/4})_{t\in[0,T]}$. We denote it by $(\tilde{I}(g)_t)_{t\in[0,T]}$. Then we can define the associated version for a general $g \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ by

$$\tilde{I}(g)(\omega) = \tilde{I}(g^M)(\omega),$$

where $g^M = g\mathbf{1}_{[-M,M]}$ if $\omega \in \{\sup_{t\in[0,T]} : |B_t^{1/4}| \le M\}$. Therefore, it remains to prove that the forward third-order integral has a Hölder-continuous version (with Hölder parameter less than $\frac{1}{4}$), when g is bounded and continuous.

We prove that the L^2-valued function $t \mapsto I(g)(t) := \int_0^t g(B_u^{1/4})\, d^{-3} B_u^{1/4}$ has a Hölder-continuous version on $[0, T]$. We need to control, for $s < t$, s, t in compact intervals,

$$\begin{aligned}
&\mathrm{E}\bigl[\bigl(I(g)(t) - I(g)(s)\bigr)^2\bigr] \\
&\quad = \mathrm{E}\biggl[\biggl(\int_s^t g(B_u^{1/4})\, d^{-3} B_u^{1/4}\biggr)^2\biggr] \\
&\quad \le \iint_{s\le u<v\le t} du\, dv\, \mathrm{E}\bigl[|g(B_u^{1/4}) g(B_v^{1/4})| \\
&\qquad\qquad \times |\mathcal{E}_1(u,v) + \mathcal{E}_2(u,v) + \mathcal{E}_3(u,v) - \lambda_{12}|\bigr],
\end{aligned}$$

where $\mathcal{E}_i(u, v)$, $i = 1, 2, 3$, are given by (5.20). Let us denote

$$\begin{aligned}
\mathcal{E}_1(u,v) &= \tilde{\mathcal{E}}_1(u,v)(B_u^{1/4})^2, \\
\mathcal{E}_2(u,v) &= \tilde{\mathcal{E}}_2(u,v) B_u^{1/4}(B_v^{1/4} - B_u^{1/4}), \\
\mathcal{E}_3(u,v) &= \tilde{\mathcal{E}}_3(u,v)(B_v^{1/4} + B_u^{1/4})(B_v^{1/4} - B_u^{1/4}).
\end{aligned}$$

We denote again $\eta = v - u$. Therefore,

$$\begin{aligned}
\tilde{\mathcal{E}}_1(u, u+\eta) &= \lambda_{11}\lambda_{12} + \lambda_{11}\lambda_{22} + \lambda_{12}^2 + \lambda_{12}\lambda_{22} \\
&= \frac{1}{2\Delta^2}\eta \frac{\sqrt{u}}{\sqrt{u+\eta} + \sqrt{u}} = \frac{1}{2\Delta^2}\eta \frac{\sqrt{u/\eta}}{\sqrt{1+u/\eta} + \sqrt{u/\eta}}, \\
\tilde{\mathcal{E}}_2(u, u+\eta) &= \lambda_{11}\lambda_{22} + \lambda_{12}^2 \\
&= \frac{1}{2\Delta^2}\bigl(u + \eta + 3\sqrt{u}\sqrt{u+\eta} - \sqrt{u}\sqrt{\eta} - \sqrt{\eta}\sqrt{u+\eta}\bigr), \\
\tilde{\mathcal{E}}_3(u, u+\eta) &= \lambda_{12}\lambda_{22} \\
&= -\frac{1}{2\Delta^2} u \biggl(1 + \frac{\sqrt{u}}{\sqrt{u+\eta} + \sqrt{\eta}}\biggr)
\end{aligned}$$

$$= -\frac{1}{2\Delta^2} u\left(1 + \frac{\sqrt{u/\eta}}{1+\sqrt{1+u/\eta}}\right) - \lambda_{12}$$
$$= \frac{1}{2\Delta}\sqrt{u}\left(1 + \frac{\sqrt{u}}{\sqrt{u+\eta}+\sqrt{\eta}}\right)$$
$$= \frac{1}{2\Delta}\sqrt{u}\left(1 + \frac{\sqrt{u/\eta}}{1+\sqrt{1+u/\eta}}\right),$$

where

$$\Delta := \sqrt{u(u+\eta)} - K_H^2(u, u+\eta)$$
$$= \frac{1}{2}\sqrt{u}\sqrt{\eta}\left(1 + \frac{\sqrt{u}}{\sqrt{u+\eta}+\sqrt{\eta}} + \frac{\sqrt{\eta}}{\sqrt{u+\eta}+\sqrt{u}}\right)$$
$$\geq \frac{1}{2}\sqrt{u}\sqrt{\eta}.$$

The functions $\psi_1(x) = \sqrt{x}/(\sqrt{x}+\sqrt{1+x})$, respectively, $\psi_2(x) = \sqrt{x}/(1+\sqrt{1+x})$, are positive increasing on $[0,+\infty[$ with limit $\frac{1}{2}$, respectively, 1, as $x \uparrow \infty$. Moreover, we see that $\sqrt{u+\eta} \leq \sqrt{u}+\sqrt{\eta}$. Therefore,

$$0 \leq \tilde{\mathcal{E}}_1(u, u+\eta) \leq \frac{1}{u}, \qquad |\tilde{\mathcal{E}}_2(u, u+\eta)| \leq \frac{8}{\eta} + \frac{4}{u} + \frac{10}{\sqrt{u}\sqrt{\eta}},$$
$$|\tilde{\mathcal{E}}_2(u, u+\eta)| \leq \frac{4}{\eta}, \qquad 0 \leq -\lambda_{12} \leq \frac{2}{\sqrt{\eta}}.$$

Hence,

$$\iint_{s\leq u<v\leq t} \mathrm{E}\big[|g(B_u^{1/4})g(B_v^{1/4})|\,|\tilde{\mathcal{E}}_1(u,v)|(B_u^{1/4})^2\big]\,du\,dv$$
$$\leq \text{const} \iint_{s\leq u\leq t, 0<\eta\leq t-s} \frac{du\,d\eta}{\sqrt{u}} = \text{const}\,(t-s)^{3/2},$$

$$\iint_{s\leq u<v\leq t} \mathrm{E}\big[|g(B_u^{1/4})g(B_v^{1/4})|\,|\tilde{\mathcal{E}}_2(u,v)|\,|B_u^{1/4}(B_v^{1/4}-B_u^{1/4})|\big]\,du\,dv$$
$$\leq \text{const} \iint_{s\leq u\leq t, 0<\eta\leq t-s} \left(8\frac{u^{1/4}}{\eta^{3/4}} + 4\frac{\eta^{1/4}}{u^{3/4}} + 10\frac{1}{u^{1/4}\eta^{1/4}}\right) du\,d\eta$$
$$= \text{const}\big(8(t^{5/4}-s^{5/4})(t-s)^{1/4}$$
$$+ 4(t^{1/4}-s^{1/4})(t-s)^{5/4} + 10(t^{3/4}-s^{3/4})(t-s)^{3/4}\big)$$
$$\leq \text{const}\,(t-s)^{3/2-\rho},$$

where $\rho > 0$,

$$\iint_{s\le u<v\le t} \mathrm{E}\big[|g(B_u^{1/4})g(B_v^{1/4})|\,|\mathcal{E}_3(u,v)|\,|(B_v^{1/4}+B_u^{1/4})(B_v^{1/4}-B_u^{1/4})|\big]\,du\,dv$$
$$\le \text{const} \iint_{s\le u\le t, 0<\eta\le t-s} \frac{du\,d\eta}{\sqrt{u}\,\eta^{3/4}} = \text{const}\,(t-s)^{5/4}$$

and

$$\iint_{s\le u<v\le t} \mathrm{E}\big[|g(B_u^{1/4})g(B_v^{1/4})|\,|\lambda_{12}|\big]\,du\,dv \le \text{const}\,(t-s)^{3/2}.$$

Therefore,

$$\mathrm{E}\big[\big(I(g)(t)-I(g)(s)\big)^2\big] \le \text{const}\,(t-s)^{1+1/2-\rho} \qquad \text{with } \rho>0.$$

The classical Kolmogorov criterion allows us to conclude the proof. □

STEP VI [Proof of (3.4) and part (d)]. It is not easy to make computations or to recognize the positivity using the right-hand side of the second moment of the third-order integrals; see (3.6). We need to give other expression of the second moment but also to compute its covariance with the integral in part (d). This will be possible when g is smooth. Using Proposition 3.6 and an obvious approximation argument, it is enough to suppose that $g \in C^1(\mathbb{R})$ with g and g' bounded.

Since the third-order integrals are continuous, to prove (3.4) we need only to verify that, for fixed $t > 0$,

$$\mathrm{E}\bigg(\int_0^t g(B_u^{1/4})\,d^{\pm 3}B_u^{1/4} \mp \tfrac{3}{2}\int_0^t g'(B_u^{1/4})\,du\bigg)^2 = 0. \tag{5.21}$$

This equality is a simple consequence of the following lemma.

LEMMA 5.3. *Let g and h be real functions, $g \in C^1(\mathbb{R})$ and h locally bounded such that g, g', h fulfill the subexponential inequality* (3.3). *The following equalities hold*:

$$\mathrm{E}\bigg\{\bigg(\int_0^t g(B_u^{1/4})\,d^{\pm 3}B_u^{1/4}\bigg)^2\bigg\} = \tfrac{9}{4}\mathrm{E}\bigg\{\bigg(\int_0^t g'(B_u^{1/4})\,du\bigg)^2\bigg\} \tag{5.22}$$

and

$$\begin{aligned}&\mathrm{E}\bigg\{\bigg(\int_0^t g(B_u^{1/4})\,d^{\pm 3}B_u^{1/4}\bigg)\bigg(\int_0^t h(B_u^{1/4})\,du\bigg)\bigg\}\\ &\quad= \mp\tfrac{3}{2}\mathrm{E}\bigg\{\bigg(\int_0^t g'(B_u^{1/4})\,du\bigg)\bigg(\int_0^t h(B_u^{1/4})\,du\bigg)\bigg\}.\end{aligned} \tag{5.23}$$

Finally, by (2.15) we also get the statement in part (d). □

This completes the proof of Theorem 3.4, and we can proceed to the proof of Lemma 5.3.

PROOF OF (5.22) IN LEMMA 5.3. To simplify the notation, we write K for $K_{1/4}(u,v)$ and Δ for $\sqrt{uv}-K^2$. Hence,

$$\lambda_{11}=\frac{\sqrt{v}}{\Delta}, \qquad \lambda_{22}=\frac{\sqrt{u}}{\Delta}, \qquad \lambda_{12}=-\frac{K}{\Delta}.$$

Let us introduce the matrix

$$M=\begin{pmatrix} u^{1/4} & 0 \\ K/u^{1/4} & \sqrt{\Delta}/u^{1/4}\end{pmatrix} \qquad \text{with } M^{-1}=\begin{pmatrix} u^{1/4} & 0 \\ -u^{-1/4}K/\sqrt{\Delta} & u^{1/4}/\sqrt{\Delta}\end{pmatrix}$$

and observe that, by (5.12), MM^* is the covariance matrix of $(B_u^{1/4}, B_v^{1/4})$. Furthermore, if N_1 and N_2 are two independent standard normal random variables, then

$$\begin{pmatrix} B_u^{1/4} \\ B_v^{1/4}\end{pmatrix} = M\begin{pmatrix} N_1 \\ N_2\end{pmatrix}.$$

After some algebraic computations, we obtain

$$\begin{aligned}
&\lambda_{11}\lambda_{12}(B_u^{1/4})^2+(\lambda_{11}\lambda_{22}+\lambda_{12}^2)B_u^{1/4}B_v^{1/4}+\lambda_{12}\lambda_{22}(B_v^{1/4})^2-\lambda_{12}\\
&\quad=\left((M^{-1})^*\begin{pmatrix} N_1 \\ N_2\end{pmatrix}\right)_1\cdot\left((M^{-1})^*\begin{pmatrix} N_1 \\ N_2\end{pmatrix}\right)_2-\left((M^{-1})^*M^{-1}\right)_{12}\\
&\quad=\frac{N_1N_2}{\sqrt{\Delta}}-\frac{K\,N_2^2}{\Delta}+\frac{K}{\Delta}.
\end{aligned}$$

Therefore, by (3.6), for $t=1$,

$$\begin{aligned}
&\mathrm{E}\left\{\left(\int_0^1 g(B_u^{1/4})\,d^{-3}B_u^{1/4}\right)^2\right\}\\
&\quad=\frac{9}{2}\iint_{0<u<v<1} du\,dv\,\mathrm{E}\bigg[g(u^{1/4}N_1)g\left(\frac{K}{u^{1/4}}N_1+\frac{\sqrt{\Delta}}{u^{1/4}}N_2\right)\\
&\qquad\qquad\times\left(\frac{N_1N_2}{\sqrt{\Delta}}-\frac{K\,N_2^2}{\Delta}+\frac{K}{\Delta}\right)\bigg]\\
&\quad=\frac{9}{2}\iint_{0<u<v<1} du\,dv\,\mathrm{E}\left[g'(u^{1/4}N_1)g'\left(\frac{K}{u^{1/4}}N_1+\frac{\sqrt{\Delta}}{u^{1/4}}N_2\right)\right]\\
&\quad=\frac{9}{4}\mathrm{E}\left\{\left(\int_0^1 g'(B_u^{1/4})\,du\right)^2\right\}.
\end{aligned}$$

The second equality is given by the following identity, for $a, b, c \in \mathbb{R}$, $a > 0$,

$$\begin{aligned} &\mathrm{E}\biggl[g(aN_1)g\biggl(\frac{b}{a}N_1 + \frac{c}{a}N_2\biggr)\biggl(\frac{1}{c}N_1N_2 - \frac{b}{c^2}(N_2^2 - 1)\biggr)\biggr] \\ &\qquad = \mathrm{E}\biggl[g'(aN_1)g'\biggl(\frac{b}{a}N_1 + \frac{c}{a}N_2\biggr)\biggr], \end{aligned} \tag{5.24}$$

which can be obtained by direct calculation, using Gaussian densities, the assumption on g and integration by parts. This concludes the proof of (5.22). □

PROOF OF (5.23) IN LEMMA 5.3. We now verify a more general covariance-type equality between the third-order integral $\int_0^1 g(B_u^{1/4})\,d^{-3}B_u^{1/4}$ with a random variable of the form $\int_0^t h(B_u^{1/4})\,du$: *Let g and h be real locally bounded functions fulfilling the subexponential inequality* (3.3). *Then*

$$\begin{aligned} &\mathrm{E}\biggl\{\biggl(\int_0^t g(B_u^{1/4})\,d^{-3}B_u^{1/4}\biggr)\biggl(\int_0^t h(B_u^{1/4})\,du\biggr)\biggr\} \\ &\qquad = -\tfrac{3}{2}\mathrm{E}\biggl\{\int_0^t dv\int_0^t du\, g(B_u^{1/4})h(B_v^{1/4})(\lambda_{11}B_u^{1/4} + \lambda_{12}B_v^{1/4})\biggr\}. \end{aligned} \tag{5.25}$$

Before verifying this result, we prove (5.23). Taking again $t = 1$, (5.25) implies that the left-hand side of (5.23) equals

$$-\frac{3}{2}\int_0^1 dv\int_0^1 du\, g(B_u^{1/4})h(B_v^{1/4})\biggl(\frac{\sqrt{v}}{\Delta}B_u^{1/4} - \frac{K}{\Delta}B_v^{1/4}\biggr), \tag{5.26}$$

where we denote again $K = K_{1/4}(u, v)$, $\Delta = \sqrt{uv} - K^2$. As in the proof of (5.22), we can write

$$B_u^{1/4} = u^{1/4}N_1, \qquad B_v^{1/4} = \frac{K}{u^{1/4}}N_1 + \frac{\sqrt{\Delta}}{u^{1/4}}N_2,$$

where N_1 and N_2 are again independent $N(0, 1)$ random variables. Therefore, (5.27) gives

$$\begin{aligned} &-\frac{3}{2}\int_0^1 dv\int_0^1 du\,\mathrm{E}\biggl\{g(u^{1/4}N_1) \\ &\qquad\qquad \times h\biggl(\frac{K}{u^{1/4}}N_1 + \frac{\sqrt{\Delta}}{u^{1/4}}N_2\biggr)\biggl[\frac{N_1}{u^{1/4}} - \frac{K}{\sqrt{\Delta}u^{1/4}}\biggr]\biggr\}. \end{aligned} \tag{5.27}$$

Similar to identity (5.24), we can establish the following, for $a, b, c \in \mathbb{R}$, $a > 0$,

$$\begin{aligned} &\mathrm{E}\biggl(g(aN_1)h\biggl(\frac{b}{a}N_1 + \frac{c}{a}N_2\biggr)\biggl(\frac{N_1}{a} - \frac{b}{ac}N_2\biggr)\biggr) \\ &\qquad = \mathrm{E}\biggl(g'(aN_1)h\biggl(\frac{b}{a}N_1 + \frac{c}{a}N_2\biggr)\biggr). \end{aligned} \tag{5.28}$$

The proof follows easily again using integration by parts. We apply (5.28) with $a = u^{1/4}$, $b = K$, $c = \sqrt{\Delta}$. Hence, (5.27) gives

$$\begin{aligned}
-\frac{3}{2}\int_0^1 dv \int_0^1 du \, \mathrm{E}\biggl\{g'(u^{1/4}N_1)h\biggl(\frac{K}{u^{1/4}}N_1 + \frac{\sqrt{\Delta}}{u^{1/4}}N_2\biggr)\biggr\} \\
= -\frac{3}{2}\int_0^1 dv \int_0^1 du \, \mathrm{E}\{g'(B_u^{1/4})h(B_v^{1/4})\},
\end{aligned}$$

which is the right-hand side of (5.23).

We come back to the proof of (5.25), and we follow a similar reasoning as for the evaluation of the second moment of the third-order integral; see part (c) of Theorem 3.4. Since $\int_0^1 g(B_u^{1/4})\, d^{-3}B_u^{1/4}$ is the limit in $L^2(\Omega)$ of $I_\varepsilon(g)$,

$$\mathrm{E}\biggl(\int_0^1 g(B_u^{1/4})\, d^{-3}B_u^{1/4} \int_0^1 h(B_v^{1/4})\, dv\biggr) \text{ is the limit of } J_\varepsilon^1 + J_\varepsilon^2,$$

where

$$\begin{aligned}
J_\varepsilon^1 &:= \frac{1}{\varepsilon}\int_0^1 dv \int_0^v du \, \mathrm{E}\bigl(g(B_u^{1/4})(B_{u+\varepsilon}^{1/4} - B_u^{1/4})^3 h(B_v^{1/4})\bigr) \\
&= \int_0^1 \int_0^v du \, \mathrm{E}\biggl(g(G_1)h(G_2)\frac{(G_3^\varepsilon)^3}{\varepsilon}\biggr), \\
J_\varepsilon^2 &:= \frac{1}{\varepsilon}\int_0^1 dv \int_0^v du \, \mathrm{E}\bigl(g(B_v^{1/4})(B_{v+\varepsilon}^{1/4} - B_v^{1/4})^3 h(B_u^{1/4})\bigr) \\
&= \int_0^1 \int_0^v du \, \mathrm{E}\biggl(g(G_2)h(G_1)\frac{(G_4^\varepsilon)^3}{\varepsilon}\biggr)
\end{aligned}$$

using the same notation as for the evaluation of the second moment in part (c). We can write

$$\begin{aligned}
J_\varepsilon^1 &= \int_0^1 \int_0^v du \, \mathrm{E}\biggl\{g(G_1)h(G_2)\mathrm{E}\biggl(\frac{(G_3^\varepsilon)^3}{\varepsilon}\bigg| G_1, G_2\biggr)\biggr\} \\
&= -\frac{3}{2}\{\mathrm{E}[g(G_1)h(G_2)(\lambda_{11}G_1 - \lambda_{12}G_2)] + o(1)\}
\end{aligned}$$

by part (a′) of Lemma 5.1, since $H = \frac{1}{4}$. Moreover, by part (c) of the same lemma, the estimates are uniform in u and v. Therefore, the Lebesgue dominated convergence theorem says that

$$\lim_{\varepsilon \downarrow 0} J_\varepsilon^1 = -\tfrac{3}{2}\int_0^1 dv \int_0^v du \, \mathrm{E}[g(B_u^{1/4})h(B_v^{1/4})(\lambda_{11}B_u^{1/4} + \lambda_{12}B_v^{1/4})].$$

Proceeding similarly for J_ε^2, using again Lemma 5.1, we obtain

$$\lim_{\varepsilon\downarrow 0} J_\varepsilon^2 = -\tfrac{3}{2}\int_0^1 dv \int_0^v du\, \mathrm{E}\big[g(B_v^{1/4})h(B_u^{1/4})(\lambda_{12}B_u^{1/4} + \lambda_{22}B_v^{1/4})\big]$$

$$= -\tfrac{3}{2}\int_0^1 dv \int_v^1 du\, \mathrm{E}\big[g(B_u^{1/4})h(B_v^{1/4})(\lambda_{12}B_v^{1/4} + \lambda_{11}B_u^{1/4})\big].$$

Finally,

$$\lim_{\varepsilon\downarrow 0}(J_\varepsilon^1 + J_\varepsilon^2) = -\tfrac{3}{2}\int_0^1 dv \int_0^1 du\, \mathrm{E}\big[g(B_u^{1/4})h(B_v^{1/4})(\lambda_{11}B_u^{1/4} + \lambda_{12}B_v^{1/4})\big],$$

which is the desired quantity. □

This completes the proof of Lemma 5.3, and we can proceed to the proof of Lemma 5.1.

PROOF OF PART (a) IN LEMMA 5.1. We write the covariance matrix of $(G_1, G_2, G_3^\varepsilon, G_4^\varepsilon)$ by blocks:

$$\Lambda_\varepsilon = \begin{pmatrix} \Lambda_{11} & \Lambda_{12}^\varepsilon \\ \Lambda_{21}^\varepsilon & \Lambda_{22}^\varepsilon \end{pmatrix}.$$

By classical Gaussian analysis, we know that the matrix A_ε and the covariance matrix of the vector Z^ε in Step IV(1) can be expressed as

$$A_\varepsilon = \Lambda_{21}^\varepsilon \Lambda_{11}^{-1} \quad \text{and} \quad K_{Z^\varepsilon} = \Lambda_{22}^\varepsilon - A_\varepsilon(\Lambda_{21}^\varepsilon)^*. \tag{5.29}$$

Here

$$\Lambda_{11} = \begin{pmatrix} u^{2H} & K_H(u,v) \\ K_H(v,u) & v^{2H} \end{pmatrix},$$

$$\Lambda_{21}^\varepsilon = \begin{pmatrix} \alpha_\varepsilon(u)u^{2H} & \gamma_\varepsilon(u,v) \\ \gamma_\varepsilon(v,u) & \alpha_\varepsilon(v)v^{2H} \end{pmatrix}, \tag{5.30}$$

$$\Lambda_{22}^\varepsilon = \begin{pmatrix} \varepsilon^{2H} & \eta_\varepsilon(u,v) \\ \eta_\varepsilon(v,u) & \varepsilon^{2H} \end{pmatrix},$$

where α_ε is given by (5.4) and

$$\gamma_\varepsilon(u,v) := \mathrm{Cov}(G_3^\varepsilon, G_2)$$
$$= \tfrac{1}{2}\big((u+\varepsilon)^{2H} - u^{2H} - |v-u-\varepsilon|^{2H} + |v-u|^{2H}\big),$$
$$\eta_\varepsilon(u,v) := \mathrm{Cov}(G_3^\varepsilon, G_4^\varepsilon)$$
$$= \tfrac{1}{2}\big(|v-u+\varepsilon|^{2H} + |v-u-\varepsilon|^{2H} - 2|v-u|^{2H}\big).$$

Also recall that $\Lambda_{11}^{-1} = (\lambda_{ij})_{i,j=1,2}$ is the inverse of the covariance matrix of (G_1, G_2) [see (5.12)]. We can see that

$$\gamma_\varepsilon(u,v) = H\big(u^{2H-1} + |v-u|^{2H-1}\big)\varepsilon + o(\varepsilon) \qquad \text{as } \varepsilon \downarrow 0 \tag{5.31}$$

and

$$\eta_\varepsilon(u,v) = H(2H-1)|v-u|^{2H-2}\varepsilon^2 + o(\varepsilon^2) \qquad \text{as } \varepsilon \downarrow 0. \tag{5.32}$$

We split the proof into several steps.

STEP 1 (Expansion of the matrix A_ε). We express its components by

$$A_\varepsilon := \begin{pmatrix} a_{11}^\varepsilon & a_{12}^\varepsilon \\ a_{21}^\varepsilon & a_{22}^\varepsilon \end{pmatrix}. \tag{5.33}$$

Using (5.6), (5.29) and (5.31), when $\varepsilon \downarrow 0$, gives

$$\begin{aligned} a_{11}^\varepsilon &= \lambda_{11}\alpha_\varepsilon(u)u^{2H} + \lambda_{12}\gamma_\varepsilon(u,v) \\ &= -\frac{\lambda_{11}}{2}\varepsilon^{2H} + H\big((\lambda_{11}+\lambda_{12})u^{2H-1} + \lambda_{12}|v-u|^{2H-1}\big)\varepsilon + o(\varepsilon). \end{aligned} \tag{5.34}$$

The asymptotics of the other coefficients a_{ij}^ε behaves similarly, since

$$\begin{aligned} a_{12}^\varepsilon &= \lambda_{12}\alpha_\varepsilon(u)u^{2H} + \lambda_{22}\gamma_\varepsilon(u,v), \\ a_{21}^\varepsilon &= \lambda_{12}\alpha_\varepsilon(v)v^{2H} + \lambda_{11}\gamma_\varepsilon(v,u), \\ a_{22}^\varepsilon &= \lambda_{22}\alpha_\varepsilon(v)v^{2H} + \lambda_{12}\gamma_\varepsilon(v,u). \end{aligned}$$

The expansion, as $\varepsilon \downarrow 0$, for the matrix A_ε becomes

$$A_\varepsilon = \begin{pmatrix} -\frac{\lambda_{11}}{2}\varepsilon^{2H} + k_{11}\varepsilon + o(\varepsilon) & -\frac{\lambda_{12}}{2}\varepsilon^{2H} + k_{12}\varepsilon + o(\varepsilon) \\ -\frac{\lambda_{12}}{2}\varepsilon^{2H} + k_{21}\varepsilon + o(\varepsilon) & -\frac{\lambda_{22}}{2}\varepsilon^{2H} + k_{22}\varepsilon + o(\varepsilon) \end{pmatrix}, \tag{5.35}$$

where $k_{ij} := k_{ij}(u,v)$, $i,j = 1,2$,

$$\begin{aligned} &\begin{pmatrix} k_{11}(u,v) & k_{12}(u,v) \\ k_{21}(u,v) & k_{22}(u,v) \end{pmatrix} \\ &\quad = H\begin{pmatrix} (\lambda_{11}+\lambda_{12})u^{2H-1} + \lambda_{12}|v-u|^{2H-1} & (\lambda_{12}+\lambda_{22})u^{2H-1} + \lambda_{22}|v-u|^{2H-1} \\ (\lambda_{12}+\lambda_{11})v^{2H-1} + \lambda_{11}|u-v|^{2H-1} & (\lambda_{22}+\lambda_{12})v^{2H-1} + \lambda_{12}|u-v|^{2H-1} \end{pmatrix}. \end{aligned} \tag{5.36}$$

STEP 2 (Expansion of the matrix K_{Z^ε}). We claim that the expansion of the matrix K_{Z^ε}, when $\varepsilon \downarrow 0$,

$$K_{Z^\varepsilon} = \begin{pmatrix} K_{Z^\varepsilon}(1,1) & K_{Z^\varepsilon}(1,2) \\ K_{Z^\varepsilon}(1,2) & K_{Z^\varepsilon}(2,2) \end{pmatrix}, \tag{5.37}$$

with

$$\begin{aligned} K_{Z^\varepsilon}(1,1) &= \varepsilon^{2H} - \frac{\lambda_{11}}{4}\varepsilon^{4H} + k_{11}\varepsilon^{1+2H} + o(\varepsilon^{1+2H}), \\ K_{Z^\varepsilon}(1,2) &= -\frac{\lambda_{12}}{4}\varepsilon^{4H} + \frac{k_{12}+k_{21}}{2}\varepsilon^{1+2H} + o(\varepsilon^{1+2H}), \\ K_{Z^\varepsilon}(2,2) &= \varepsilon^{2H} - \frac{\lambda_{22}}{4}\varepsilon^{4H} + k_{22}\varepsilon^{1+2H} + o(\varepsilon^{1+2H}). \end{aligned} \tag{5.38}$$

We compute K_{Z^ε} explicitly. Clearly, the computations for $K_{Z^\varepsilon}(1,1)$ and $K_{Z^\varepsilon}(2,2)$ are similar. Using (5.29)–(5.32) and (5.35), for $\varepsilon \downarrow 0$,

$$\begin{aligned}
K_{Z^\varepsilon}(1,1) &= \varepsilon^{2H} - a_{11}^\varepsilon \alpha_\varepsilon(u) u^{2H} - a_{12}^\varepsilon \gamma_\varepsilon(u,v) \\
&= \varepsilon^{2H} - \varepsilon^{4H}\left(-\frac{\lambda_{11}}{2} + k_{11}\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\right) \\
&\qquad \times \left(-\frac{1}{2} + H u^{2H-1}\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\right) \\
&\quad - \varepsilon^{1+2H}\left(-\frac{\lambda_{12}}{2} + k_{12}\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\right) \\
&\qquad \times \big(H(u^{2H-1} + |v-u|^{2H-1}) + o(1)\big) \\
&= \varepsilon^{2H} - \varepsilon^{4H}\left(\frac{\lambda_{11}}{4} - \left(\frac{\lambda_{11}}{2} H u^{2H-1} + \frac{k_{11}}{2}\right)\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\right) \\
&\quad - \varepsilon^{1+2H}\left(-\frac{\lambda_{12}}{2} H(u^{2H-1} + |v-u|^{2H-1}) + o(1)\right) \\
&= \varepsilon^{2H} - \frac{\lambda_{11}}{4}\varepsilon^{4H} + k_{11}\varepsilon^{1+2H} + o(\varepsilon^{1+2H}),
\end{aligned}$$

whereas

$$\begin{aligned}
K_{Z^\varepsilon}(1,2) &= \eta_\varepsilon(u,v) - a_{12}^\varepsilon \alpha_\varepsilon(v) v^{2H} - a_{11}^\varepsilon \gamma_\varepsilon(v,u) \\
&= \varepsilon^2\big(H(2H-1)|v-u|^{2H-2} + o(1)\big) \\
&\quad - \varepsilon^{4H}\left(-\frac{\lambda_{12}}{2} + k_{12}\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\right) \\
&\qquad \times \left(-\frac{1}{2} + H v^{2H-1}\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\right) \\
&\quad - \varepsilon^{1+2H}\left(-\frac{\lambda_{11}}{2} + k_{11}\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\right) \\
&\qquad \times \big(H(v^{2H-1} + |v-u|^{2H-1}) + o(1)\big) \\
&= -\frac{\lambda_{12}}{4}\varepsilon^{4H} + \frac{k_{12}+k_{21}}{2}\varepsilon^{1+2H} + o(\varepsilon^{1+2H}).
\end{aligned}$$

STEP 3 (The law of the vector Z^ε). Using (5.37) and (5.38), we observe that the Gaussian vector Z^ε can be written as

$$\begin{pmatrix} Z_1^\varepsilon \\ Z_2^\varepsilon \end{pmatrix} \overset{\text{(law)}}{=} \begin{pmatrix} \nu(\varepsilon) N_1 \\ \mu(\varepsilon) N_1 + \theta(\varepsilon) N_2 \end{pmatrix}, \tag{5.39}$$

where N_1 and N_2 are independent standard normal random variables, also independent of G_1 and G_2. Moreover, for $\varepsilon \downarrow 0$,

$$
\begin{aligned}
\nu(\varepsilon) &= \varepsilon^H\left(1 - \frac{\lambda_{11}}{8}\varepsilon^{2H} + c_1\varepsilon + o(\varepsilon)\right),\\
\mu(\varepsilon) &= \varepsilon^{3H}\left(-\frac{\lambda_{12}}{4} + c_2\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\right),\\
\theta(\varepsilon) &= \varepsilon^H\left(1 - \frac{\lambda_{22}}{8}\varepsilon^{2H} + c_3\varepsilon + o(\varepsilon)\right),
\end{aligned}
\tag{5.40}
$$

where $c_i := c_i(u, v)$, $i = 1, 2, 3$,

$$
c_1(u, v) := \begin{cases} \dfrac{k_{11}}{2}, & \text{if } H > \dfrac{1}{4},\\[2mm] \dfrac{k_{11}}{2} - \dfrac{\lambda_{11}^2}{128}, & \text{if } H = \dfrac{1}{4}, \end{cases}
$$

$$
c_2(u, v) := \begin{cases} \dfrac{k_{12} + k_{21}}{2}, & \text{if } H > \dfrac{1}{4},\\[2mm] \dfrac{k_{12} + k_{21}}{2} - \dfrac{\lambda_{11}\lambda_{12}}{32}, & \text{if } H = \dfrac{1}{4}, \end{cases}
$$

and

$$
c_3(u, v) := \begin{cases} \dfrac{k_{22}}{2}, & \text{if } H > \dfrac{1}{4},\\[2mm] \dfrac{k_{22}}{2} + \dfrac{\lambda_{12}^2}{32} - \dfrac{\lambda_{22}^2}{128}, & \text{if } H = \dfrac{1}{4}. \end{cases}
$$

Indeed, when $\varepsilon \downarrow 0$,

$$
\begin{aligned}
\nu(\varepsilon) &= \sqrt{K_{Z^\varepsilon}(1, 1)}\\
&= \varepsilon^H\left(1 - \frac{\lambda_{11}}{4}\varepsilon^{2H} + k_{11}\varepsilon + o(\varepsilon)\right)^{1/2}\\
&= \varepsilon^H\left(1 - \frac{\lambda_{11}}{8}\varepsilon^{2H} + \frac{k_{11}}{2}\varepsilon - \frac{\lambda_{11}^2}{128}\varepsilon^{4H} + o(\varepsilon)\right)\\
&= \begin{cases} \varepsilon^H\left(1 - \dfrac{\lambda_{11}}{8}\varepsilon^{2H} + \dfrac{k_{11}}{2}\varepsilon + o(\varepsilon)\right), & \text{if } H > \dfrac{1}{4},\\[2mm] \varepsilon^H\left(1 - \dfrac{\lambda_{11}}{8}\varepsilon^{2H} + \left(\dfrac{k_{11}}{2} - \dfrac{\lambda_{11}^2}{128}\right)\varepsilon + o(\varepsilon)\right), & \text{if } H = \dfrac{1}{4}, \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\mu(\varepsilon) &= \frac{K_{Z^\varepsilon}(1,2)}{\nu(\varepsilon)} \\
&= \frac{\varepsilon^{4H}(-\lambda_{12}/4 + ((k_{12}+k_{21})/2)\varepsilon^{1-2H} + o(\varepsilon^{1-2H}))}{\varepsilon^{H}(1-(\lambda_{11}/8)\varepsilon^{2H} + c_1\varepsilon + o(\varepsilon))} \\
&= \varepsilon^{3H}\biggl(-\frac{\lambda_{12}}{4} + \frac{k_{12}+k_{21}}{2}\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\biggr) \\
&\quad \times \biggl(1 + \frac{\lambda_{11}}{8}\varepsilon^{2H} - c_1\varepsilon + \frac{\lambda_{11}^2}{64}\varepsilon^{4H} + o(\varepsilon)\biggr) \\
&= \varepsilon^{3H}\biggl(-\frac{\lambda_{12}}{4} - \frac{\lambda_{11}\lambda_{12}}{32}\varepsilon^{2H} + \frac{k_{12}+k_{21}}{2}\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\biggr) \\
&= \begin{cases} \varepsilon^{3H}\biggl(-\dfrac{\lambda_{12}}{4} + \dfrac{k_{12}+k_{21}}{2}\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\biggr), & \text{if } H > \dfrac{1}{4}, \\ \varepsilon^{3H}\biggl(-\dfrac{\lambda_{12}}{4} + \biggl(\dfrac{k_{12}+k_{21}}{2} - \dfrac{\lambda_{11}\lambda_{12}}{32}\biggr)\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\biggr), & \text{if } H = \dfrac{1}{4}, \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
\theta(\varepsilon) &= \sqrt{K_{Z^\varepsilon}(2,2) - \mu^2(\varepsilon)} \\
&= \biggl(\varepsilon^{2H} - \frac{\lambda_{22}}{4}\varepsilon^{4H} + k_{22}\varepsilon^{1+2H} + o(\varepsilon^{1+2H}) \\
&\qquad - \varepsilon^{6H}\biggl(-\frac{\lambda_{12}}{4} + c_2\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\biggr)^2\biggr)^{1/2} \\
&= \biggl(\varepsilon^{2H} - \frac{\lambda_{22}}{4}\varepsilon^{4H} + k_{22}\varepsilon^{1+2H} - \frac{\lambda_{12}^2}{16}\varepsilon^{6H} + o(\varepsilon^{1+2H})\biggr)^{1/2} \\
&= \varepsilon^{H}\biggl(1 - \frac{\lambda_{22}}{4}\varepsilon^{2H} + k_{22}\varepsilon - \frac{\lambda_{12}^2}{16}\varepsilon^{4H} + o(\varepsilon)\biggr)^{1/2} \\
&= \varepsilon^{H}\biggl(1 - \frac{\lambda_{22}}{8}\varepsilon^{2H} + \frac{k_{22}}{2}\varepsilon - \biggl(\frac{\lambda_{12}^2}{32} + \frac{\lambda_{22}^2}{128}\biggr)\varepsilon^{4H} + o(\varepsilon)\biggr) \\
&= \begin{cases} \varepsilon^{H}\biggl(1 - \dfrac{\lambda_{22}}{8}\varepsilon^{2H} + \dfrac{k_{22}}{2}\varepsilon + o(\varepsilon)\biggr), & \text{if } H > \dfrac{1}{4}, \\ \varepsilon^{H}\biggl(1 - \dfrac{\lambda_{22}}{8}\varepsilon^{2H} + \biggl(\dfrac{k_{22}}{2} - \dfrac{\lambda_{12}^2}{32} - \dfrac{\lambda_{22}^2}{128}\biggr)\varepsilon + o(\varepsilon)\biggr), & \text{if } H = \dfrac{1}{4}. \end{cases}
\end{aligned}
$$

STEP 4 [Law of the vector $(G_3^\varepsilon, G_4^\varepsilon)$]. We claim that, for $\varepsilon \downarrow 0$,

$$\begin{pmatrix} G_3^\varepsilon \\ G_4^\varepsilon \end{pmatrix} \overset{\text{(law)}}{=} \begin{pmatrix} N_1\varepsilon^H + Q_1\varepsilon^{2H} - \frac{\lambda_{11}}{8}N_1\varepsilon^{3H} + R_1\varepsilon + o(\varepsilon) \\ N_2\varepsilon^H + Q_2\varepsilon^{2H} - (\frac{\lambda_{12}}{4}N_1 + \frac{\lambda_{22}}{8}N_2)\varepsilon^{3H} + R_2\varepsilon + o(\varepsilon) \end{pmatrix}, \tag{5.41}$$

where

$$R_1 := k_{11}G_1 + k_{12}G_2 \quad \text{and} \quad R_2 := k_{21}G_1 + k_{22}G_2.$$

Indeed, using (5.33), (5.35), (5.39) and (5.40), when $\varepsilon \downarrow 0$, we get

$$\begin{aligned} G_3^\varepsilon &= a_{11}^\varepsilon G_1 + a_{12}^\varepsilon G_2 + Z_1^\varepsilon \\ &\overset{\text{(law)}}{=} \Big(-\frac{\lambda_{11}}{2}\varepsilon^{2H} + k_{11}\varepsilon + o(\varepsilon)\Big)G_1 + \Big(-\frac{\lambda_{12}}{2}\varepsilon^{2H} + k_{12}\varepsilon + o(\varepsilon)\Big)G_2 \\ &\quad + \varepsilon^H\Big(1 - \frac{\lambda_{11}}{8}\varepsilon^{2H} + c_1\varepsilon + o(\varepsilon)\Big)N_1, \\ G_4^\varepsilon &= a_{21}^\varepsilon G_1 + a_{22}^\varepsilon G_2 + Z_2^\varepsilon \\ &\overset{\text{(law)}}{=} \Big(-\frac{\lambda_{12}}{2}\varepsilon^{2H} + k_{21}\varepsilon + o(\varepsilon)\Big)G_1\Big(-\frac{\lambda_{22}}{2}\varepsilon^{2H} + k_{22}\varepsilon + o(\varepsilon)\Big)G_2 \\ &\quad + \varepsilon^{3H}\Big(-\frac{\lambda_{12}}{4} + c_2\varepsilon^{1-2H} + o(\varepsilon^{1-2H})\Big)N_1 \\ &\quad + \varepsilon^H\Big(1 - \frac{\lambda_{22}}{8}\varepsilon^{2H} + c_3\varepsilon + o(\varepsilon)\Big)N_2. \end{aligned}$$

STEP 5 (Evaluation of the law of $G_3^\varepsilon G_4^\varepsilon$). As a consequence of the previous step,

$$\begin{aligned} G_3^\varepsilon G_4^\varepsilon &\overset{\text{(law)}}{=} \varepsilon^{2H}\Big(N_1 + Q_1\varepsilon^H - \frac{\lambda_{11}}{8}N_1\varepsilon^{2H} + R_1\varepsilon^{1-H} + o(\varepsilon^{1-H})\Big) \\ &\qquad \times \Big(N_2 + Q_2\varepsilon^H - \Big(\frac{\lambda_{12}}{4}N_1 + \frac{\lambda_{22}}{8}N_2\Big)\varepsilon^{2H} + R_2\varepsilon^{1-H} + o(\varepsilon^{1-H})\Big) \\ &\overset{\text{(law)}}{=} \varepsilon^{2H}\Big(N_1N_2 + (N_1Q_2 + N_2Q_1)\varepsilon^H \\ &\qquad + \Big(Q_1Q_2 - \frac{\lambda_{12}}{4}N_1^2 - \frac{\lambda_{11}+\lambda_{22}}{8}N_1N_2\Big)\varepsilon^{2H} + o(\varepsilon^{2H})\Big) \\ &\overset{\text{(law)}}{=} \varepsilon^{2H}(N_1N_2 + S_\varepsilon), \end{aligned}$$

where

$$S_\varepsilon \overset{\text{(law)}}{=} \varepsilon^H\Big(N_1Q_2 + N_2Q_1 + \Big(Q_1Q_2 - \frac{\lambda_{12}}{4}N_1^2 - \frac{\lambda_{11}+\lambda_{22}}{8}N_1N_2\Big)\varepsilon^H + o(\varepsilon^H)\Big).$$

STEP 6 [Evaluation of the law of $(G_3^\varepsilon G_4^\varepsilon)^3$]. We observe that, when $\varepsilon \downarrow 0$,

$$S_\varepsilon^2 \overset{\text{(law)}}{=} \varepsilon^{2H}\big((N_1 Q_2 + N_2 Q_1)^2 + o(1)\big) \quad \text{and} \quad S_\varepsilon^3 \overset{\text{(law)}}{=} o(\varepsilon^{3H}).$$

Hence,

$$\begin{aligned}(G_3^\varepsilon G_4^\varepsilon)^3 &\overset{\text{(law)}}{=} \{\varepsilon^{2H}(N_1N_2 + S_\varepsilon)\}^3\\ &\overset{\text{(law)}}{=} \varepsilon^{6H}(N_1^3N_2^3 + 3N_1^2N_2^2S_\varepsilon + 3N_1N_2S_\varepsilon^2 + S_\varepsilon^3)\\ &\overset{\text{(law)}}{=} \varepsilon^{6H}\bigg\{N_1^3N_2^3 + 3N_1^2N_2^2[N_1Q_2 + N_2Q_1]\varepsilon^H\\ &\qquad + \bigg[9N_1^2N_2^2Q_1Q_2 - \frac{3\lambda_{12}}{4}N_1^4N_2^2\\ &\qquad - 3\frac{\lambda_{11}+\lambda_{22}}{8}N_1^3N_2^3\\ &\qquad + 3N_1^3N_2Q_2^2 + 3N_1N_2^3Q_1^2\bigg]\varepsilon^{2H} + o(\varepsilon^{2H})\bigg\}.\end{aligned}$$

STEP 7 [Computation of the conditional expectation in (5.14)]. Consequently, for $\varepsilon \downarrow 0$,

$$\begin{aligned}\frac{(G_3^\varepsilon)^3(G_4^\varepsilon)^3}{\varepsilon^2} &\overset{\text{(law)}}{=} \varepsilon^{6H-2}\bigg\{N_1^3N_2^3 + [3N_1^3N_2^2Q_2 + 3N_1^2N_2^3Q_1]\varepsilon^H\\ &\qquad + \bigg[9N_1^2N_2^2Q_1Q_2 - \frac{3\lambda_{12}}{4}N_1^4N_2^2\\ &\qquad - 3\frac{\lambda_{11}+\lambda_{22}}{8}N_1^3N_2^3\\ &\qquad + 3N_1^3N_2Q_2^2 + 3N_1N_2^3Q_1^2\bigg]\varepsilon^{2H} + o(\varepsilon^{2H})\bigg\}.\end{aligned}$$

Since N_1 and N_2 are independent standard normal random variables, also independent of G_1 and G_2, we obtain the conditional expectation in (5.14). □

PROOF OF (b) OF LEMMA 5.1. The proof is similar as for (a). We will only provide the most significant arguments in several steps. The asymptotics for $\varepsilon \downarrow 0$, $\delta \downarrow 0$ of some functions of (ε, δ) in fractional powers can be done using a Maple procedure. Recall that the Hurst index is $H = \frac{1}{4}$.

STEP 1 (Linear regression). We can write

$$\begin{pmatrix} G_3^\varepsilon \\ G_4^\delta \end{pmatrix} = A_{\varepsilon,\delta}\begin{pmatrix} G_1 \\ G_2 \end{pmatrix} + \begin{pmatrix} Z_1^{\varepsilon,\delta} \\ Z_2^{\varepsilon,\delta} \end{pmatrix}, \tag{5.42}$$

with

$$A_{\varepsilon,\delta} = \Lambda_{12}^{\varepsilon,\delta}\Lambda_{11}^{-1} \quad \text{and} \quad K_{Z^{\varepsilon,\delta}} = \Lambda_{22}^{\varepsilon,\delta} - A_{\varepsilon,\delta}(\Lambda_{12}^{\varepsilon,\delta})^*. \tag{5.43}$$

Here

$$\Lambda_{12}^{\varepsilon,\delta} = \begin{pmatrix} \alpha_\varepsilon(u)\sqrt{u} & \gamma_\varepsilon(u,v) \\ \gamma_\delta(v,u) & \alpha_\delta(v)\sqrt{v} \end{pmatrix}, \qquad \Lambda_{22}^{\varepsilon,\delta} = \begin{pmatrix} \varepsilon^{1/2} & \eta_{\varepsilon,\delta}(u,v) \\ \eta_{\varepsilon,\delta}(u,v) & \delta^{1/2} \end{pmatrix}, \tag{5.44}$$

with

$$\begin{aligned} \eta_{\varepsilon,\delta}(u,v) &= \operatorname{Cov}(G_3^\varepsilon, G_4^\delta) \\ &= \tfrac{1}{2}\big(|v-u+\delta|^{1/2} + |v-u-\varepsilon|^{1/2} - |v-u|^{1/2} - |v-u+\delta-\varepsilon|^{1/2}\big). \end{aligned}$$

Therefore, when $\varepsilon \downarrow 0$, $\delta \downarrow 0$,

$$\eta_{\varepsilon,\delta}(u,v) = -\frac{\varepsilon\delta}{8|v-u|^{3/2}} + o\big((\varepsilon+\delta)^2\big). \tag{5.45}$$

STEP 2 (Expansion and computations for the matrix $A_{\varepsilon,\delta}$). We can write

$$A_{\varepsilon,\delta} := \begin{pmatrix} a_{11}^\varepsilon & a_{12}^\varepsilon \\ a_{21}^\delta & a_{22}^\delta \end{pmatrix},$$

with

$$\begin{aligned} a_{11}^\varepsilon &= \lambda_{11}\alpha_\varepsilon(u)\sqrt{u} + \lambda_{12}\gamma_\varepsilon(u,v), \qquad & a_{12}^\varepsilon &= \lambda_{12}\alpha_\varepsilon(u)\sqrt{u} + \lambda_{22}\gamma_\varepsilon(u,v), \\ a_{21}^\delta &= \lambda_{12}\alpha_\delta(v)\sqrt{v} + \lambda_{11}\gamma_\delta(v,u), \qquad & a_{22}^\delta &= \lambda_{22}\alpha_\delta(v)\sqrt{v} + \lambda_{12}\gamma_\delta(v,u). \end{aligned}$$

Hence, as in Step 1 of part (a), as $\varepsilon \downarrow 0$, $\delta \downarrow 0$,

$$A_{\varepsilon,\delta} = \begin{pmatrix} -\dfrac{\lambda_{11}}{2}\varepsilon^{1/2} + k_{11}\varepsilon + o(\varepsilon) & -\dfrac{\lambda_{12}}{2}\varepsilon^{1/2} + k_{12}\varepsilon + o(\varepsilon) \\ -\dfrac{\lambda_{12}}{2}\delta^{1/2} + k_{21}\delta + o(\delta) & -\dfrac{\lambda_{22}}{2}\delta^{1/2} + k_{22}\delta + o(\delta) \end{pmatrix}, \tag{5.46}$$

where the k_{ij} are given by (5.36).

STEP 3 (Computations related to matrix $K_{Z^{\varepsilon,\delta}}$). We can write

$$K_{Z^{\varepsilon,\delta}} = \begin{pmatrix} K_{Z^{\varepsilon,\delta}}(1,1) & K_{Z^{\varepsilon,\delta}}(1,2) \\ K_{Z^{\varepsilon,\delta}}(1,2) & K_{Z^{\varepsilon,\delta}}(2,2) \end{pmatrix}, \tag{5.47}$$

where, if $\varepsilon \downarrow 0$, $\delta \downarrow 0$, we have

$$\begin{aligned} K_{Z^{\varepsilon,\delta}}(1,1) &= \varepsilon^{1/2} - \frac{\lambda_{11}}{4}\varepsilon + k_{11}\varepsilon^{3/2} + o(\varepsilon^{3/2}), \\ K_{Z^{\varepsilon,\delta}}(2,2) &= \delta^{1/2} - \frac{\lambda_{22}}{4}\delta + k_{22}\delta^{3/2} + o(\delta^{3/2}), \\ K_{Z^{\varepsilon,\delta}}(1,2) &= -\frac{\lambda_{12}}{4}\varepsilon^{1/2}\delta^{1/2} + \frac{k_{12}}{2}\varepsilon\delta^{1/2} + \frac{k_{21}}{2}\varepsilon^{1/2}\delta + o\big((\varepsilon+\delta)^2\big). \end{aligned} \tag{5.48}$$

Indeed,

$$K_{Z^{\varepsilon,\delta}}(1,1) = \varepsilon^{1/2} - a_{11}^{\varepsilon}\alpha_{\varepsilon}(u)\sqrt{u} - a_{12}^{\varepsilon}\gamma_{\varepsilon}(u,v)$$

and

$$K_{Z^{\varepsilon,\delta}}(2,2) = \delta^{1/2} - a_{22}^{\delta}\alpha_{\delta}(v)\sqrt{v} - a_{21}^{\delta}\gamma_{\delta}(v,u).$$

Hence, the expansions of these two coefficients are similar as in Step 3 of part (a). The expansion of the remaining element behaves differently. Indeed, for $\varepsilon \downarrow 0$, $\delta \downarrow 0$,

$$\begin{aligned}
K_{Z^{\varepsilon,\delta}}(1,2) &= \eta_{\varepsilon,\delta}(u,v) - a_{12}^{\varepsilon}\alpha_{\delta}(v)\sqrt{v} - a_{11}^{\varepsilon}\gamma_{\delta}(v,u) \\
&= -\frac{\varepsilon\delta}{8|v-u|^{3/2}} + o\big((\varepsilon+\delta)^2\big) \\
&\quad - \varepsilon^{1/2}\delta^{1/2}\bigg(-\frac{\lambda_{12}}{2} + k_{12}\varepsilon^{1/2} + o(\varepsilon^{1/2})\bigg) \\
&\quad \times \bigg(-\frac{1}{2} + \frac{1}{4\sqrt{v}}\delta^{1/2} + o(\delta^{1/2})\bigg) \\
&\quad - \varepsilon^{1/2}\delta\bigg(-\frac{\lambda_{11}}{2} + k_{11}\varepsilon^{1/2} + o(\varepsilon^{1/2})\bigg) \\
&\quad \times \bigg(\frac{1}{4\sqrt{v}} + \frac{1}{4\sqrt{|u-v|}} + o(1)\bigg) \\
&= -\frac{\lambda_{12}}{4}\varepsilon^{1/2}\delta^{1/2} + \frac{k_{12}}{2}\varepsilon\delta^{1/2} + \frac{k_{21}}{2}\varepsilon^{1/2}\delta + o\big((\varepsilon+\delta)^2\big).
\end{aligned}$$

STEP 4 [Law of the vector $(Z_1^{\varepsilon,\delta}, Z_2^{\varepsilon,\delta})$]. Computations give

$$\begin{pmatrix} Z_1^{\varepsilon,\delta} \\ Z_2^{\varepsilon,\delta} \end{pmatrix} \overset{\text{(law)}}{=} \begin{pmatrix} \nu(\varepsilon)N_1 \\ \mu(\varepsilon,\delta)N_1 + \theta(\varepsilon,\delta)N_2 \end{pmatrix}, \tag{5.49}$$

where N_1 and N_2 are independent standard normal random variables, also independent of G_1 and G_2, and where

$$\nu(\varepsilon) = \varepsilon^{1/4} - \frac{\lambda_{11}}{8}\varepsilon^{3/4} + o(\varepsilon^{3/4}),$$

$$\mu(\varepsilon,\delta) = -\frac{\lambda_{12}}{4}\varepsilon^{1/4}\delta^{1/2} + o\big((\varepsilon^{1/2}+\delta^{1/2})^2\big),$$

$$\begin{aligned}
\theta(\varepsilon,\delta) &= \delta^{1/4} + \frac{\lambda_{12}}{8}\varepsilon^{1/4}\delta^{1/4} - \frac{\lambda_{22}}{8}\delta^{3/4} - \frac{\lambda_{12}^2}{128}\varepsilon^{1/2}\delta^{1/4} \\
&\quad + \bigg(\frac{\lambda_{11}\lambda_{12}}{64} - \frac{k_{12}}{4} + \frac{\lambda_{12}^3}{1024}\bigg)\varepsilon^{3/4}\delta^{1/4} \\
&\quad + \bigg(\frac{\lambda_{12}\lambda_{22}}{64} - \frac{k_{21}}{4}\bigg)\varepsilon^{1/4}\delta^{3/4} + o(\varepsilon^{1/2}+\delta^{1/2}).
\end{aligned} \tag{5.50}$$

Indeed, $\nu(\varepsilon)$ is given by the first equality in (5.40), when $\varepsilon \downarrow 0$. The other coefficients are given by

$$\mu(\varepsilon,\delta) = \frac{K_{Z^{\varepsilon,\delta}}(1,2)}{\nu(\varepsilon)},$$
$$\theta(\varepsilon,\delta) = \sqrt{K_{Z^{\varepsilon,\delta}}(2,2) - \mu^2(\varepsilon,\delta)}$$

and we use the results in the previous step and the Maple procedure. Hence, we have

$$Z_1^{\varepsilon,\delta} \overset{\text{(law)}}{=} N_1\varepsilon^{1/4} - \frac{\lambda_{11}}{8}N_1\varepsilon^{3/4} + o(\varepsilon^{3/4}),$$

$$\begin{aligned}Z_2^{\varepsilon,\delta} \overset{\text{(law)}}{=}\ & N_2\delta^{1/4} + \frac{\lambda_{12}}{8}N_2\varepsilon^{1/4}\delta^{1/4} \\ & - \frac{\lambda_{12}^2}{128}N_2\varepsilon^{1/2}\delta^{1/4} - \frac{\lambda_{12}}{4}N_1\varepsilon^{1/4}\delta^{1/2} - \frac{\lambda_{22}}{8}N_2\delta^{3/4} \\ & + \left(\frac{\lambda_{11}\lambda_{12}}{64} - \frac{k_{12}}{4} + \frac{\lambda_{12}^3}{1024}\right)N_2\varepsilon^{3/4}\delta^{1/4} \\ & + \left(\frac{\lambda_{12}\lambda_{22}}{64} - \frac{k_{21}}{4}\right)N_2\varepsilon^{1/4}\delta^{3/4} + o\big((\varepsilon^{1/2}+\delta^{1/2})^2\big).\end{aligned}$$

STEP 5 [Law of the vector $(G_3^\varepsilon, G_4^\delta)$]. Using the first line of (5.41), (5.50) and (5.48), when $\varepsilon \downarrow 0$, $\delta \downarrow 0$, we obtain

$$G_3^{\varepsilon} \overset{\text{(law)}}{=} N_1\varepsilon^{1/4} + Q_1\varepsilon^{1/2} - \frac{\lambda_{11}}{8}N_1\varepsilon^{3/4} + o(\varepsilon^{3/4}),$$

$$\begin{aligned}G_4^{\delta} \overset{\text{(law)}}{=}\ & N_2\delta^{1/4} + Q_2\delta^{1/2} - \frac{\lambda_{22}}{8}N_2\delta^{3/4} + R_2\delta \\ & + \frac{\lambda_{12}}{8}N_2\varepsilon^{1/4}\delta^{1/4} - \frac{\lambda_{12}^2}{128}N_2\varepsilon^{1/2}\delta^{1/4} - \frac{\lambda_{12}}{4}N_1\varepsilon^{1/4}\delta^{1/2} \\ & + \left(\frac{\lambda_{11}\lambda_{12}}{64} - \frac{k_{12}}{4} + \frac{\lambda_{12}^3}{1024}\right)N_2\varepsilon^{3/4}\delta^{1/4} \\ & + \left(\frac{\lambda_{12}\lambda_{22}}{64} - \frac{k_{21}}{4}\right)N_2\varepsilon^{1/4}\delta^{3/4} + o\big((\varepsilon^{1/2}+\delta^{1/2})^2\big),\end{aligned} \tag{5.51}$$

with Q_1 and Q_2 given by (5.13) and R_2 is as in Step 4 of part (b).

STEP 6 (Computation of the law of $G_3^\varepsilon G_4^\delta$). From (5.51), when $\varepsilon \downarrow 0$, $\delta \downarrow 0$, we get

$$\begin{aligned} G_3^\varepsilon G_4^\delta \overset{\text{(law)}}{=} {} & N_1 N_2 \varepsilon^{1/4}\delta^{1/4} + \left(\frac{\lambda_{12}}{8} N_1 N_2 + Q_1 N_2\right)\varepsilon^{1/2}\delta^{1/4} + N_1 Q_2 \varepsilon^{1/4}\delta^{1/2} \\ & + \left(-\frac{\lambda_{12}^2}{128} N_1 N_2 - \frac{\lambda_{11}}{8} N_1 N_2 + \frac{\lambda_{12}}{8} Q_1 N_2\right)\varepsilon^{3/4}\delta^{1/4} \\ & + \left(-\frac{\lambda_{12}}{4} N_1^2 + Q_1 Q_2\right)\varepsilon^{1/2}\delta^{1/2} \\ & - \frac{\lambda_{22}}{8} N_1 N_2 \varepsilon^{1/4}\delta^{3/4} + o\big((\varepsilon^{1/2}+\delta^{1/2})^2\big). \end{aligned}$$

STEP 7 [Computation of the conditional expectation in (5.16)]. When $\varepsilon \downarrow 0$, $\delta \downarrow 0$, it follows that

$$\begin{aligned} & \frac{(G_3^\varepsilon)^3 (G_4^\delta)^3}{\varepsilon\delta} \\ & \quad \overset{\text{(law)}}{=} N_1^3 N_2^3 \varepsilon^{-1/4}\delta^{-1/4} \\ & \qquad + 3\left(\frac{\lambda_{12}}{8} N_1^3 N_2^3 + Q_1 N_1^2 N_2^3\right)\delta^{-1/4} + 3 Q_2 N_1^3 N_2^2 \varepsilon^{-1/4} \\ & \qquad + 3\left(\left(\frac{\lambda_{12}^2}{128} - \frac{\lambda_{11}}{8}\right) N_1^3 N_2^3 + Q_1^2 N_1 N_2^3 + \frac{3\lambda_{12}}{8} Q_1 N_1^2 N_2^3\right)\varepsilon^{1/4}\delta^{-1/4} \\ & \qquad + 3\left(-\frac{\lambda_{22}}{8} N_1^3 N_2^3 + Q_2^2 N_1^3 N_2\right)\varepsilon^{-1/4}\delta^{1/4} \\ & \qquad + \frac{3\lambda_{12}}{4} Q_2 N_1^3 N_2^2 - \frac{3\lambda_{12}}{4} N_1^4 N_2^2 + 9 Q_1 Q_2 N_1^2 N_2^2 + o(1). \end{aligned}$$

Since N_1 and N_2 are independent standard normal random variables, independent of G_1 and G_2, we finally deduce the conditional expectation in (5.16). □

PROOF OF (a′) IN LEMMA 5.1. Using (5.9), we recall that

$$G_3^\varepsilon = \left[A_\varepsilon \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}\right]_1 + Z_1^\varepsilon, \qquad G_4^\varepsilon = \left[A_\varepsilon \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}\right]_2 + Z_2^\varepsilon.$$

Therefore, the left-hand side of the first equality in (5.15) can be written as

$$\begin{aligned} \mathrm{E}\left(\frac{(G_3^\varepsilon)^3}{\varepsilon}\middle| G_1, G_2\right) &= \mathrm{E}\left(\left[A_\varepsilon \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}\right]_1 + Z_1^\varepsilon\right)^3 \\ &= \left[A_\varepsilon \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}\right]_1^3 + 3\left[A_\varepsilon \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}\right]_1 \mathrm{E}[(Z_1^\varepsilon)^2] \\ &= (a_{11}^\varepsilon G_1 + a_{12}^\varepsilon G_2)^3 + 3(a_{11}^\varepsilon G_1 + a_{12}^\varepsilon G_2) K_{Z^\varepsilon}(1,1), \end{aligned}$$

according to the notation in (5.33), (5.34) and (5.37). Recall that, by (5.35) and (5.36),

$$a_{11}^{\varepsilon} = -\frac{\lambda_{11}}{2}\varepsilon^{2H} + k_{11}\varepsilon + o(\varepsilon),$$

where $k_{11} = H(\lambda_{11} + \lambda_{12})u^{2H-1} - \lambda_{12}H|u - v|^{2H-1}$,

$$a_{12}^{\varepsilon} = -\frac{\lambda_{12}}{2}\varepsilon^{2H} + k_{12}\varepsilon + o(\varepsilon),$$

where $k_{12} = H(\lambda_{12} + \lambda_{22})u^{2H-1} - \lambda_{22}H|u - v|^{2H-1}$, and, by (5.38),

$$K_{Z^{\varepsilon}}(1, 1) = \varepsilon^{2H} - \frac{\lambda_{11}}{4}\varepsilon^{4H} + o(\varepsilon).$$

Hence, we obtain

$$\begin{aligned}
\mathrm{E}&\left(\frac{(G_3^{\varepsilon})^3}{\varepsilon}\middle|G_1, G_2\right)\\
&= \left\{\left[-\frac{\lambda_{11}}{2}\varepsilon^{2H} + k_{11}\varepsilon + o(\varepsilon)\right]G_1 + \left[-\frac{\lambda_{12}}{2}\varepsilon^{2H} + k_{12}\varepsilon + o(\varepsilon)\right]G_2\right\}^3\\
&\quad + 3\left\{\left[-\frac{\lambda_{11}}{2}\varepsilon^{2H} + k_{11}\varepsilon + o(\varepsilon)\right]G_1 + \left[-\frac{\lambda_{12}}{2}\varepsilon^{2H} + k_{12}\varepsilon + o(\varepsilon)\right]G_2\right\}\\
&\qquad \times \left(\varepsilon^{2H} - \frac{\lambda_{11}}{4}\varepsilon^{4H} + o(\varepsilon)\right)\\
&= 3\varepsilon^{4H}\left[-\lambda_{11}\frac{G_1}{2} - \lambda_{12}\frac{G_2}{2}\right] + o(\varepsilon).
\end{aligned}$$

This gives (5.15). □

PROOF OF (c) IN LEMMA 5.1. We need to show that the asymptotics in (5.14)–(5.16) are uniform in u and v. We do the job for (5.14), the others behaving similarly. It is enough to analyze the uniformity of the expansions on $\{1 < u, 1 < v - u\}$ of $\alpha_{\varepsilon}(u)$, $\gamma_{\varepsilon}(u, v)$ and $\eta_{\varepsilon}(u, v)$, when $\varepsilon \downarrow 0$, because the other asymptotics are obtained in terms of these. When $\varepsilon \downarrow 0$, by (5.4), we have

$$\begin{aligned}
\alpha_{\varepsilon}(u) &= \frac{1}{2u^{2H}}\left((u + \varepsilon)^{2H} - u^{2H} - \varepsilon^{2H}\right)\\
&= \frac{1}{2}\left(\left(1 + \frac{\varepsilon}{u}\right)^{2H} - 1 - \left(\frac{\varepsilon}{u}\right)^{2H}\right)\\
&= -\frac{1}{2}\left(\frac{\varepsilon}{u}\right)^{2H} + H\frac{\varepsilon}{u} + o\left(\frac{\varepsilon}{u}\right),
\end{aligned}$$

which provides a uniform expansion on $\{u > 1\}$. Similarly, when $\varepsilon \downarrow 0$, one obtains

$$\begin{aligned}\gamma_\varepsilon(u,v) &= \frac{1}{2}\big((u+\varepsilon)^{2H} - u^{2H} - |v-u-\varepsilon|^{2H} + (v-u)^{2H}\big)\\ &= \frac{1}{2}\left[u^{2H}\left(\left(1+\frac{\varepsilon}{u}\right)^{2H} - 1\right) - (v-u)^{2H}\left(\left|1-\frac{\varepsilon}{v-u}\right|^{2H} - 1\right)\right]\\ &= H(u^{2H-1} + |v-u|^{2H-1})\varepsilon + o(\varepsilon),\end{aligned}$$

uniformly on $\{1 < u, 1 < v-u\}$, and, when $\varepsilon \downarrow 0$,

$$\begin{aligned}\eta_\varepsilon(u,v) &= \frac{1}{2}\big((v-u+\varepsilon)^{2H} + |v-u-\varepsilon|^{2H} - 2(v-u)^{2H}\big)\\ &= \frac{(v-u)^{2H}}{2}\left[\left(1+\frac{\varepsilon}{v-u}\right)^{2H} + \left|1-\frac{\varepsilon}{v-u}\right|^{2H} - 2\right]\\ &= H(2H-1)|v-u|^{2H-2}\varepsilon^2 + o(\varepsilon^2),\end{aligned}$$

uniformly on $\{1 < v-u\}$. □

PROOF OF (d) IN LEMMA 5.1. We look for the homogeneity degree of all quantities used so far. For a function $f = f(\varepsilon,u,v)$, we shall denote

$$\deg_{\varepsilon,u,v}(f) =: p \Leftrightarrow f(\kappa\varepsilon,\kappa u,\kappa v) = \kappa^p f(\varepsilon,u,v),$$

where we make the convention that

$$\gamma(\varepsilon,u,v) := \gamma_\varepsilon(u,v), \qquad K_Z(i,j)(\varepsilon,u,v) := K_{Z^\varepsilon}(i,j)(u,v).$$

We have

$$\begin{aligned}\deg_{\varepsilon,u}(\alpha) &= 0 && [\text{by } (5.4)],\\ \deg_{\varepsilon,u,v}(\lambda_{ij}) &= -2H && [\text{by } (5.12)],\\ \deg_{\varepsilon,u,v}(\gamma) &= 2H && [\text{by } (5.31)],\\ \deg_{\varepsilon,u,v}(\eta) &= 2H && [\text{by } (5.32)],\\ \deg_{\varepsilon,u,v}(a_{ij}) &= 0 && [\text{by } (5.29), (5.30) \text{ and } (5.33)],\\ \deg_{\varepsilon,u,v}(K_Z(i,j)) &= 2H && [\text{by } (5.29), (5.30) \text{ and } (5.37)],\end{aligned}$$

$$\deg_{\varepsilon,u,v}(\nu) = \deg_{\varepsilon,u,v}(\mu) = \deg_{\varepsilon,u,v}(\theta) = H \qquad [\text{by } (5.39)].$$

From this, (5.9) and (5.33), recalling that $G_1(u) = B_u^H$, $G_2(v) = B_v^H$, we deduce that

$$\begin{aligned}G_3^{\kappa\varepsilon}(\kappa u) &= a_{11}^{\kappa\varepsilon}(\kappa u,\kappa v)G_1(\kappa u) + a_{12}^{\kappa\varepsilon}(\kappa u,\kappa v)G_2(\kappa v) + Z_1^{\kappa\varepsilon}(\kappa u,\kappa v)\\ &\overset{\text{(law)}}{=} a_{11}^{\varepsilon}(u,v)\kappa^H G_1(u) + a_{12}^{\varepsilon}(u,v)\kappa^H G_2(v) + \kappa^H Z_1^{\varepsilon}(u,v)\\ &\overset{\text{(law)}}{=} \kappa^H G_3^{\varepsilon}(u),\end{aligned}$$

and, in a similar way, $G_4^{\kappa\varepsilon}(\kappa v) = \kappa^H G_4^{\varepsilon}(v)$. Therefore, (5.17) is proved. On the other hand, using (3.7) and (5.13), we obtain

$$\begin{aligned}
&9Q_1(\kappa u, \kappa v)Q_2(\kappa u, \kappa v) - \tfrac{9}{4}\lambda_{12}(\kappa u, \kappa v)\\
&\quad = \tfrac{9}{4}\big[\big(\lambda_{11}(\kappa u, \kappa v)G_1(\kappa u) + \lambda_{12}(\kappa u, \kappa v)G_2(\kappa v)\big)\\
&\qquad \times \big(\lambda_{12}(\kappa u, \kappa v)G_1(\kappa u) + \lambda_{22}(\kappa u, \kappa v)G_2(\kappa v)\big) - \lambda_{12}(\kappa u, \kappa v)\big]\\
&\quad \overset{\text{(law)}}{=} \tfrac{9}{4}\big[\kappa^{-2H}\big(\lambda_{11}(u, v)G_1(u) + \lambda_{12}(u, v)G_2(v)\big)\\
&\qquad \times \big(\lambda_{12}(u, v)G_1(u) + \lambda_{22}(u, v)G_2(v)\big) - \kappa^{-2H}\lambda_{12}(u, v)\big]
\end{aligned}$$

and, consequently, (5.18) is also proved. □

This completes the proof of Lemma 5.1.

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