The Annals of Probability
2003, Vol. 31, No. 3, 1615–1654

ATTRACTING EDGE PROPERTY FOR A CLASS OF REINFORCED RANDOM WALKS

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Using martingale techniques and comparison with the generalized Urn scheme, it is shown that the edge reinforced random walk on a graph of bounded degree, with the *weight function* $W(k) = k^\rho$, $\rho > 1$, traverses (crosses) a random *attracting* edge at all large times. If the graph is a triangle, the above result is in agreement with a conjecture of Sellke.

1. Introduction. Consider a connected graph $\mathcal{G}$ with the set of vertices $V = V(\mathcal{G})$, and the set of (unoriented) edges $E = E(\mathcal{G})$. Two vertices v, v' are *adjacent* ($v \sim v'$ in symbols) if there exists an edge, denoted by $[v, v'] = [v', v]$, connecting them. For any vertex v of $\mathcal{G}$, let $A(v) \subset V$ be the set of adjacent vertices.

Let $W(k) > 0, k \geq 0$ be the *weight* function. The edge reinforced random walk on $\mathcal{G}$ records a random motion of a particle along the vertices of $\mathcal{G}$ with the following properties:

(i) if currently at vertex $v \in \mathcal{G}$, in the next step the particle jumps to one of the adjacent vertices,

(ii) the probability of jump to v' is *W-proportional* to the number of previous traversals of the edge $[v, v']$.

More precisely, let the initial *edge weights* be X_0^e for all $e \in E$. Let I_n be a V-valued random variable, recording the position of the particle at time n, $n \geq 0$. For concreteness, set $X_0^e = 1$, $e \in E$, and $I_0 = v_0$ for some $v_0 \in V$. A *traversal* of edge e occurs at time $n + 1$ if $e = [I_n, I_{n+1}]$. Denote by $X_n^e - X_0^e$ the total number of traversals of edge e until time n. Let $\mathcal{F}_n$ be the filtration $\sigma\{(I_k, X_k^e, e \in E), k = 0, \ldots, n\}$. Let $\delta_{[v,v']}$ be the Kronecker symbol, that is, $\delta_{[v,v']}(e) = 1$ for $e = [v, v']$ and $\delta_{[v,v']}(e) = 0$ otherwise. The *edge reinforced random walk* on $\mathcal{G}$ with weight function W is a Markov chain $(I, X) = \{(I_n, X_n^e, e \in E), n \geq 0\}$ with the following conditional transition probabilities: on the event $\{I_n = v\}$, for $v' \in A(v)$,

$$\begin{aligned} &P\big(I_{n+1} = v',\ X_{n+1}^e = X_n^e + \delta_{[v,v']}(e), e \in \mathcal{G} | \mathcal{F}_n\big) \\ &\qquad = \frac{W(X_n^{[v,v']})}{\sum_{u \in A(v)} W(X_n^{[v,u]})}. \end{aligned} \tag{1}$$

Received February 2001; revised October 2002.

[1]Supported by an NSF Mathematical Sciences Postdoctoral Research Fellowship.

AMS 2000 subject classifications. 60J15, 60J10.

Key words and phrases. Reinforced walk, supermartingale, coupling, urn.

It is easily seen that the edge reinforced random walk is well defined for any weight function W, where $W(k) > 0, k \geq 0$.

The edge reinforced random walk (ERRW) was originally introduced by Coppersmith and Diaconis (cf. [3]) as a model of exploring an unknown city. Paper [3] considers only the reinforcement weights which increase linearly in the number of edge traversals. Davis [2] introduces ERRW on $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$ in a setting where the reinforcement weights are given by a nondecreasing function $W : \{1, \dots\} \mapsto (0, \infty)$. Theorem 3.2 in [2] implies that if (a) $\sum_k 1/W(k) = \infty$, then with probability 1, every vertex in $\mathbb{Z}$ is visited by the walk infinitely often, else if (b) $\sum_k 1/W(k) < \infty$, then almost surely there exists some random integer i such that the walk visits only i and $i+1$ at all large times. Therefore, on infinite graphs, the reinforced walks may explore only finite subgraphs almost surely. The same never holds for a classical random walk on an infinite graph. Sellke [12] studies the long-term behavior of reinforced random walks on $\mathbb{Z}^d$ in terms of weights W. In particular, [12] shows a natural generalization of the above result from [2]: if

$$\sum_k \frac{1}{W(k)} < \infty, \tag{2}$$

then almost surely there exists some random *attracting* edge on $\mathbb{Z}^d$ which is traversed by the walk at all large times. The same paper conjectures that the above property holds for edge reinforced walks on a triangle. The goal of this paper is to show that in the special case of power function reinforcement weights, condition (2) implies the existence of the attracting edge with probability 1. Here the only assumption on the graph is that each vertex has at most $D(\mathcal{G})$ adjacent vertices (edges), for some $D(\mathcal{G}) < \infty$. So the graph $\mathcal{G}$ is either finite, or infinite with bounded degree. Special cases are the infinite lattices.

Let $\mathcal{G}_\infty = \{e \in E : \sup_n X_n^e = \infty\}$ be the (random) graph spanned by all edges in $\mathcal{G}$ that are traversed by the walk infinitely often. The main result of this paper is the following.

THEOREM 1. *If $W(k) = k^\rho$, $\rho > 1$, and $\sup_e X_0^e < \infty$, the edge reinforced random walk on $\mathcal{G}$ traverses a random attracting edge at all large times a.s., that is,*

$$\begin{aligned} &P\left(\exists\, e \in E \text{ such that } \sup_{e' \neq e} \sup_n X_n^{e'} < \infty\right) \\ &\qquad = P(\mathcal{G}_\infty \text{ has only one edge}) = 1. \end{aligned} \tag{3}$$

REMARKS. A simple calculation of Borel–Cantelli type shows that (2) is a necessary and sufficient condition for $P(\mathcal{G}_\infty$ has only one edge$) > 0$. It is easy to see that Theorem 1 holds if the weight function increases very fast, for example, $W(k) = 10^{2^k}$ (Andrzej Zuk, personal communication).

Toth [14] studies the asymptotic behavior of edge occupation times for ERRW on $\mathbb{Z}$ under the reinforcement with weights $W(k) = k^\rho$, $\rho < 1$. Keane and Rolles [7] determine asymptotics of the joint occupation times of edges and cycles, for a linearly reinforced ERRW on a finite graph. Pemantle [8] determines the recurrence/transience phase transition for linearly reinforced ERRW on a tree. The analogous *vertex reinforced random walk* (VRRW) was initially studied by Pemantle [9]. Pemantle and Volkov [11] showed that a linearly reinforced VRRW on $\mathbb{Z}$ visits only finitely many vertices, almost surely, and conjectured that the size of the set of vertices visited infinitely often is five with probability 1. This was recently proved by Tarrès [13]. These results are remarkable, since the analogous linearly reinforced ERRW on $\mathbb{Z}$ visits all vertices infinitely often, for example, by [2], Theorem 3.2. Volkov [15] proved that for a large class of graphs ($\mathbb{Z}^d$ in particular) the linearly reinforced VRRW visits only finitely many vertices with positive probability. The question of whether linearly reinforced ERRW on $\mathbb{Z}^d$, $d \geq 2$ is recurrent or transient remains open. The "once reinforced random walk" on $\mathcal{G}$ corresponds to the weight function $W(1) = 1$, $W(k) = c$, $k \geq 2$, for some $c > 1$. Davis [2] asked whether the above walk returns to its starting point with probability 1. Durrett, Kesten and Limic [5] study once reinforced random walks on regular trees.

In the next section the problem on the general graph is decoupled into analogous problems (cf. Proposition 1) on finite cycles. Proposition 1 and Theorem 1 are proved in Section 5. Section 3 considers the problem on a very simple graph, gives some intuition, and introduces a useful supermartingale. Section 4 consists of a sequence of lemmas needed in the proof of Proposition 1, and Section 6 contains the proof of conditional probability estimates used in Section 4. Throughout this paper $C(p_1, p_2, \dots)$ or $c(p_1, p_2, \dots)$ denote finite positive constants that depend only on the parameters $p_1, p_2, \dots$ inside the parentheses, and whose exact value is not important.

Many arguments in this paper depend on the assumption $W(k) = k^\rho$, $\rho > 1$. The remark at the end of Section 5 gives a detailed list of the "critical" places where special properties of power functions are used.

2. A useful construction. The (discrete time) edge reinforced random walk (I, X) on G is a skeleton of the continuous time reinforced random walk $(\tilde{I}, \tilde{X})$ defined below. We take this useful construction from Sellke [12] who attributes it to Rubin and refers to the Appendix of [2]. An analogous construction appears in [1].

Let the initial weights be $\tilde{X}_0^e = X_0^e$ $(= 1)$ on all edges, and let the initial position of $\tilde{I}$ be equal to $v_0 \in V$. The process $(\tilde{I}, \tilde{X})$ moves by instantaneous jumps at transition times $0 = T_0 < T_1 < T_2 < \cdots \leq \infty$. If $I_{T_j} = v$ then consider mutually independent (and independent of $\mathcal{F}_{T_j}$) exponential (rate 1) random variables

$(E_j^e, e = [v,u], u \in A(v))$. Then $T_{j+1} = T_j + D_j$, where

$$D_j = \frac{E_j^{e'}}{W(\tilde{X}_{T_j}^{e'})} = \min_{u \in A(v)} \frac{E_j^{[v,u]}}{W(\tilde{X}_{T_j}^{[v,u]})}$$

for some $e' = [v, v']$, $v' \in A(v)$. Moreover, $(\tilde{I}_t, \tilde{X}_t) = (\tilde{I}_{T_j}, \tilde{X}_{T_j})$ for $t \in [T_j, T_{j+1})$ and $\tilde{I}_{T_{j+1}} = v'$, $\tilde{X}_{T_{j+1}}^e = \tilde{X}_{T_j}^e + \delta_{e'}(e)$, $e \in E$.

Due to the elementary properties of exponentials the skeleton process $\{(\tilde{I}_{T_j}, \tilde{X}_{T_j}), j \geq 0\}$ is a version of the edge reinforced random walk (I, X) from Section 1. Note that the continuous time process $(\tilde{I}, \tilde{X})$ may (and typically will) traverse certain edge(s) infinitely often in a finite amount of time.

Say that the process $\tilde{I}$ is "in contact" with edge $e = [u, v]$ whenever $\tilde{I}$ equals u or v. Define

$$Y_k^e = \int_{[0,\infty)} \mathbb{1}_{\{\tilde{I}_t = u \text{ or } \tilde{I}_t = v\}} \mathbb{1}_{\{\tilde{X}_t^e = k\}} \, dt$$

to be the total amount of time that $\tilde{I}$ is in contact with e while $X_\cdot^e = k$. Then the total time $T^e = \sum_k Y_k^e = \int_{[0,\infty)} \mathbb{1}_{\{\tilde{I}_t = u \text{ or } \tilde{I}_t = v\}} \, dt$ of $\tilde{I}$ in contact with edge e equals the total amount of time that $(\tilde{I}_s, s \in [0, \infty))$ spends at the endpoints of e.

Due to the memoryless property of exponentials, $Y_k^e \leq Z_k^e$ where Z_k^e is an exponential [rate $W(k)$] random variable. Due to independence of $\{E_j^e, e \in E, j \geq 1\}$, it is easy to see that one can construct a coupling of $\{Y_k^e, e \in E, k \geq 0\}$ and a family $\{Z_k^e, k \geq 1, e \in E\}$ of independent exponential random variables [where Z_k^e has rate $W(k)$, $k \geq 1$, $e \in E$] so that $Y_k^e = Z_k$ on event {edge e is traversed at least k times}, for details see [12]. Since $E \sum_k Z_k^e = \sum_k 1/W(k) < \infty$ by (2), we get that $T^e \leq \sum_k Z_k^e$ is finite almost surely. Moreover, on the event $\{\sup_t \tilde{X}_t^e > k\}$, where e is traversed by $\tilde{I}$ more than k times, we have $Y_k^e = Z_k^e$. The last observation is the main ingredient in the slick proof of Sellke ([12], Theorem 3) showing assertion (3) for $\mathcal{G}$ the d-dimensional integer lattice, under the minimal assumption (2).

Let $\mathcal{G}_1$ be the random subgraph of $\mathcal{G}$ spanned by all edges that are visited at least once by the walk. Clearly $\mathcal{G}_\infty \subset \mathcal{G}_1$ almost surely. It is not difficult to see (and proved in [12], Lemma 4) that (2) implies

$$P(\mathcal{G}_1 \text{ is a finite graph}) = 1, \tag{4}$$

whenever $\mathcal{G}$ has bounded degree. This implies $P(\mathcal{G}_\infty$ is a finite and connected graph$) = 1$, as shown in [12].

A *cycle* of *length* k ($k \geq 3$) is a k-tuple of vertices $(v_1, \ldots, v_k)$, such that $v_1 \sim v_2 \sim \cdots \sim v_k \sim v_1$. There are no repeated vertices in a cycle. Call a cycle of even (resp. odd) length an *even* (*resp. odd*) *cycle*.

LEMMA 1. *Let* (I, X) *be the edge reinforced random walk on* $\mathcal{G}$ *with weights* W *satisfying* (2). *Then*:

(i) $P(\mathcal{G}_\infty$ *does not contain an even cycle*$) = 1$, *and*
(ii) $P(\mathcal{G}_\infty$ *is a tree*$) = P(\mathcal{G}_\infty$ *has only one edge*$)$.

As pointed out by Stas Volkov (personal communication), this is a part of the argument in [12], proof of Theorem 3. For reader's benefit we give the argument for $P(\mathcal{G}_\infty$ does not contain a square$) = 1$ (the proof for longer even cycles is similar) and for (ii). Let $e_1 = [v_1, v_2], e_2 = [v_2, v_3], e_3 = [v_3, v_4], e_4 = [v_4, v_1]$ be the edges of a square Σ such that

$$P(\Sigma \subset \mathcal{G}_\infty) > 0. \tag{5}$$

Since (I, X) is the skeleton of $(\tilde{I}, \tilde{X})$, (5) implies that $\tilde{I}$ traverses all four edges of Σ infinitely often, with positive probability. Let T^{e_i} denote the total time of $\tilde{I}$ in contact with edge e_i. Now the total time T that $\tilde{I}$ spends at the set of vertices $\{v_1, v_2, v_3, v_4\}$ of Σ can be obtained from T^{e_i}, $1 \le i \le 4$ in two (symmetric) ways. Since $T =$ (total time at v_1 or v_2) $+$ (total time at v_3 or v_4), then $T = T^{e_1} + T^{e_3}$, and similarly, $T = T^{e_2} + T^{e_4}$. On the event in (5), $\sum_{k=1}^\infty Y_k^{e_i} = \sum_{k=1}^\infty Z_k^{e_i}$ by the observation preceding the lemma. Therefore, on the same event,

$$T = \sum_{k=1}^\infty Z_k^{e_1} + \sum_{k=1}^\infty Z_k^{e_3} = \sum_{k=1}^\infty Z_k^{e_2} + \sum_{k=1}^\infty Z_k^{e_4}. \tag{6}$$

Since two independent continuous random variables are equal with probability 0, (5) is impossible.

PROOF OF (ii). Suppose that Σ is a tree with at least two edges such that

$$P(\Sigma = \mathcal{G}_\infty) > 0. \tag{7}$$

Let v be a vertex in Σ such that there are at least two [and at most $D(\mathcal{G})$] different edges in Σ incident to v. Call these edges $e_1 = [v, u_1], e_2 = [v, u_2], \dots, e_k = [v, u_k]$, $u_i \ne u_j$ for $1 \le i \ne j \le k$. Relation (7) implies existence of time t and integers $m_1, m_2, \dots, m_k$ such that

$$P\left(\{\Sigma = \mathcal{G}_\infty\} \cap \{\tilde{I}_{t-} \ne \tilde{I}_t = v\} \cap \bigcap_{s \ge 0} \{\tilde{I}_{t+s} \in \Sigma\} \cap \bigcap_{i=1}^k \{\tilde{X}_t^{[v,u_i]} = m_i\}\right) > 0.$$

However, on the event above, the time $T^{v,t} := \int_0^\infty \mathbb{1}_{\{\tilde{I}_t + s = v\}}\, ds$ in contact with vertex v on the interval $[t, \infty)$ equals both $\sum_{l=0}^\infty Z_{m_1+2l}^{[v,u_1]}$ and $\sum_{l=0}^\infty Z_{m_2+2l}^{[v,u_2]}$, and these are again two independent continuous random variables which are equal with probability 0, a contradiction.

It is important to note that the above argument easily generalizes to the case of initial state X_0^e, where $\{X_0^e, e \in E\}$ is a bounded set, but it depends heavily on the fact that the cycle Ξ has even length. $\square$

LEMMA 2. *Under the assumptions of Lemma* 1,

$$P(\mathcal{G}_\infty \textit{ contains an odd cycle}) = P\big(\mathcal{G}_\infty \textit{ contains a unique } (\textit{odd}) \textit{ cycle}\big).$$

PROOF. This is a consequence of the fact that any two co-existing cycles inside of $\mathcal{G}_\infty$ would either be disjoint but connected by a finite path in $\mathcal{G}_\infty$ (since $\mathcal{G}_\infty$ is a finite connected graph), or would have exactly one vertex in common, or would have at least two vertices in common.

The first case is impossible by the following argument. Let $\Xi_1 \cup \Xi_2 \cup \Xi_3 \subset \mathcal{G}_\infty$, where $\Xi_1 = (1, 2, \dots, \ell_1)$, $\Xi_2 = (\ell_1 + \ell, \ell_1 + \ell + 1, \dots, \ell_1 + \ell + \ell_2 - 1)$ are odd cycles, $\ell_1, \ell_2 \geq 3$, and Ξ_3 is a path $1 \sim \ell_1 + 1 \sim \cdots \sim \ell_1 + \ell$ of length $\ell \geq 1$. Consider a combined path

$$\begin{aligned}1 \mapsto 2 \mapsto \cdots \mapsto \ell_1 \mapsto 1 \mapsto \ell_1 + 1 \mapsto \cdots \mapsto \ell_1 + \ell \mapsto \ell_1 + \ell + 1 \mapsto \cdots \\ \mapsto \ell_1 + \ell + \ell_2 - 1 \mapsto \ell_1 + \ell \mapsto \ell_1 + \ell - 1 \mapsto \cdots \mapsto \ell_1 + 1 \mapsto 1,\end{aligned}$$

and let $e_j, 1 \leq j \leq \ell_1 + \ell_2 + 2\ell$, be the jth edge on the above path. Note that all edges $e \in E(\Xi_1 \cup \Xi_2)$ appear once, while all edges $e \in E(\Xi_3)$ appear twice on the above path. Moreover, since ℓ_1, ℓ_2 are both odd numbers, the sets $E_1 = \{e_j, 1 \leq j \leq \ell_1 + \ell_2 + 2\ell, j \text{ odd}\}$ and $E_2 = \{e_j, 1 \leq j \leq \ell_1 + \ell_2 + 2\ell, j \text{ even}\}$ are disjoint sets of edges. To show this suppose $e \in E_1 \cap E_2$. Then it must be $e \in \{[1, \ell_1 + 1], [\ell_1 + 1, \ell_1 + 2], \dots, [\ell_1 + \ell - 1, \ell_1 + \ell]\}$. However, $[1, \ell_1 + 1]$ equals to both $e_{\ell_1+1} \in E_2$ and $e_{\ell_1+\ell_2+2\ell} \in E_2$ and similarly, $[\ell_1 + k, \ell_1 + k + 1]$, $1 \leq k \leq \ell - 1$, equals both e_{ℓ_1+k+1} and $e_{\ell_1+\ell_2+2\ell-k}$, where $\ell_1 + k + 1$ and $\ell_1 + \ell_2 + 2\ell - k$ have the same parity. The identity analogous to (6),

$$\sum_{e \in E_1} \sum_{k=1}^{\infty} Z_k^e = \sum_{e \in E_2} \sum_{k=1}^{\infty} Z_k^e \qquad \text{on } \{\Xi_1 \cup \Xi_2 \cup \Xi_3 \subset \mathcal{G}_\infty\}$$

implies the statement.

The second case ($\ell = 0$) is again impossible by a similar argument. To illustrate this fact, consider a graph Ξ composed of two triangles with a common vertex v in Figure 1.

One can express the total time $T^v = \int_0^\infty \mathbb{1}_{\{\tilde{I}_t = v\}}\, dt$ in contact with vertex v as both $(T^{e_2} + T^{e_3} - T^{e_1})/2$ and $(T^{e_4} + T^{e_5} - T^{e_6})/2$. This again implies that

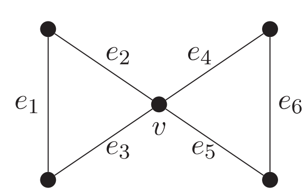

FIG. 1.

$\{\Xi \subset \mathcal{G}_\infty\}$ is a subset of an event on which two independent continuous random variables are equal.

In the last case, where the two odd cycles have at least two vertices in common, a subgraph of the union of the two cycles would be an even cycle, which is impossible by Lemma 1. The details are left to the reader. $\square$

COROLLARY 1. *Under the assumptions of Lemma* 1,

$$P(\mathcal{G}_\infty \textit{ contains a vertex of degree } \geq 3) = 0.$$

PROOF. The proof is similar to the one for Lemma 1(ii). From the previous results, note that there are only two possibilities for a vertex of $\mathcal{G}_\infty$ to have three or more adjacent edges: either none of the adjacent edges belongs to a cycle in $\mathcal{G}_\infty$, or two of the adjacent edges belong to the unique cycle in $\mathcal{G}_\infty$ and all the other edges do not belong to a cycle in $\mathcal{G}_\infty$. The first case above occurs with probability 0 by the argument in Lemma 1(ii). The second case is similar. The following example illustrates the argument. Assume that $\mathcal{G}_\infty$ contains the graph Σ in Figure 2 with positive probability.

Then, as before, there would exist positive integers m_e, m_{e_i}, $i = 1, 2, 3$, and a time $t \geq 0$ such that

$$P\left(\{\Sigma \subset \mathcal{G}_\infty\} \cap \{\tilde{I}_{t-} \neq \tilde{I}_t = v\} \cap \bigcap_{s>0}\{\tilde{I}_{t+s} \in \mathcal{G}_\infty\} \cap \bigcap_{i=1}^{k}\{\tilde{X}_t^{[v,u_i]} = m_i\}\right) > 0.$$

Recall the total time $T^{v,t}$ in contact with vertex v from the proof of Lemma 1(ii). On the event above (due to almost sure absence of other cycles) we have both $T^{v,t} = (\sum_{k\geq m_{e_2}} Z_k^{e_2} + \sum_{k\geq m_{e_3}} Z_k^{e_3} - \sum_{k\geq m_{e_1}} Z_k^{e_1})/2$ and $T^{v,t} = \sum_{k\geq 0} Z^e_{m_e+2k}$ which is again impossible by independence and continuity of Z's, therefore $P(\{e, e_1, e_2, e_3\} \subset \mathcal{G}_\infty) = 0$. $\square$

The next statement follows directly from Lemma 1(ii).

COROLLARY 2. *Let $\mathcal{G}$ be a cycle of length ℓ. Under the assumptions of Lemma* 1 *we have* $\{\mathcal{G}_\infty$ *has only one edge*$\} = \{\mathcal{G}_\infty \neq \mathcal{G}\}$ *almost surely.*

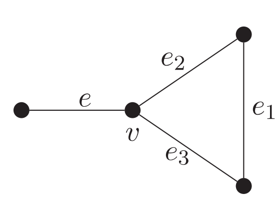

FIG. 2.

REMARKS. In the notation $e = [v, w]$ it is implicit that at most one edge connects any two vertices. One can apply the reasoning of Lemma 1 and Corollary 1 to conclude that, even in the setting where loops $[v, v]$ and multiple edges connecting two vertices exist in $\mathcal{G}$, as long as $D(\mathcal{G}) < \infty$, Lemma 1(i) holds (so cycles of length 2 are not contained in $\mathcal{G}_\infty$ either) and $\mathcal{G}_\infty$ contains subgraphs spanned by two edges $[v, v]$, $[v, w]$ with probability 0.

It is clear that property (4), Lemmas 1, 2 and Corollaries 1, 2 continue to hold (with arguments changed slightly) under a weaker assumption $\sup_{e\in\mathcal{G}} X_0^e < \infty$. So in order to show assertion (3), it is sufficient to show that $\mathcal{G}_\infty$ does not include an odd cycle. Theorem 1 is a consequence of the following proposition.

PROPOSITION 1. *Let $W(k) = k^\rho$ for some fixed $\rho > 1$. Let $\mathcal{G}$ be a cycle of length ℓ. Then*

$$P(\mathcal{G}_\infty \textit{ has only one edge}) = 1. \tag{8}$$

It is plausible that the statement of the proposition (and Theorem 1) remains true under the assumption (2) only.

Both Proposition 1 and Theorem 1 are proved in Section 5.

3. Preliminaries: ERRW on two edges. We first consider a much simpler process, the edge reinforced random walk "on two edges." The graph $\mathcal{G}$ has now three vertices $0, 1, 2$ and two edges $[0, 1] = [1, 0]$ and $[0, 2] = [2, 0]$. Process (I, X) is the same as in Section 1, except we denote $X^{[0,1]}$ by X^1 and $X^{[0,2]}$ by X^2. The transition probabilities are given in (1) when $I_n = 0$, but the particle now deterministically goes back from vertex 1 (and 2) to 0 in one step.

PROPOSITION 2. *If $W(k) = k^\rho$ with $\rho > 1$, the edge reinforced random walk on $\mathcal{G}$ traverses the attracting edge for all large times a.s., equivalently,*

$$P\left(\min_{i=1,2} \sup_{n\geq 1} X_n^i < \infty\right) = 1. \tag{9}$$

REMARK. The above proposition is only a special case of Lemma 1(ii). We sketch here a longer proof since we are going to use some of its ingredients later in the proof of Proposition 1.

From now on assume $W(k) = k^\rho$ with $\rho > 1$.

LEMMA 3. *If $I_0 = 0$, and X_0^1, X_0^2 are positive integers then the process $X_{2n}^1 X_{2n}^2/(X_{2n}^1 + X_{2n}^2)^2$, $n \geq 0$, is a supermartingale.*

PROOF. Calculate

$$
\begin{aligned}
E\biggl(&\frac{X^1_{2n+2}X^2_{2n+2}}{(X^1_{2n+2}+X^2_{2n+2})^2}\biggm|\mathcal{F}_{2n}\biggr)\\
&=\frac{(X^1_{2n}+2)X^2_{2n}}{(X^1_{2n}+X^2_{2n}+2)^2}\frac{(X^1_{2n})^\rho}{(X^1_{2n})^\rho+(X^2_{2n})^\rho}\\
&\quad+\frac{X^1_{2n}(X^2_{2n}+2)}{(X^1_{2n}+X^2_{2n}+2)^2}\frac{(X^2_{2n})^\rho}{(X^1_{2n})^\rho+(X^2_{2n})^\rho}\\
&=\frac{X^1_{2n}X^2_{2n}}{(X^1_{2n}+X^2_{2n})^2}\frac{(X^1_{2n}+X^2_{2n})^2}{(X^1_{2n}+X^2_{2n}+2)^2}\\
&\quad\times\frac{(X^1_{2n})^\rho+(X^2_{2n})^\rho+2(X^1_{2n})^{\rho-1}+2(X^2_{2n})^{\rho-1}}{(X^1_{2n})^\rho+(X^2_{2n})^\rho}.
\end{aligned}
\tag{10}
$$

It suffices to check that

$$
\frac{(X^1_{2n}+X^2_{2n})^2}{(X^1_{2n}+X^2_{2n}+2)^2}\frac{(X^1_{2n})^\rho+(X^2_{2n})^\rho+2(X^1_{2n})^{\rho-1}+2(X^2_{2n})^{\rho-1}}{(X^1_{2n})^\rho+(X^2_{2n})^\rho}\le 1,\qquad n\ge 1,
$$

which is true, since the maximum of the quantity

$$
\frac{(x^1+x^2)^2}{(x^1+x^2+2)^2}\frac{(x^1)^\rho+(x^2)^\rho+2(x^1)^{\rho-1}+2(x^2)^{\rho-1}}{(x^1)^\rho+(x^2)^\rho}
$$

is obtained for $x^1=x^2=(x^1+x^2)/2$ [perhaps the simplest way to check this is by setting $x^1=\alpha x$, $x^2=(1-\alpha)x$ and by maximizing in α], and since $(1+2/x)(x/(x+1))^2<1$ if $x>0$. $\square$

It immediately follows from the supermartingale convergence theorem that $(X^1_nX^2_n)/(X^1_n+X^2_n)^2$ has an almost sure nonnegative limit $\in[0,1/4]$, so it must be

$$
\lim_n\frac{X^1_n}{n}=X\qquad\text{a.s.}\tag{11}
$$

for some random $X\in[0,1]$. Then, intuitively, along each path X^1_{2n} is asymptotically equal to a Binomial$(n,X^\rho/(X^\rho+(1-X)^\rho))$ variable multiplied by 2, and therefore

$$
X=\text{ long run average }=P(\text{success})=\frac{X^\rho}{X^\rho+(1-X)^\rho}.\tag{12}
$$

So the only possible values X may take are 0, 1/2 and 1, and provided 1/2 can be excluded, one is a step closer to (9). One can show that indeed the limit X in (11) may take only values 0 or 1 and that the events $\{X=0\}$ and $\{\max_{n\ge2}X^1_n<\infty\}$ are equal; see [6] for similar arguments.

In the next section we will need the following corollary.

COROLLARY 3. *For any two $c_1 < c_2 < 1$,*

$$(13)\qquad P\left(\sup_{k\ge 1}\frac{X^1_{n+k}}{X^1_{n+k}+X^2_{n+k}}\le\frac{c_2}{2}\,\middle|\,\frac{X^1_n}{X^1_n+X^2_n}\le\frac{c_1}{2}, I_n=0\right)\ge 1-\frac{c_1(2-c_1)}{c_2(2-c_2)},$$

$$(14)\qquad P\left(\sup_k X^1_k<\infty\,\middle|\,\frac{X^1_n}{X^1_n+X^2_n}\le\frac{c_1}{2}, I_n=0\right)\ge 1-\frac{c_1(2-c_1)}{c_2(2-c_2)}.$$

PROOF. If Z_k is a nonnegative supermartingale then by optional stopping

$$(15)\qquad P\left(\sup_{k\ge 0} Z_{k+n}\le b|\mathcal{F}_n\right)\mathbb{1}_{\{Z_n\le a\}}\ge\left(1-\frac{a}{b}\right)^+\mathbb{1}_{\{Z_n\le a\}}.$$

Fix $c < 1$. Note that $X^1_n/(X^1_n + X^2_n) \le c/2$ implies $X^1_n X^2_n/(X^1_n + X^2_n)^2 \le c(2-c)/4$, and also $X^1_n/(X^1_n + X^2_n) < 1/2$ and $\sup_{k\ge 0} X^1_{n+2k}X^2_{n+2k}/(X^1_{n+2k} + X^2_{n+2k})^2 \le c(2-c)/4$ imply $\sup_{k\ge 0} X^1_{n+k}/(X^1_{n+k} + X^2_{n+k}) < c/2$. Since $X^1_{n+2k}X^2_{n+2k}/(X^1_{n+2k} + X^2_{n+2k})^2$ is a nonnegative supermartingale, and since $\sup_{k\ge 1} X^1_{n+k}/(X^1_{n+k} + X^2_{n+k})$ is realized at $n + 2k$ for some k, (15) and previous observations give statement (13). Furthermore, by Proposition 2 (or the remark following it), on the event $\{\sup_{k\ge n} X^1_k/(X^1_k + X^2_k) \le c_2/2 < 1/2\}$ we know that vertex 1 will be visited only finitely often, yielding (14). □

4. Attracting edge for ERRW on a cycle. Let $\mathcal{G}$ be a cycle of length ℓ. Denote its vertices by $\{1, 2, \dots, \ell\}$ and its edges by $\{[1, 2], [2, 3], \dots, [\ell, 1]\}$. The statement of Proposition 1 is true for even ℓ by Lemma 1, so we may assume that $\ell \ge 3$ is an odd number, although this is not necessary for the argument below. Without loss of generality, assume the initial configuration $X^{[1,2]}_0, X^{[2,3]}_0, \dots, X^{[\ell,1]}_0$ is given at the initial time $X^{[1,2]}_0 + X^{[2,3]}_0 + \cdots + X^{[\ell,1]}_0$. Intuitively, in order to show Proposition 1, one hopes to show first that the ERRW on $\mathcal{G}$ eventually stops traversing one of the edges with probability 1, and then apply Lemma 1(ii). The actual analysis is done in terms of searching for a "candidate" attracting edge. The addition in brackets $[\cdot,\cdot]$ is always done modulo ℓ. It is easy to check that if $X^{[1,2]}_0$ is large, and all the initial weights are comparable ($X^{[i-1,i]}_0 \approx X^{[i,i+1]}_0$) then only after a very large number of steps it may be true that one edge is traversed many more times than the others. Indeed, the attracting edge appears gradually, as shown by the sequence of lemmas below.

It seems difficult to show a priori that $(X^{[1,2]}_n/n, X^{[2,3]}_n/n, \dots, X^{[\ell,1]}_n/n)$ has a limit as $n \to \infty$. Lemmas 7 and 9 imply that with probability 1, the vector of rescaled weights $(X^{[1,2]}_n/n, X^{[2,3]}_n/n, \dots, X^{[\ell,1]}_n/n)$ does not converge to

$(1/\ell, 1/\ell, \dots, 1/\ell)$. Lemma 11 says there exists an edge such that, up to some future time, the particle traverses this edge many more times than one of its neighbors. Lemmas 13 and 14 use coupling arguments to show that eventually there exists one edge which is traversed at least $2^{1/(\rho-1)}$ times as both of its neighbors, and finally that its neighbors end up being traversed only finitely many times, with probability uniformly bounded away from 0.

For $r > 0$, let

$$\tau_0^m(r) := \inf\left\{k \geq m : \max_i |X_k^{[i-1,i]} - X_k^{[i,i+1]}| > r\sqrt{k}\right\},$$

$$\tau_1^m(r) := \inf\left\{k \geq m : \max_i |X_k^{[i-1,i]} - X_k^{[i,i+1]}| > rk\right\},$$

$$\tau_2^m(r) := \inf\left\{k \geq m : \frac{X_k^{[j,j-1]}}{X_k^{[j,j+1]}} \notin \left(\frac{1}{r}, r\right) \text{ for some } j \in \{1, \dots, \ell\}\right\}.$$

For a fixed integer m, let $T(0) := m$ and

$$T(n) = \inf\{k > T(n-1) : I_k = 1\} \tag{16}$$

be the successive times of return to vertex 1. If $\{I_k \neq 1, \forall\, k > T(n-1)\}$ for some n, then set $T(n) = T(n+1) = \cdots = \infty$. Since $I_{T(n)} = 1$ on the event $\{T(n) < \infty\}$, in the next step the particle moves to $I_{T(n)+1} \in \{2, \ell\}$, and $X^{[1, I_{T(n)+1}]}$ increases by 1. If $T(n+1) < \infty$ then $[I_k, I_{k+1}] \notin \{[1,2], [1,\ell]\}$ for all $T(n) < k < T(n+1) - 1$, and $[I_{T(n+1)-1}, I_{T(n+1)}] \in \{[1,2], [1,\ell]\}$. Call $\{(I_k, X_k) : T(n) < k \leq T(n+1)\}$ an *excursion of* (I, X) *away from* vertex 1. Let $M > 0$ be a large number to be fixed later. Recall the filtration $\mathcal{F}_n = \sigma\{(I_k, X_k^e, e \in E), k = 0, \dots, n\}$, $n \geq 1$. For $j \in \{2, \ell\}$, let

$$\begin{aligned} p_j(n) = P\big(&\{I_{T(n+1)-1} = j, T(n+1) < \tau_2^m(2M)\} \\ &\cup \{T(n+1) \geq \tau_2^m(2M)\} \big| \mathcal{F}_{T(n)}, I_{T(n)+1} = j\big). \end{aligned} \tag{17}$$

Probability $p_j(n)$ is an analytically tractable analogue of the quantity

$$q_j(n) = P\big(I_{T(n+1)-1} = j, T(n+1) < \infty \big| \mathcal{F}_{T(n)}, I_{T(n)+1} = j\big), \tag{18}$$

the probability that the last edge in the nth excursion equals $[1, j]$, given that the first edge equals $[1, j]$. The $p_j(n)$'s in (17) are more useful in this proof than the $q_j(n)$'s due to the fact that $\tau_2(2M)$ may happen when $X_\cdot^{[j,j-1]}/X_\cdot^{[j,j+1]} \notin (1/2M, 2M)$ for some $j \neq 1$ during some excursion away from 1. Although this is a good event [since $\tau_2(M) < \tau_2(2M)$], the estimates in Lemmas 4–6 apply only in the intermediate regime where all the weights are "balanced." The supermartingale calculations ahead are based on estimates for $p_j(n)$.

The following three technical lemmas will be proved in Section 6. If (I, X) were a classical (nonreinforced) random walk with fixed edge weights $(X_{T(n)}^\cdot)^\rho$

then $q_j(n) = P(I_{T(n+1)-1} = j | I_{T(n)+1} = j)$ and by reversibility

$$\frac{1 - q_2(n)}{1 - q_\ell(n)} = \left(\frac{X^{[1,\ell]}_{T(n)}}{X^{[1,2]}_{T(n)}}\right)^\rho.$$

Similarly, the estimates in Lemmas 5 and 6 have direct analogs in the classical setting.

LEMMA 4. *Let $\iota \in (0, 1/2)$ be fixed.*

(i) *For each fixed $M \in (0, \infty)$, we have*

$$\frac{1 - p_2(n)}{1 - p_\ell(n)} = \left(\frac{X^{[1,\ell]}_{T(n)}}{X^{[1,2]}_{T(n)}}\right)^\rho \left(1 + O\left(\frac{1}{T(n)^{1-\iota}}\right)\right)$$

on the event $\{\tau_2^m(M) > T(n)\}$.

(ii) *For any fixed $\varepsilon > 0$, there exists $n(M, \varepsilon)$ such that for all $m \ge n(M, \varepsilon)$ on the event $\{\tau_2^m(M) > T(n)\} \cap \{X^{[1,2]}_{T(n)} - X^{[1,\ell]}_{T(n)} > \varepsilon\sqrt{T(n)}\}$,*

$$p_\ell(n) \le p_2(n).$$

LEMMA 5. *For each $\delta > 0$ there is an $\varepsilon(\delta) > 0$ so that for any $\varepsilon_1 \in (0, \varepsilon(\delta))$ there exists $n_1(\delta, \varepsilon_1) < \infty$ with the property: if $m \ge n_1(\delta, \varepsilon_1)$ then on the event $\{\tau_1^m(\varepsilon_1) > T(n)\}$ we have*

$$\frac{\ell - 1}{\ell}(1 - \delta) \le p_j(n), \qquad j = 2, \ell.$$

The next lemma gives a stronger statement on the event $\{\tau_0^m(K) > T(n)\}$.

LEMMA 6. *For any $K < \infty$ on the event $\{\tau_0^m(K) > T(n)\}$, we have:*

(i) $$\frac{\ell - 1}{\ell}\big(1 - c_1(n)\big) \le p_j(n) \le \frac{\ell - 1}{\ell}\big(1 + c_1(n)\big), \qquad j = 2, \ell,$$

(ii) $$\frac{1}{2}\big(1 - c_2(n)\big) \le \frac{(X^{[1,j]}_{T(n)})^\rho}{(X^{[1,2]}_{T(n)})^\rho + (X^{[1,\ell]}_{T(n)})^\rho} \le \frac{1}{2}\big(1 + c_2(n)\big), \qquad j = 2, \ell,$$

where the nonnegative sequences $c_1(n), c_2(n)$ converge to 0 as $n \to \infty$.

LEMMA 7. *For any fixed K, $0 < K < \infty$,*

$$\inf_m P\big(\tau_0^m(K) < \infty | \mathcal{F}_m\big) > 0.$$

PROOF. Fix some K and let m be the initial time large enough so that $\tau_0^m(K) < \tau_2^m(M)$. The reason why such m exists is the following: for each n,

$\sum_{i=1}^{\ell} X_n^{[i,i+1]} = n$, and so if $|X_n^{[i,i+1]} - X_n^{[i,i-1]}| \le K\sqrt{n} + 1 < (K+1)\sqrt{n}$ for all i, then by telescoping

$$X_n^{[i,i+1]} \in \left[\frac{n - C(\ell)(K+1)\sqrt{n}}{\ell}, \frac{n + C(\ell)(K+1)\sqrt{n}}{\ell}\right] \tag{19}$$

for some finite positive constant $C(\ell)$, in particular, $X_n^{[i,i-1]}/X_n^{[i,i+1]} \approx 1$ for all n large $n \le \tau_0^m(K)$. Without loss of generality assume $I_m = 1$, and observe the process at times $T(n)$ of return to vertex 1, as defined above Lemma 4. Denote by $N(k)$ the number of returns to vertex 1 in the time interval $\{m, m+1, \dots, m+k\}$, so that $N(T(k)) = k$. For $k \ge 0$ such that $T(k) < \tau_0^m(K)$ define

$$Y_{k+1}^1 = \begin{cases} \Big(\big(X_{T(k+1)}^{[1,2]} - X_{T(k+1)}^{[1,\ell]}\big) - \big(X_{T(k)}^{[1,2]} - X_{T(k)}^{[1,\ell]}\big)\Big), \\ \qquad\qquad\qquad\qquad\qquad\qquad T(k+1) < \tau_2^m(2M), \\ 2, \qquad T(k+1) \ge \tau_2^m(2M) \quad \text{and} \quad I_{T(k)+1} = 2, \\ -2, \qquad T(k+1) \ge \tau_2^m(2M) \quad \text{and} \quad I_{T(k)+1} = \ell. \end{cases}$$

Note that Y_{k+1}^1 takes values in the set $\{-2, 0, 2\}$. Moreover, on $\{T(k) < \tau_0^m(K)\}$,

$$P\big(Y_{k+1}^1 = 2 | \mathcal{F}_{T(k)}\big) = \frac{(X_{T(k)}^{[1,2]})^\rho}{(X_{T(k)}^{[1,2]})^\rho + (X_{T(k)}^{[1,\ell]})^\rho} p_2(k),$$

$$P\big(Y_{k+1}^1 = -2 | \mathcal{F}_{T(k)}\big) = \frac{(X_{T(k)}^{[1,\ell]})^\rho}{(X_{T(k)}^{[1,2]})^\rho + (X_{T(k)}^{[1,\ell]})^\rho} p_\ell(k).$$

Let $(Y_k^2, k \ge 1)$ be a sequence of independent and identically distributed random variables, and independent of $\mathcal{F}_\infty = \bigcup_n \mathcal{F}_n$, such that the distribution of Y_1^2 is also concentrated on the set $\{-2, 0, 2\}$, and $P(Y_1^2 = -2) = P(Y_1^2 = 2) = (\ell - 1)/(2\ell)$. Define

$$Y_{k+1} = Y_{k+1}^1 \mathbb{1}_{\{T(k) < \tau_0^m(K)\}} + Y_{k+1}^2 \mathbb{1}_{\{T(k) \ge \tau_0^m(K)\}}.$$

Due to Lemma 6,

$$\left| \operatorname{var}\big(Y_{k+1} | \mathcal{F}_{T(k)}\big) - 4\frac{\ell - 1}{\ell} \right| \to 0 \qquad \text{almost surely.} \tag{20}$$

Moreover, on event $\{\tau_0^m(K) > T(k)\}$,

$$
\begin{aligned}
&E(Y_{k+1}|\mathcal{F}_{T(k)}) \\
&\quad = E(Y_{k+1}^1|\mathcal{F}_{T(k)}), \\
&\quad = 2p_2(k)\frac{(X_{T(k)}^{[1,2]})^\rho}{(X_{T(k)}^{[1,2]})^\rho + (X_{T(k)}^{[1,\ell]})^\rho} - 2p_\ell(k)\frac{(X_{T(k)}^{[1,\ell]})^\rho}{(X_{T(k)}^{[1,2]})^\rho + (X_{T(k)}^{[1,\ell]})^\rho} \qquad (21) \\
&\quad = \frac{2(p_2(k) - p_\ell(k))(X_{T(k)}^{[1,2]})^\rho}{(X_{T(k)}^{[1,2]})^\rho + (X_{T(k)}^{[1,\ell]})^\rho} + \frac{2p_\ell(k)((X_{T(k)}^{[1,2]})^\rho - (X_{T(k)}^{[1,\ell]})^\rho)}{(X_{T(k)}^{[1,2]})^\rho + (X_{T(k)}^{[1,\ell]})^\rho}.
\end{aligned}
$$

Fix $\iota \in (0, 1/2)$. Note that, by Lemma 4(i) on event $\{\tau_0^m(K) > T(k)\} \subset \{\tau_2^m(M) > T(k)\}$,

$$
p_2(k) - p_\ell(k) = (1 - p_\ell(k))\left(1 - \frac{(X_{T(k)}^{[1,\ell]})^\rho}{(X_{T(k)}^{[1,2]})^\rho}\left(1 + O\left(\frac{1}{T(k)^{1-\iota}}\right)\right)\right), \qquad (22)
$$

so (21) becomes

$$
\begin{aligned}
&E(Y_{k+1}|\mathcal{F}_{T(k)})\mathbb{1}_{\{\tau_0^m(K)>T(k)\}} \\
&\quad = \left(\frac{2(X_{T(k)}^{[1,2]})^\rho - 2(X_{T(k)}^{[1,\ell]})^\rho}{(X_{T(k)}^{[1,2]})^\rho + (X_{T(k)}^{[1,\ell]})^\rho} + O\left(\frac{1}{T(k)^{1-\iota}}\right)\right)\mathbb{1}_{\{\tau_0^m(K)>T(k)\}}.
\end{aligned}
$$

Since on $\{\tau_0^m(K) > T(k)\}$ we have $|X_{T(k)}^{[1,2]} - X_{T(k)}^{[1,\ell]}| \le K\sqrt{T(k)}$, therefore $|(X_{T(k)}^{[1,2]})^\rho - (X_{T(k)}^{[1,\ell]})^\rho| \le (\max\{X_{T(k)}^{[1,2]}, X_{T(k)}^{[1,\ell]}\})^{\rho-1}\rho K\sqrt{T(k)}$ and

$$
|E(Y_{k+1}|\mathcal{F}_{T(k)})| \le \frac{C(K,\ell,\rho)}{\sqrt{T(k)}} \qquad (23)
$$

for some fixed $C(K,\ell,\rho) < \infty$. Here we use the fact that if $T(k) < \tau_0^m(K)$ then $X_{T(k)}^{[i,i+1]} \approx T(k)/\ell$ for each i [see also (19) and (26)]. Moreover, note that on $\{\tau_0^m(K) = \infty\}$ we have $X_{T(k)}^{[i,i+1]}/T(k) \to 1/\ell$ for all i, due to $\sum_{i=1}^{\ell} X_{T(k)}^{[i,i+1]} = T(k)$. Since the number of visits to vertex 1 in any time interval differs by at most 1 from the half of the total number of traversals of adjacent edges $[1,2]$ and $[1,\ell]$ in this interval, we have

$$
\frac{k}{T(k)} = \frac{N(T(k))}{T(k)} \to \frac{1}{\ell} \qquad \text{on } \{\tau_0^m(K) = \infty\} \text{ almost surely.}
$$

Also due to (23), on $\{\tau_0^m(K) = \infty\}$,

$$
\left|\sum_{k=1}^{l} E(Y_k|\mathcal{F}_{T(k-1)})\right| \le 2C(K,\ell,\rho)\sqrt{T(l)}.
$$

Asymptotic behavior of the variance (20) and the central limit theorem for martingales (Durrett [4]) give

$$\frac{\sum_{k=1}^{l}(Y_k - E(Y_k|\mathcal{F}_{T(k-1)}))}{\sqrt{l}} \to \mathcal{N}\left(0, 4\left(1-\frac{1}{\ell}\right)\right),$$

therefore, on $\{\tau_0^m(K)=\infty\} \subset \bigcap_k \bigcap_{i=1}^{\ell}\{|X_{T(k)}^{[i,i+1]} - X_{T(k)}^{[i,i-1]}| \le K\sqrt{T(k)}, T(k) < \tau_0^m(K)\}$,

$$\begin{aligned} K &> \frac{X_{T(l)}^{[1,2]} - X_{T(l)}^{[1,\ell]}}{\sqrt{T(l)}} = \frac{\sum_{k=1}^{l} Y_k + X_m^{[1,2]} - X_m^{[1,\ell]}}{\sqrt{T(l)}} \\ &= \frac{\sum_{k=1}^{l}(Y_k - E(Y_k|\mathcal{F}_{T(k-1)}))}{\sqrt{l}} \frac{\sqrt{l}}{\sqrt{T(l)}} \\ &\quad + \frac{X_m^{[1,2]} - X_m^{[1,\ell]}}{\sqrt{T(l)}} + \frac{\sum_{k=1}^{l} E(Y_k|\mathcal{F}_{T(k-1)})}{\sqrt{T(l)}}, \end{aligned}$$

where the last expression behaves asymptotically as a normal random variable $\mathcal{N}$ with expectation 0 and variance $4(\ell-1)/\ell^2$, plus a term bounded by $2C(K,\ell,\rho)$ in absolute value. In particular,

$$\begin{aligned} \sup_m P(\tau_0^m(K)=\infty|\mathcal{F}_m) &\le P(\mathcal{N} < K + 2C(K,\ell,\rho)) \\ &= c(K,\ell,\rho) < 1, \end{aligned}$$

which implies the statement. $\square$

From now on assume that $K \in [20,\infty)$ is large so that

$$1 - 64\ell^4 \sum_{i=0}^{\infty} \frac{1}{2^{3i/2}} \frac{1}{K^{3/2}} > 0 \quad \text{and} \quad \prod_{i=0}^{\infty}\left(1 - \frac{1}{(2^i K)^{1/4}}\right) > 2/3. \tag{24}$$

Without loss of generality suppose $\tau_0^m(K) = \inf\{k \ge m : |X_k^{[1,2]} - X_k^{[1,\ell]}| > K\sqrt{k}\}$, let $m_0 = \tau_0^m(K)$, and assume $I_{m_0} = 1$ and $X_{m_0}^{[1,2]} > X_{m_0}^{[1,\ell]}$.

Fix $\varepsilon_1 > 0$ such that

$$\varepsilon_1 \ell < 1/4, \tag{25}$$

and $\varepsilon_1 < \varepsilon(1/10)$ where $\varepsilon(1/10)$ is from Lemma 5. Note that then $(1+\varepsilon_1\ell)/(1-\varepsilon_1\ell) < 2$. Further assume that the initial m is large enough so that

$$\tau_0^m(K) < \tau_1^m(\varepsilon_1) < \tau_1^m(2\varepsilon_1) < \tau_2^m(M) < \tau_2^m(2M),$$

provided all the stopping times above (except maybe the last one) are finite. This can be done without loss of generality by (19) and by noting that if $T(n) \le \tau_1^m(\varepsilon_1)$, then $X_{T(n)}^{[1,2]} + X_{T(n)}^{[2,3]} + \cdots + X_{T(n)}^{[\ell-1,\ell]} = T(n)$ implies

$$\frac{T(n)}{\ell}(1-\varepsilon_1\ell) \le X_{T(n)}^{[i,i+1]} \le \frac{T(n)}{\ell}(1+\varepsilon_1\ell), \qquad i \in \{1,\ldots,\ell\}, \tag{26}$$

and therefore

$$\frac{X^{[1,2]}_{T(n)}}{X^{[1,\ell]}_{T(n)}} \le \frac{1+\varepsilon_1\ell}{1-\varepsilon_1\ell}. \tag{27}$$

The above implies $\tau_1^m(\varepsilon_1) = \tau_1^{m_0}(\varepsilon_1)$, $\tau_2^m(r) = \tau_2^{m_0}(r)$, $r = M, 2M$, and both notations will be used below. In order to show that $\{\tau_1^m(\varepsilon_1) < \infty\}$ happens with probability uniformly bounded away from 0, (cf. Lemma 9) we will construct a supermartingale analogous to the one in Lemma 3. Let $T(0) := m_0$ and $T(n)$ be defined in (16).

Let $f(x,y) = \log(x) + \log(y) - 2\log(x+y) + \log(4) = \log(1-(x-y)^2/(x+y)^2)$, and consider

$$Y_{k+1} = \begin{cases} f\big(X^{[1,2]}_{T(k+1)}, X^{[1,\ell]}_{T(k+1)}\big), & T(k+1) < \tau_2^m(2M), \\ f\big(X^{[1,2]}_{T(k)}+2, X^{[1,\ell]}_{T(k)}\big), & T(k+1) \ge \tau_2^m(2M) \text{ and } I_{T(k)+1} = 2, \\ f\big(X^{[1,2]}_{T(k)}, X^{[1,\ell]}_{T(k)}+2\big), & T(k+1) \ge \tau_2^m(2M) \text{ and } I_{T(k)+1} = \ell. \end{cases} \tag{28}$$

Define

$$\begin{aligned} \tau^+(K) &= \inf\big\{n \ge 0 : X^{[1,2]}_{T(n)} - X^{[1,\ell]}_{T(n)} > 2K\sqrt{T(n)}\big\}, \\ \tau^-(K) &= \inf\big\{n \ge 0 : X^{[1,2]}_{T(n)} - X^{[1,\ell]}_{T(n)} < K\sqrt{T(n)}/(2\ell)\big\}, \\ \tilde\tau &= \tilde\tau(\varepsilon_1) = \inf\big\{n \ge 0 : T(n) \ge \tau_1^{m_0}(\varepsilon_1)\big\} \end{aligned}$$

and

$$\tau(K) := \tau^+(K) \wedge \tau^-(K) \wedge \tilde\tau. \tag{29}$$

Set

$$\mathcal{H}_n = \mathcal{F}_{T(n)\wedge\tau_2^m(2M)}. \tag{30}$$

LEMMA 8. *Let $\varepsilon_1 < \min\{1/(4\ell), \varepsilon(1/10)\}$. For sufficiently large m ($\le m_0$) we have $(Y_{n\wedge\tau(K)}, n \ge 0)$ is a $\mathcal{H}$-supermartinagale for any $K \ge 20$. Moreover*:

(i) *On $\{n < \tau^-(K) \wedge \tilde\tau\}$,*

$$E(\Delta Y_n | \mathcal{H}_n) \le -\frac{1}{3}\frac{(X^{[1,2]} - X^{[1,\ell]})((X^{[1,2]})^{\rho-1} - (X^{[1,\ell]})^{\rho-1})}{(X^{[1,2]} + X^{[1,\ell]})((X^{[1,2]})^{\rho} + (X^{[1,\ell]})^{\rho})}, \tag{31}$$

$$E\big[\big(\Delta Y_{n\wedge\tau(K)}\big)^2 | \mathcal{H}_n\big] \le \frac{32\ell^4K^2}{T(n)^3}. \tag{32}$$

(ii) *On* $\{\tau_0^m(K) < \infty\}$,

$$P\biggl(\tau^+(K) \wedge \tilde{\tau} < \tau^-(K), \sup_{n\geq 0} Y_{n\wedge\tau(K)} \leq \biggl(1 - \frac{1}{K^{1/4}}\biggr) Y_1 \bigg| \mathcal{F}_{\tau_0^m(K)}\biggr) \geq 1 - \frac{64\ell^4}{K^{3/2}}. \tag{33}$$

Before showing the lemma we do some preliminary calculations. In order to calculate $E(\Delta Y_n|\mathcal{H}_n) = E(Y_{n+1} - Y_n|\mathcal{H}_n)$, approximate

$$\log(X + b) - \log(X) \approx \frac{b}{X} - \frac{b^2}{2X^2} \tag{34}$$

where X equals $X^{[1,2]}_{T(n)}$, $X^{[1,\ell]}_{T(n)}$ or $X^{[1,2]}_{T(n)} + X^{[1,\ell]}_{T(n)}$ and $b \in \{1, 2\}$, and note that

$$\left|\log(X + b) - \log(X) - \left(\frac{b}{X} - \frac{b^2}{2X^2}\right)\right| \leq \frac{16}{3X^3}.$$

The following are ingredients in later calculations:

$$\frac{2}{X} - \frac{4}{X+Y} = \frac{2(Y - X)}{X(X+Y)}, \tag{35}$$

$$\frac{2}{Y} - \frac{4}{X+Y} = \frac{2(X - Y)}{Y(X+Y)}, \tag{36}$$

$$\frac{1}{X} + \frac{1}{Y} - \frac{4}{X+Y} = \frac{(X - Y)^2}{XY(X+Y)}. \tag{37}$$

Moreover,

$$\frac{4}{2X^2} - \frac{8}{2(X+Y)^2} = \frac{2[(X+Y)^2 - 2X^2]}{X^2(X+Y)^2}, \tag{38}$$

$$\frac{4}{2Y^2} - \frac{8}{2(X+Y)^2} = \frac{2[(X+Y)^2 - 2Y^2]}{Y^2(X+Y)^2}, \tag{39}$$

$$\frac{1}{2X^2} + \frac{1}{2Y^2} - \frac{8}{2(X+Y)^2} = \frac{(X+Y)^2(X^2+Y^2) - 8X^2Y^2}{2X^2Y^2(X+Y)^2}. \tag{40}$$

It is easy to check that (40) is always nonnegative and that (38) and (39) are positive as long as $Y < X < 2Y$. Recall the assumption $X^{[1,2]}_{T(0)} > X^{[1,\ell]}_{T(0)}$, and note that $X^{[1,2]}_{T(n)} > X^{[1,\ell]}_{T(n)}$ and $X^{[1,2]}_{T(n)} < 2X^{[1,\ell]}_{T(n)}$, as long as $n < \tau^-(K) \wedge \tilde{\tau}$, for ε_1 small enough so that the right-hand side in (27) is less than 2.

By definition, $\Delta Y_{n\wedge\tau(K)} = Y_{(n+1)\wedge\tau(K)} - Y_{n\wedge\tau(K)}$. If $n + 1 \leq \tau(K)$ this simplifies to $\Delta Y_n = f(X^{[1,2]}_{T(n+1)}, X^{[1,\ell]}_{T(n+1)}) - f(X^{[1,2]}_{T(n)}, X^{[1,\ell]}_{T(n)})$. If $n < \tau - (K) \wedge \tilde{\tau}$, then on the event $\{I_{T(n)+1} = 2\} \cap (\{I_{T(n+1)-1} = 2, T(n+1) < \tau_2^m(2M)\} \cup \{T(n+1) \geq \tau_2^m(2M)\})$,

$$\Delta Y_n = f\big(X^{[1,2]}_{T(n)} + 2, X^{[1,\ell]}_{T(n)}\big) - f\big(X^{[1,2]}_{T(n)}, X^{[1,\ell]}_{T(n)}\big), \tag{41}$$

and similarly, on $\{I_{T(n)+1} = \ell\} \cap (\{I_{T(n+1)-1} = \ell, T(n+1) < \tau_2^m(2M)\} \cup \{T(n+1) \geq \tau_2^m(2M)\})$, one has

$$\Delta Y_n = f\big(X^{[1,2]}_{T(n)}, X^{[1,\ell]}_{T(n)} + 2\big) - f\big(X^{[1,2]}_{T(n)}, X^{[1,\ell]}_{T(n)}\big), \tag{42}$$

while on

$$\begin{aligned}&\big(\{I_{T(n)+1} = \ell\} \cap \{I_{T(n+1)-1} = 2, T(n+1) < \tau_2^m(2M)\}\big)\\ &\qquad \cup \big(\{I_{T(n)+1} = 2\} \cap \{I_{T(n+1)-1} = \ell, T(n+1) < \tau_2^m(2M)\}\big),\end{aligned}$$

$$\Delta Y_n = f\big(X^{[1,2]}_{T(n)} + 1, X^{[1,\ell]}_{T(n)} + 1\big) - f\big(X^{[1,2]}_{T(n)}, X^{[1,\ell]}_{T(n)}\big). \tag{43}$$

Recall $p_j(n)$ defined in (17). Using approximation (34), calculations (35)–(40) with $X = X^{[1,2]}_{T(n)} = X^{[1,2]}$, $Y = X^{[1,\ell]}_{T(n)} = X^{[1,\ell]}$, (41)–(43), and the above observations, one can estimate on $\{n < \tau^-(K) \wedge \tilde{\tau}\}$:

$$\begin{aligned}
&E(\Delta Y_n | \mathcal{H}_n)\\
&\quad \leq \frac{(X^{[1,2]})^\rho}{(X^{[1,2]})^\rho + (X^{[1,\ell]})^\rho}\bigg(p_2(n)\frac{2(X^{[1,\ell]} - X^{[1,2]})}{X^{[1,2]}(X^{[1,2]} + X^{[1,\ell]})}\\
&\qquad\qquad + (1 - p_2(n))\frac{(X^{[1,\ell]} - X^{[1,2]})^2}{X^{[1,2]}X^{[1,\ell]}(X^{[1,2]} + X^{[1,\ell]})}\bigg)\\
&\qquad + \frac{(X^{[1,\ell]})^\rho}{(X^{[1,2]})^\rho + (X^{[1,\ell]})^\rho}\bigg(p_\ell(n)\frac{2(X^{[1,2]} - X^{[1,\ell]})}{X^{[1,\ell]}(X^{[1,2]} + X^{[1,\ell]})}\\
&\qquad\qquad + (1 - p_\ell(n))\frac{(X^{[1,\ell]} - X^{[1,2]})^2}{X^{[1,2]}X^{[1,\ell]}(X^{[1,2]} + X^{[1,\ell]})}\bigg)\\
&\qquad + \frac{16}{3}\bigg(\frac{1}{(X^{[1,2]})^3} + \frac{1}{(X^{[1,\ell]})^3}\bigg).
\end{aligned} \tag{44}$$

PROOF OF LEMMA 8. (i) Fix $\iota \in (0, 1/2)$. For K satisfying (24), assume $m \geq n(M, K/(2\ell))$, where $n(M, K/(2\ell))$ is from Lemma 4(ii). Since $\tau_0^m(K) \equiv m_0 \geq m$ this gives $p_\ell(n) \leq p_2(n)$. Rearrange the terms in (44) to obtain that on $\{n < \tau^-(K) \wedge \tilde{\tau}\}$,

$$
\begin{aligned}
E(\Delta Y_n|\mathcal{H}_n) \le{} & \frac{16}{3}\left(\frac{1}{(X^{[1,2]})^3} + \frac{1}{(X^{[1,\ell]})^3}\right) \\
& + \frac{X^{[1,2]} - X^{[1,\ell]}}{(X^{[1,2]} + X^{[1,\ell]})((X^{[1,2]})^\rho + (X^{[1,\ell]})^\rho)} \\
& \times \Big[-2p_\ell(n)\big((X^{[1,2]})^{\rho-1} - (X^{[1,\ell]})^{\rho-1}\big) \\
& \quad - 2\big(p_2(n) - p_\ell(n)\big)\big(X^{[1,2]}\big)^{\rho-1} + \big(X^{[1,2]} - X^{[1,\ell]}\big) \\
& \quad \times \left(\frac{(X^{[1,2]})^{\rho-1}}{X^{[1,\ell]}}(1 - p_2(n)) + \frac{(X^{[1,\ell]})^{\rho-1}}{X^{[1,2]}}(1 - p_\ell(n))\right)\Big].
\end{aligned} \tag{45}
$$

Recall assumption $\varepsilon_1 < \varepsilon(1/10)$ and further assume that $m \ge n_1(1/10, \varepsilon_1)$ in Lemma 5 so that

$$
p_\ell(n) \ge \frac{\ell - 1}{\ell}(1 - 1/10) \ge 3/5, \qquad \ell \ge 3. \tag{46}
$$

Let

$$
\begin{aligned}
H_1 &= -p_\ell(n)\Big(\big(X^{[1,2]}\big)^{\rho-1} - \big(X^{[1,\ell]}\big)^{\rho-1}\Big), \\
H_2 &= -2\big(p_2(n) - p_\ell(n)\big)\big(X^{[1,2]}\big)^{\rho-1}
\end{aligned}
$$

and

$$
H_3 = \big(X^{[1,2]} - X^{[1,\ell]}\big)\left(\frac{(X^{[1,2]})^{\rho-1}}{X^{[1,\ell]}}(1 - p_2(n)) + \frac{(X^{[1,\ell]})^{\rho-1}}{X^{[1,2]}}(1 - p_\ell(n))\right).
$$

We claim that on $\{n < \tau^-(K) \wedge \tilde{\tau}\}$,

$$
H_1 + H_2 + H_3 \le 0. \tag{47}
$$

Due to Lemma 4(i), relation (22) holds so $H_2 + H_3$ equals the product of $\frac{1-p_\ell(n)}{X^{[1,2]}}$ and

$$
\begin{aligned}
& -2\Big(\big(X^{[1,2]}\big)^\rho - \big(X^{[1,\ell]}\big)^\rho\big(1 + \delta(n)\big)\Big) \\
& \quad + \big(2 + \delta(n)\big)\big(X^{[1,2]} - X^{[1,\ell]}\big)\big(X^{[1,\ell]}\big)^{\rho-1},
\end{aligned} \tag{48}
$$

where $|\delta(n)| = O(1/T(n)^{1-\iota})$. If $\delta(n) \le 0$, $H_2 + H_3 \le 0$ and (47) follows. If $\delta(n) > 0$, we have (recall $X^{[1,2]} > X^{[1,\ell]}$)

$$
H_2 + H_3 \le \frac{\delta(n)(1 - p_\ell(n))}{X^{[1,2]}}\Big(2\big(X^{[1,\ell]}\big)^\rho + \big(X^{[1,2]} - X^{[1,\ell]}\big)\big(X^{[1,\ell]}\big)^{\rho-1}\Big),
$$

and it suffices to show that both

$$
H_1/2 + \frac{\delta(n)(1 - p_\ell(n))}{X^{[1,2]}}2\big(X^{[1,\ell]}\big)^\rho \le 0
$$

and

$$H_1/2 + \frac{\delta(n)(1-p_\ell(n))}{X^{[1,2]}}\big(X^{[1,2]} - X^{[1,\ell]}\big)\big(X^{[1,\ell]}\big)^{\rho-1} \le 0.$$

For the first inequality, use (46) and note that, on the event $\{n < \tau^-(K) \wedge \tilde{\tau}\}$, the intermediate value theorem implies $1 - (X^{[1,\ell]}/X^{[1,2]})^{\rho-1} > (1 - X^{[1,\ell]}/X^{[1,2]}) \times (\rho-1)/2$, and use the fact that on the same event we have $\delta(n) = O(1/(X^{[1,2]})^{1-\iota})$, for $\iota < 1/2$, and $X^{[1,2]} - X^{[1,\ell]} > K\sqrt{X^{[1,2]}}/(2\ell)$. The second inequality is easier, and follows by a similar argument. Now (45) and (47) give, on $\{n < \tau^-(K) \wedge \tilde{\tau}\}$,

$$\begin{aligned} E(\Delta Y_n|\mathcal{H}_n) &\le -\frac{3}{5}\frac{(X^{[1,2]} - X^{[1,\ell]})((X^{[1,2]})^{\rho-1} - (X^{[1,\ell]})^{\rho-1})}{(X^{[1,2]} + X^{[1,\ell]})((X^{[1,2]})^{\rho} + (X^{[1,\ell]})^{\rho})} \\ &\quad + \frac{16}{3}\left(\frac{1}{(X^{[1,2]})^3} + \frac{1}{(X^{[1,\ell]})^3}\right) \\ &\le -\frac{1}{3}\frac{(X^{[1,2]} - X^{[1,\ell]})((X^{[1,2]})^{\rho-1} - (X^{[1,\ell]})^{\rho-1})}{(X^{[1,2]} + X^{[1,\ell]})((X^{[1,2]})^{\rho} + (X^{[1,\ell]})^{\rho})} \qquad (49) \\ &= -\frac{1}{3}\frac{\sqrt{1-e^{Y_n}}((X^{[1,2]})^{\rho-1} - (X^{[1,\ell]})^{\rho-1})}{((X^{[1,2]})^{\rho} + (X^{[1,\ell]})^{\rho})}. \qquad (50) \end{aligned}$$

Inequality (49) holds on $\{n < \tau^-(K) \wedge \tilde{\tau}\}$ for m_0 large since

$$\frac{(X^{[1,2]} - X^{[1,\ell]})((X^{[1,2]})^{\rho-1} - (X^{[1,\ell]})^{\rho-1})}{(X^{[1,2]} + X^{[1,\ell]})((X^{[1,2]})^{\rho} + (X^{[1,\ell]})^{\rho})} \ge \frac{C(\rho, M, \ell)}{(X^{[1,2]})^2}$$

for some $C(\rho, M, \ell) \in (0, \infty)$, proving (31) and the supermartingale property of $Y_{n\wedge\tau(K)}$. Equality (50) holds by definition of Y_n, and it will be used later.

In order to estimate $E[(\Delta Y_{n\wedge\tau(K)})^2|\mathcal{H}_n]$ again use (35)–(40). Bound the square of the right-hand side in each of these relations by using that both the lower bound in (26), and $|X^{[1,2]} - X^{[1,\ell]}| < 2K\sqrt{T(n)}$, hold on $\{n < \tau^-(K) \wedge \tilde{\tau}\}$. For example, the square of (35) can be bounded using (25) by $\frac{4(2K\sqrt{T(n)})^2}{(3T(n)/(4\ell))^2(3T(n)/(2\ell))^2}$ which amounts to $\frac{16}{(9/8)^2}\ell^4K^2/T(n)^3 < 16\ell^4K^2/T(n)^3$. Similar bounds apply to (36) and (37). For (38)–(40) use in addition that on $\{n < \tau^-(K) \wedge \tilde{\tau}\}$ we have both $X^{[1,2]} + X^{[1,\ell]} \le 2X^{[1,2]}$ and $X^{[1,2]} + X^{[1,\ell]} \le (1 + \frac{1+\varepsilon_1\ell}{1-\varepsilon_1\ell})X^{[1,\ell]} \le (1+5/3)X^{[1,\ell]}$, and note that the squares of the terms in (38)–(40) are therefore bounded by $(4^2)9\ell^4/T(n)^4$ which is bounded by $\ell^4K^2/T(n)^3$ for $m \ge 144$. Clearly, the squares of error terms $16/(3X^{[1,\ell]})$, $16/(3X^{[1,\ell]})$, $16/(3X^{[1,2]})$, and $16/(3X^{[1,\ell]} + 3X^{[1,2]})$ are of even smaller order on the same event. All of the above calculations and the fact $(a+b)^2 \le 2(a^2+b^2)$ easily imply (32).

(ii) Let $\delta_1 = 1/K^{1/4}$. Due to (49), $Y_{n\wedge\tau(K)}$ is a supermartinagale started at time 0 with initial value

$$Y_0 := \log\left(\frac{4X^{[1,2]}_{T(0)}X^{[1,\ell]}_{T(0)}}{(X^{[1,2]}_{T(0)} + X^{[1,\ell]}_{T(0)})^2}\right) = \log\left(1 - \frac{(X^{[1,2]}_{T(0)} - X^{[1,\ell]}_{T(0)})^2}{(X^{[1,2]}_{T(0)} + X^{[1,\ell]}_{T(0)})^2}\right) \le -\frac{K^2}{T(0)}.$$

Write $Y_{n\wedge\tau(K)} = Y_0 + Z_{n\wedge\tau(K)} + D_{n\wedge\tau(K)}$ where Z denotes the martingale part, D denotes the drift, and $Z_0 = 0 = D_0$. Using (35)–(40) one can bound the martingale variance increments by

$$E\big[(\Delta Z_{n\wedge\tau(K)})^2\big|\mathcal{H}_n\big] \le E\big[(\Delta Y_{n\wedge\tau(K)})^2\big|\mathcal{H}_n\big] \le \frac{32\ell^4K^2}{T(n)^3}.$$

The first inequality above is true since $E(\Delta D_{n\wedge\tau(K)}\Delta Z_{n\wedge\tau(K)}|\mathcal{H}_n) = 0$, and the second inequality is the assertion (32). Therefore the total variance of $Z_{\cdot\wedge\tau(K)}$ is bounded by $64\ell^4K^2/(T(0))^2$ and

$$\begin{aligned}
&P\Big(\sup_{n\ge0} Y_{n\wedge\tau(K)} > -(1-\delta_1)\frac{K^2}{T(0)}\Big)\\
&\quad\le P\Big(\sup_{n\ge0} Y_{n\wedge\tau(K)} > (1-\delta_1)Y_0\Big)\\
&\quad\le P\Big(\sup_{n\ge0} Y_{n\wedge\tau(K)} > Y_0 + \delta_1\frac{K^2}{T(0)}\Big)\\
&\quad\le P\Big(\sup_{n\ge0} Z_{n\wedge\tau(K)} > \delta_1\frac{K^2}{T(0)}\Big)\\
&\quad\le 64\ell^4/(\delta_1^2K^2) = 64\ell^4/K^{3/2}.
\end{aligned}\tag{51}$$

Due to

$$-\frac{K^2(1-\delta_1)}{T(0)} = -\frac{K^2(1-\delta_1)}{T(k)}\frac{T(k)}{T(0)},\tag{52}$$

we have $\{\sup_{n\ge0} Y_{n\wedge\tau(K)} \le -(1-\delta_1)K^2/T(0)\} \subset \{\tau(K) < \infty\}$, since the opposite would imply both $Y_k \le -10K^2\ell^2/T(k)$ and

$$\begin{aligned}
Y_k &\ge -2\frac{(X_{T(k)}^{[1,2]} - X_{T(k)}^{[1,\ell]})^2}{(X_{T(k)}^{[1,2]} + X_{T(k)}^{[1,\ell]})^2} \ge -2\Big(\frac{2\ell}{3T(k)}\Big)^2(X_{T(k)}^{[1,2]} - X_{T(k)}^{[1,\ell]})^2\\
&\ge -\frac{32\ell^2K^2}{9T(k)}
\end{aligned}\tag{53}$$

for some $k < \tau(K) = \infty$, a contradiction. For the calculation in (53) use

$$X_{T(k)}^{[1,2]} + X_{T(k)}^{[1,\ell]} \ge \frac{2T(k)}{\ell}(1-\varepsilon_1\ell) \ge \frac{3T(k)}{2\ell}\tag{54}$$

by (26), and

$$\log(1-x) \ge -2x \qquad \text{for } 0 < x < 2/3.\tag{55}$$

Due to (52), it is also true that $\{\sup_{n\ge0} Y_{n\wedge\tau(K)} \le -(1-\delta_1)K^2/T(0)\} \subset \{\tau^+(K) \wedge \tilde\tau < \tau^-(K)\}$, since on $\{\sup_{n\ge0} Y_{n\wedge\tau(K)} \le -(1-\delta_1)K^2/T(0)\} \cap \{\tau^-(K) <$

$\tau^+(K) \wedge \tilde{\tau}\}$ it would be

$$-\frac{2K^2}{9T(\tau(K))} \le \log\left(1 - \frac{(X^{[1,2]}_{T(\tau(K))} - X^{[1,\ell]}_{T(\tau(K))})^2}{(X^{[1,2]}_{T(\tau(K))} + X^{[1,\ell]}_{T(\tau(K))})^2}\right) \le -\frac{K^2(1-\delta_1)}{T(\tau(K))},$$

which is impossible for $K \ge 20$ ($\delta_1 \le 1/2$). For the above calculation again use (54) and (55) with $k = \tau(K) = \tau^-(K)$. This implies (33), and the lemma follows. $\square$

LEMMA 9. *If* (25) *and* $\varepsilon_1 < \varepsilon(1/10)$ *hold, then*

$$P\big(\tau_1^m(\varepsilon_1) < \infty | \mathcal{F}_{\tau_0^m(K)}\big)\mathbb{1}_{\{\tau_0^m(K)<\infty\}} \ge c^0(m, \varepsilon_1, K, \ell)\mathbb{1}_{\{\tau_0^m(K)<\infty\}},$$

where $\inf_{m\ge\bar{m}} c^0(m, \varepsilon_1, K, \ell) > 0$ *for* $\bar{m} = \max\{n(M, K/(2\ell)), n_1(1/10, \varepsilon_1)\}$.

PROOF. Fix ε_1 satisfying assumptions of the lemma and assume $m \ge \bar{m}$. Recall we assumed K satisfies (24). Use estimate (33) with $K, 2K, \dots, 2^l K, \dots$ in place of K to obtain that for each finite l on the event $\{\tau_0^m(K) < \infty\}$

$$\begin{aligned} &P\Bigg(\bigcap_{j=0}^{l}\{\tau^+(2^jK) \wedge \tilde{\tau} < \tau^-(2^jK)\} \\ &\qquad \cap \left\{\sup_{n\ge0} Y_{n\wedge\tau(2^lK)} \le \prod_{j=0}^{l}\left(1 - \frac{1}{(2^jK)^{1/4}}\right)Y_0\right\}\Bigg|\mathcal{F}_{\tau_0^m(K)}\Bigg) \\ &\qquad \ge 1 - \sum_{j=0}^{l}\frac{64\ell^4}{(2^jK)^{3/2}}. \end{aligned}$$

Let $\eta(1) = \frac{2K^2}{3T(0)}$, and define events $A^1_n = \{Y_k \le -\eta(1), 0 \le k \le n\}$ and $A^1_\infty = \{Y_k \le -\eta(1), k \ge 0\}$. Since $\bigcap_{l\ge0}\{\tau^+(2^lK) < \tau^-(2^lK) \wedge \tilde{\tau}\} \subset \{\lim_{l\to\infty}\tau^+(2^lK) = \tau(2^lK) \to \infty\}$, and $\bigcap_{j=0}^{l}\{\tau^+(2^jK) \wedge \tilde{\tau} < \tau^-(2^jK)\} \cap \{\sup_{n\ge0} Y_{n\wedge\tau(2^lK)} \le \prod_{j=0}^{l}(1 - \frac{1}{(2^jK)^{1/4}})Y_0\} \subset \{\tau_1^{m_0}(\varepsilon_1) < \tau(2^lK)\} \cup A^1_{\tau(2^lK)}$ which converges to $\{\tau_1^{m_0}(\varepsilon_1) < \infty\} \cup A^1_\infty$, as $l \to \infty$ we have, on $\{\tau_0^m(K) < \infty\}$,

$$\begin{aligned} &P\big(\{\tau_1^m(\varepsilon_1) < \infty\} \cup A^1_\infty \big| \mathcal{F}_{\tau_0^m(K)}\big) \\ &\quad \ge \liminf_{l\to\infty} P\Bigg(\bigcap_{j=0}^{l}\{\tau^+(2^jK) \wedge \tilde{\tau} < \tau^-(2^jK)\} \\ &\qquad\qquad \cap \left\{\sup_{n\ge0} Y_{n\wedge\tau(2^lK)} \le \prod_{j=0}^{l}(1 - \frac{1}{(2^jK)^{1/4}})Y_0\right\}\Bigg|\mathcal{F}_{\tau_0^m(K)}\Bigg) \\ &\quad \ge 1 - \sum_{i=0}^{\infty}\frac{64\ell^4}{2^{3i/2}}\frac{1}{K^{3/2}} > 0. \end{aligned} \tag{56}$$

Assertion (56) joint with next lemma implies the claim of Lemma 9. $\square$

LEMMA 10. $P(A^1_\infty \cap \{\tau_1^{m_0}(\varepsilon_1) = \infty\}) = 0$.

PROOF. Clearly $\{\tau_1^{m_0}(\varepsilon_1) = \infty\} \subset \{\tilde{\tau} = \infty\}$ and note that on A^1_∞ we have

$$\frac{X^{[1,2]}_{T(k)} - X^{[1,\ell]}_{T(k)}}{X^{[1,2]}_{T(k)} + X^{[1,\ell]}_{T(k)}} \geq \sqrt{1 - e^{-\eta(1)}} \geq \sqrt{\frac{\eta(1)}{2}}, \qquad \text{w.l.o.g. } \eta(1) < 1$$

so that $X^{[1,2]}_{T(k)} - X^{[1,\ell]}_{T(k)} \geq (X^{[1,2]}_{T(k)} + X^{[1,\ell]}_{T(k)})\sqrt{\frac{\eta(1)}{2}} \geq \frac{T(k)K}{\ell\sqrt{3T(0)}} > K\sqrt{T(k)}/(2\ell)$, using (25) and (26) as usual. Therefore

$$A^1_\infty \cap \{\tau_1^{m_0}(\varepsilon_1) = \infty\} \subset \{\tilde{\tau} \wedge \tau^-(K) = \infty\}. \tag{57}$$

Note that (31) implies that $Y_{n\wedge\tilde{\tau}\wedge\tau^-(K)}$ is a supermartingale. Furthermore, a little algebra shows that for any $x > y > 1$ such that $(x - y)/(x + y) \geq \sqrt{1 - e^{-\eta(1)}}$, we have $(x^{\rho-1} - y^{\rho-1})/(x^\rho + y^\rho) \geq c(\rho, \eta(1))/x$ for some $c(\rho, \eta(1)) > 0$, so that due to (50) on $A^1_\infty \cap \{\tau_1^{m_0}(\varepsilon_1) = \infty\}$, we have

$$E(\Delta Y_n | \mathcal{H}_n) \leq -\frac{c(\rho, \eta(1))\sqrt{1 - e^{-\eta(1)}}}{3X^{[1,2]}_{T(n)}}. \tag{58}$$

Since $X^{[1,2]}_{T(n+1)} \leq X^{[1,2]}_{T(n)} + 2$, the above sequence is not summable in n. At the same time, Y_n on $A^1_\infty \cap \{\tau_1^{m_0}(\varepsilon_1) = \infty\}$ takes values in $[-8\varepsilon_1^2\ell^2, -\eta(1)]$. The lower bound is again obtained from an analogue of (54) and (55). Due to (25), (26) and (35)–(40) the overshoot (and undershoot) of $Y_{n\wedge\tilde{\tau}\wedge\tau^-(K)}$ is bounded by $4\ell/T(0)$. This together with (57) and (58) and the supermartingale property of $Y_{n\wedge\tilde{\tau}\wedge\tau^-(K)}$ implies the statement of the lemma. Namely if $a < b < c$ are fixed numbers, and if $Z_n = Z_0 + M_n + D_n$ is a supermartingale such that $Z_0 = b$, $D_0 = M_0 = 0$ and such that the drift D_n diverges to $-\infty$ (in our case $D_n \leq -d\log n$ for some $d > 0$), and if undershoot of Z at a is bounded by u we have $(a - u)P_b(\tau_a \wedge \tau_c \leq n) + aP_b(\tau_a \wedge \tau_c > n) \leq E_b[Z_{\tau_a\wedge\tau_c\wedge n}]$ and

$$E_b[Z_{\tau_a\wedge\tau_c\wedge n}] \leq b - d\log n P_b(\tau_a \wedge \tau_c > n),$$

implying $E_b(\tau_a \wedge \tau_c > n) \to 0$ as $n \to \infty$. $\square$

In the continuing search for the attracting edge, the following Lemmas 11 and 13 show that once $X_n^{[j,j-1]}$ and $X_n^{[j,j+1]}$, the weights on two adjacent edges are more than $\varepsilon_1 T(n) > \varepsilon_1(X_n^{[j,j-1]} + X_n^{[j,j+1]})$ apart, then with probability uniformly bounded away from 0 one of the edges will be traversed many more times than both of its neighbors. This happens in two stages.

Recall M is a large number, introduced before Lemma 7, and

$$\tau_2^m(M) := \inf\left\{k \geq m : \frac{X_k^{[j,j-1]}}{X_k^{[j,j+1]}} \notin \left(\frac{1}{M}, M\right) \text{ for some } j \in \{1, \dots, \ell\}\right\}.$$

LEMMA 11. $P(\tau_2^m(M) < \infty | \mathcal{F}_{\tau_1^m(\varepsilon_1)}) \mathbb{1}_{\{\tau_1^m(\varepsilon_1)<\infty\}} \geq c^1(m, \varepsilon_1, K, \ell) \times \mathbb{1}_{\{\tau_1^m(\varepsilon_1)<\infty\}}$, *where* $\liminf_m c^1(m, \varepsilon_1, K, \ell) > 0$.

PROOF. Without loss of generality assume $I_{\tau_1^m(\varepsilon_1)} = 1$ and $X_{\tau_1^m(\varepsilon_1)}^{[1,2]} - X_{\tau_1^m(\varepsilon_1)}^{[1,\ell]} > \varepsilon_1 \tau_1^m(\varepsilon_1)$. This time let $T(0) = \tau_1^m(\varepsilon_1)$, and define $T(n)$ by (16). Recall definitions (28) and (30) of Y and $\mathcal{H}_n$, and in analogy to (29) define

$$\sigma^- = \sigma^-(\varepsilon_1/2) = \inf\{n \geq 0 : X_{T(n)}^{[1,2]} - X_{T(n)}^{[1,\ell]} \leq \varepsilon_1 T(n)/2\},$$
$$\tilde{\sigma} = \tilde{\sigma}(M) = \inf\{n \geq 0 : T(n) \geq \tau_2(M)\}$$

and

$$\sigma(M, \varepsilon_1/2) := \sigma^- \wedge \tilde{\sigma}. \tag{59}$$

Lemma 11 is a direct consequence of (62). $\square$

LEMMA 12. *For sufficiently large initial time* m, *we have*:

(i) *On* $\{\tau_1^m(\varepsilon_1) \leq n < \sigma(M, \frac{\varepsilon_1}{2})\}$,

$$E(\Delta Y_n | \mathcal{H}_n) \leq -\frac{(X^{[1,2]} - X^{[1,\ell]})((X^{[1,2]})^{\rho-1} - (X^{[1,\ell]})^{\rho-1})}{(M^\rho + 1)(X^{[1,2]} + X^{[1,\ell]})(X^{[1,2]})^\rho + (X^{[1,\ell]})^\rho}. \tag{60}$$

(ii) *Therefore*,

$$P\left(\sigma\left(M, \frac{\varepsilon_1}{2}\right) < \infty \middle| \mathcal{F}_{\tau_1^m(\varepsilon_1)}\right) \mathbb{1}_{\{\tau_1^m(\varepsilon_1)<\infty\}} = \mathbb{1}_{\{\tau_1^m(\varepsilon_1)<\infty\}} \tag{61}$$

and

$$P(\tilde{\sigma} < \sigma^- | \mathcal{F}_{\tau_1^m(\varepsilon_1)}) \mathbb{1}_{\{\tau_1^m(\varepsilon_1)<\infty\}} > c(m, \varepsilon_1, K, \ell) \mathbb{1}_{\{\tau_1^m(\varepsilon_1)<\infty\}}, \tag{62}$$

where $\liminf_m c(m, \varepsilon_1, K, \ell) > 0$.

PROOF. Here we use approximation

$$\log(X + b) - \log(X) = \frac{b}{X},$$

where

$$\left|\log(X + b) - \log(X) - \frac{b}{X}\right| \leq \frac{2}{X^2}.$$

The analog of expression (45), obtained using (35)–(37) only, is

$$
\begin{aligned}
&E(\Delta Y_n|\mathcal{H}_n)\\
&\quad\le 2\biggl(\frac{1}{(X^{[1,2]})^2}+\frac{1}{(X^{[1,\ell]})^2}\biggr)\\
&\qquad+\frac{X^{[1,2]}-X^{[1,\ell]}}{(X^{[1,2]}+X^{[1,\ell]})((X^{[1,2]})^\rho+(X^{[1,\ell]})^\rho)}\\
&\qquad\times\Bigl[-2p_\ell(n)\bigl((X^{[1,2]})^{\rho-1}-(X^{[1,\ell]})^{\rho-1}\bigr)\\
&\qquad\quad-2\bigl(p_2(n)-p_\ell(n)\bigr)(X^{[1,2]})^{\rho-1}\\
&\qquad\quad+\bigl(X^{[1,2]}-X^{[1,\ell]}\bigr)\biggl(\frac{(X^{[1,2]})^{\rho-1}}{X^{[1,\ell]}}(1-p_2(n))\\
&\qquad\qquad+\frac{(X^{[1,\ell]})^{\rho-1}}{X^{[1,2]}}(1-p_\ell(n))\biggr)\Bigr].
\end{aligned}
\tag{63}
$$

Recall the definition of H_i, $1\le i\le 3$, above relation (47). We next show that for m large enough, and for $\tau_1^m(\varepsilon_1)\le n<\sigma(M,\varepsilon_1/2)$, we have $H_2+H_3\le 0$. By (48) this amounts to showing

$$
\begin{aligned}
&-2\Bigl((X^{[1,2]})^\rho-(X^{[1,\ell]})^\rho(1+\delta(n))\Bigr)\\
&\qquad+\bigl(2+\delta(n)\bigr)\bigl(X^{[1,2]}-X^{[1,\ell]}\bigr)\bigl(X^{[1,\ell]}\bigr)^{\rho-1}\le 0,
\end{aligned}
\tag{64}
$$

or equivalently,

$$
2\ge\bigl(2+\delta(n)\bigr)\biggl(\frac{X^{[1,\ell]}}{X^{[1,2]}}\biggr)^{\rho-1}+\delta(n)\biggl(\frac{X^{[1,\ell]}}{X^{[1,2]}}\biggr)^{\rho},
\tag{65}
$$

where $|\delta(n)|=O(1/T(n)^{1-\iota})$. If $X^{[1,2]}/X^{[1,\ell]}>\frac{1+\varepsilon_1/2}{1-\varepsilon_1/2}$, which follows from $X^{[1,2]}-X^{[1,\ell]}>\varepsilon_1T(n)/2>\varepsilon_1(X^{[1,2]}+X^{[1,\ell]})/2$, for $\delta=\delta(n)>0$ sufficiently small so that $2\ge(2+\delta)\left(\frac{1-\varepsilon_1/2}{1+\varepsilon_1/2}\right)^{\rho-1}+\delta\left(\frac{1-\varepsilon_1/2}{1+\varepsilon_1/2}\right)^{\rho}$ indeed $H_2+H_3\le 0$. Since $p_\ell(n)\ge 1/(M^\rho+1)$ (cf. proof of Lemma 15 in Section 6), in analogy to (49), on the event $\{\tau_1^m(\varepsilon_1)\le n<\sigma(M,\varepsilon_1/2)\}$,

$$
\begin{aligned}
E(\Delta Y_n|\mathcal{H}_n)&\le\frac{-2p_\ell(n)(X^{[1,2]}-X^{[1,\ell]})((X^{[1,2]})^{\rho-1}-(X^{[1,\ell]})^{\rho-1})}{(X^{[1,2]}+X^{[1,\ell]})(X^{[1,2]})^\rho+(X^{[1,\ell]})^\rho}\\
&\qquad+2\biggl(\frac{1}{(X^{[1,2]})^2}+\frac{1}{(X^{[1,\ell]})^2}\biggr)\\
&\le\frac{-p_\ell(n)(X^{[1,2]}-X^{[1,\ell]})((X^{[1,2]})^{\rho-1}-(X^{[1,\ell]})^{\rho-1})}{(X^{[1,2]}+X^{[1,\ell]})(X^{[1,2]})^\rho+(X^{[1,\ell]})^\rho}\\
&\le\frac{-1}{M^\rho+1}\frac{(X^{[1,2]}-X^{[1,\ell]})((X^{[1,2]})^{\rho-1}-(X^{[1,\ell]})^{\rho-1})}{(X^{[1,2]}+X^{[1,\ell]})(X^{[1,2]})^\rho+(X^{[1,\ell]})^\rho}.
\end{aligned}
\tag{66}
$$

The second inequality above is due to the fact that, on event $\{n < \sigma(M, \varepsilon_1/2)\}$,

$$\frac{(X^{[1,2]} - X^{[1,\ell]})((X^{[1,2]})^{\rho-1} - (X^{[1,\ell]})^{\rho-1})}{(X^{[1,2]} + X^{[1,\ell]})(X^{[1,2]})^{\rho} + (X^{[1,\ell]})^{\rho}} \geq \frac{c(M, \ell, \varepsilon_1, \rho)}{X^{[1,2]}} \tag{67}$$

for some $c(M, \ell, \varepsilon_1, \rho) > 0$. Therefore, assertion (i) holds and $Y_{n\wedge\sigma(M,\varepsilon_1/2)}$ is a supermartinagale, started at time $\tau_1^m(\varepsilon_1)$ with initial value $\leq \log(1 - \varepsilon_1^2)$. Since the overshoot (undershoot) bounded by $4/m_0$ [cf. (35)–(37)]. Since $X^{[1,2]}_{T(n+1)} - X^{[1,2]}_{T(n)} \leq 2$, due to (67) the drift in (66) is not summable on event $\{\sigma(M, \varepsilon_1/2) = \infty\}$ and the overshoot of Y is bounded, assertion (61) holds. Assertion (62) now follows by Wald inequality for supermartingales. In this computation it may be useful to introduce auxiliary supermartingale $\tilde{Y}_{n\wedge\sigma(M,\varepsilon_1/2)}$ (that bounds $Y_{n\wedge\sigma(M,\varepsilon_1/2)}$ from below) defined as $Y_{n\wedge\sigma(M,\varepsilon_1/2)}$ on event $\{n < \sigma(M, \varepsilon_1/2)\} \cup \{\sigma^- = \sigma(M, \varepsilon_1/2) \leq n\}$, and defined as $\log(1/(M+1)) - 4/m_0$ on the opposite event $\{\tilde{\sigma}(M) = \sigma(M, \varepsilon_1/2) \leq n\}$. The details are left to the reader. $\square$

After time $\tau_2(M)$ controlling $p_j(n)$ [or $q_j(n)$] seems difficult, and the rest of the argument is based on coupling with the process on two edges. For $i \in \{1, \dots, \ell\}$ let

$$\tau_3^{m,i} := \inf\{k \geq m : X_k^{[i,i+1]} > (2^{1/(\rho-1)} + \varepsilon)(X_k^{[i+1,i+2]} + X_k^{[i,i-1]})\}. \tag{68}$$

LEMMA 13. *For M sufficiently large $P(\min_i \tau_3^{m,i} < \infty | \mathcal{F}_{\tau_2^m(M)}) \times \mathbb{1}_{\{\tau_2^m(M)<\infty\}} > c^2(m, \varepsilon_1, K, \ell)\mathbb{1}_{\{\tau_2^m(M)<\infty\}}$, where $\liminf_m c^2(m, \varepsilon_1, K, \ell) > 0$.*

PROOF. To reduce notation write τ_2^m instead of $\tau_2^m(M)$. Without loss of generality, suppose $X^{[1,2]}_{\tau_2^m} > M X^{[1,\ell]}_{\tau_2^m}$, and $I_{\tau_2^m} \in \{1, \ell\}$. Let $N = 2X^{[1,\ell]}_{\tau_2^m} + (X^{[\ell,\ell-1]}_{\tau_2^m} + X^{[1,2]}_{\tau_2^m})$. Observe the random walk $(I_n, X_n), n \geq \tau_2^m$ at times $\xi_N = \tau_2^m$, $\xi_n = \inf\{k \geq \xi_{n-1} : I_k \in \{1, \ell\}\}$, $n > N$, of return to end vertices $\{1, \ell\}$ of edge $[1, \ell]$. Let

$$W_n^1 = 2X^{[1,\ell]}_{\xi_n} \quad \text{and} \quad W_n^2 = X^{[\ell-1,\ell]}_{\xi_n} + X^{[1,2]}_{\xi_n}, \qquad n \geq N.$$

Note that $W_n^1 - W_{n-1}^1, W_n^2 - W_{n-1}^2 \in \{0, 2\}$, and $W_n^2 - W_{n-1}^2 = 2 - (W_n^1 - W_{n-1}^1)$. Let

$$\gamma_n = X^{[1,\ell]}_{\xi_n}/(W_n^1/2 + W_n^2) = X^{[1,\ell]}_{\xi_n}/(X^{[1,2]}_{\xi_n} + X^{[\ell,\ell-1]}_{\xi_n} + X^{[1,\ell]}_{\xi_n}),$$

$$\alpha_n = \max\{X^{[\ell-1,\ell]}_{\xi_n}/(W_n^1/2 + W_n^2), X^{[1,2]}_{\xi_n}/(W_n^1/2 + W_n^2)\},$$

$$\beta_n = 1 - \alpha_n - \gamma_n = \min\{X^{[\ell-1,\ell]}_{\xi_n}/(W_n^1/2 + W_n^2), X^{[1,2]}_{\xi_n}/(W_n^1/2 + W_n^2)\}.$$

Note that

$$\frac{(W_n^2)^\kappa}{(W_n^2)^\kappa+(W_n^1)^\kappa} \tag{69}$$
$$\le \min\left\{\frac{(X_{\xi_n}^{[1,2]})^\rho}{(X_{\xi_n}^{[1,2]})^\rho+(X_{\xi_n}^{[1,\ell]})^\rho}, \frac{(X_{\xi_n}^{[\ell-1,\ell]})^\rho}{(X_{\xi_n}^{[\ell-1,\ell]})^\rho+(X_{\xi_n}^{[1,\ell]})^\rho}\right\},$$

if and only if

$$\left(\frac{\beta_n}{\gamma_n}\right)^\rho \ge \left(\frac{1-\gamma_n}{2\gamma_n}\right)^\kappa. \tag{70}$$

Let $\sigma^1 := \inf\{n \ge N : (70) \text{ does not hold}\}$. Let $((J_n, V_n), n \ge N)$ be a reinforced random walk on two edges from Section 3 with weight function $W_{J,V}(k) = k^\kappa = k^{\sqrt{\rho}}$, started at the initial time N with $J_N = 0$, and $V_N^{[0,i]} = W_N^i$, $i \in \{1,2\}$. Instead of $\kappa = \sqrt{\rho}$ one could take any κ such that $1 < \kappa < \rho$. Due to estimate (13) in Corollary 3,

$$P\left(V_n^{[0,1]} \le \frac{2}{\sqrt{M}} V_n^{[0,2]}\ \forall n \ge N \,\middle|\, V_N^{[0,1]} \le \frac{2}{M} V_N^{[0,2]}\right) \ge 1 - \frac{4}{\sqrt{M}}. \tag{71}$$

Let

$$\sigma^* := \inf\{n \ge N : \xi_n = \infty\}. \tag{72}$$

The importance of conditions (69) and (70) lies in the fact that there exist versions of $((I_n, X_n), n \ge N)$ and $((J_n, V_n), n \ge N)$ on the same probability space such that $W_{N+k}^1 + W_{N+k}^2 = V_{N+2k}^{[0,1]} + V_{N+2k}^{[0,2]}$, $0 \le k \le \sigma^1 \wedge \sigma^* - N$ and

$$W_{N+k}^1 \le V_{N+2k}^{[0,1]}, \qquad 0 \le k \le \sigma^1 \wedge \sigma^* - N. \tag{73}$$

This joint version is easily constructed step by step since one can make $W_{N+k}^1 - W_{N+k-1}^1 \le V_{N+2k}^1 - V_{N+2(k-1)}^1$ as long as (69) holds with $n = N + k$. On $\{\tau_2^m < \infty\}$ let $P^*(\cdot) = P(\cdot | \mathcal{F}_{\tau_2^m})$. Note that

$$P^*\big(\{\sigma^* < \infty, \sigma^* \le \sigma^1\} \cup \{\sigma^1 < \sigma^*\} \cup \{\sigma^1 = \sigma^* = \infty\}\big) = 1. \tag{74}$$

In words, either the particle stops returning to $\{1, \ell\}$ in finite time before the coupling condition (70) breaks, or the coupling (70) and (73) breaks in finite time before the particle stops returning to $\{1, \ell\}$, or (70), (73) never breaks and the particle returns to $\{1, \ell\}$ infinitely often. Due to Corollary 2,

$$\begin{aligned}
&\left\{\min_i \tau_3^{m,i} < \infty, \sigma^* < \infty, \sigma^* \le \sigma^1\right\} \\
&\quad \supset \{\mathcal{G}_\infty \ne \mathcal{G}, \sigma^* < \infty, \sigma^* \le \sigma^1\} \\
&\quad = \{\sigma^* < \infty, \sigma^* \le \sigma^1\},
\end{aligned}$$

and due to (73) and Proposition 2 (see also proof of Corollary 3),

$$\Big\{\min_i \tau_3^{m,i} < \infty, \sigma^* = \sigma^1 = \infty\Big\}$$
$$\supset \bigg\{\sigma^* = \sigma^1 = \infty, \sup_{n\ge N} \frac{V_n^{[0,1]}}{V_n^{[0,2]}} \le 2/\sqrt{M}\bigg\}.$$

Therefore

$$\begin{aligned} &P^*\Big(\min_i \tau_3^{m,i} < \infty\Big) \\ &\quad = P^*\Big(\min_i \tau_3^{m,i} < \infty, \sigma^* < \infty, \sigma^* \le \sigma^1\Big) \\ &\qquad + P^*\Big(\min_i \tau_3^{m,i} < \infty, \sigma^1 < \sigma^*\Big) \\ &\qquad + P^*\Big(\min_i \tau_3^{m,i} < \infty, \sigma^1 = \sigma^* = \infty\Big) \\ &\quad \ge P^*(\sigma^* < \infty, \sigma^* \le \sigma^1) \\ &\qquad + P^*\Big(\sigma^1 = \sigma^* = \infty, \sup_{n\ge N} V_n^{[0,1]}/V_n^{[0,2]} \le 2/\sqrt{M}\Big) \\ &\qquad + E^*\Big(E\big(\mathbb{1}_{\{\min_i \tau_3^{m,i} < \infty\}} \big| \mathcal{F}_{\xi_{\sigma^1}}\big) \\ &\qquad\qquad \times \mathbb{1}_{\{\sigma^1 < \sigma^*, \sup_{n\in[N,\sigma^1]} V_n^{[0,1]}/V_n^{[0,2]} \le 2/\sqrt{M}\}}\Big), \end{aligned} \tag{75}$$

where $E^*(\cdot) = E(\cdot|\mathcal{F}_{\tau_2^m})$. Note that by the above construction on $\{\sigma^1 < \sigma^*\} \cap \{\sup_{n\ge N} V_n^{[0,1]}/V_n^{[0,2]} \le 2/\sqrt{M}\}$ we have $W_{\sigma^1}^1 \le \frac{2}{\sqrt{M}} W_{\sigma^1}^2$. This implies $\gamma_{\sigma^1} \le \frac{2}{\sqrt{M}}(\alpha_{\sigma^1} + \beta_{\sigma^1})/2 \le \frac{2}{\sqrt{M}}\alpha_{\sigma^1}$, and also by the definition of σ^1,

$$\beta_{\sigma^1} < \left(\frac{1}{2}\right)^{\kappa/\rho} \gamma_{\sigma^1}^{(\rho-\kappa)/\rho} \le \gamma_{\sigma^1}^{(1-1/\sqrt{\rho})} \le \left(\frac{2}{\sqrt{M}}\right)^{(1-1/\sqrt{\rho})}.$$

Since $\alpha_{\sigma^1} + \beta_{\sigma^1} + \gamma_{\sigma^1} = 1$, the above inequalities imply

$$\begin{aligned} &\alpha_{\sigma_1} \ge 1 - 2\bigg(\frac{2}{\sqrt{M}}\bigg)^{(1-1/\sqrt{\rho})} \qquad \text{and} \\ &\max\{\beta_{\sigma^1}, \gamma_{\sigma^1}\} \le \frac{(2/\sqrt{M})^{(1-1/\sqrt{\rho})}}{1 - 2(2/\sqrt{M})^{(1-1/\sqrt{\rho})}} \alpha_{\sigma^1}. \end{aligned} \tag{76}$$

Assume for a moment that $\ell = 3$. Then if

$$\frac{(2/\sqrt{M})^{(1-1/\sqrt{\rho})}}{1 - 2(2/\sqrt{M})^{(1-1/\sqrt{\rho})}} < (2^{1/(1-\rho)} + \varepsilon)^{-1},$$

we see that $E^*(E(\mathbb{1}_{\{\min_i \tau_3^{m,i}<\infty\}}|\mathcal{F}_{\xi_{\sigma^1}}) = 1$ since $\tau_3^{m,i}$ happens at σ^1 (if not before) for i such that $\alpha_{\sigma^1} = X^{[i,i+1]}/(W^1_{\sigma^1}/2 + W^2_{\sigma^1})$. So if $\ell = 3$, the last term in (75) is at least $P^*(\sigma^1 < \sigma^*,\ \sup_{n\in[N,\sigma^1]} V_n^{[0,1]}/V_n^{[0,2]} \le 2/\sqrt{M}) \ge P^*(\sigma^1 < \sigma^*) - P^*(\sigma^1 < \sigma^*, \sup_{n\ge N} V_n^{[0,1]}/V_n^{[0,2]} > 2/\sqrt{M})$. Therefore by (71), (74) and (75), on $\{\tau_2^m < \infty\}$ we have

$$P^*\Big(\min_i \tau_3^{m,i} < \infty\Big) \ge 1 - 4/\sqrt{M}. \tag{77}$$

If $\ell > 3$ continue the coupling on $\{\sigma^1 < \sigma^*,\ \sup_{n\ge N} V_n^{[0,1]}/V_n^{[0,2]} \le 2/\sqrt{M}\}$ in order to bound $E^*(E(\mathbb{1}_{\{\min_i \tau_3^{m,i}<\infty\}}|\mathcal{F}_{\xi_{\sigma^1}})$ as follows. Set $M_0 = M$, and define inductively

$$\frac{1}{M_{i+1}} = \frac{(2/\sqrt{M_i})^{(1-1/\sqrt{\rho})}}{1 - 2(2/\sqrt{M_i})^{(1-1/\sqrt{\rho})}}, \qquad 0 \le i \le \ell - 3. \tag{78}$$

Then (76) says

$$\max\Big\{X_{\xi_{\sigma^1}}^{[\ell,1]}, X_{\xi_{\sigma^1}}^{[\ell-1,\ell]}\Big\} \le \frac{1}{M_1} X_{\xi_{\sigma^1}}^{[1,2]}.$$

Now define

$$N_1 = \Big(X_{\xi_{\sigma^1}}^{[1,\ell]} + X_{\xi_{\sigma^1}}^{[\ell,\ell-1]}\Big) + \Big(X_{\xi_{\sigma^1}}^{[1,2]} + X_{\xi_{\sigma^1}}^{[\ell-1,\ell-2]}\Big).$$

Observe the random walk $(I_n, X_n), n \ge \xi_{\sigma^1}$, at times $\xi_n = \inf\{k \ge \xi_{n-1} : I_k \in \{1, \ell - 1\}\}, n > N_1$, of return to the vertices $\{1, \ell - 1\}$. Let

$$W_n^1 = X_{\xi_n}^{[1,\ell]} + X_{\xi_n}^{[\ell,\ell-1]} \quad \text{and} \quad W_n^2 = X_{\xi_n}^{[\ell-1,\ell-2]} + X_{\xi_n}^{[1,2]}, \qquad n \ge N_1.$$

Note that again $W_n^1 - W_{n-1}^1, W_n^2 - W_{n-1}^2 \in \{0, 2\}$, $W_n^2 - W_{n-1}^2 = 2 - (W_n^1 - W_{n-1}^1)$. Let $\beta_n = X_{\xi_n}^{[\ell-1,\ell]}/(W_n^1 + W_n^2)$, $\gamma_n = X_{\xi_n}^{[1,\ell]}/(W_n^1 + W_n^2)$, and $\alpha_n = \max\{X_{\xi_n}^{[\ell-1,\ell-2]}/(W_n^1 + W_n^2), X_{\xi_n}^{[1,2]}/(W_n^1 + W_n^2)\}$, and $\delta_n = 1 - \alpha_n - \beta_n - \gamma_n = \min\{X_{\xi_n}^{[\ell-1,\ell-2]}/(W_n^1 + W_n^2), X_{\xi_n}^{[1,2]}/(W_n^1 + W_n^2)\}$. The condition

$$\begin{aligned} &\frac{(W_n^2)^\kappa}{(W_n^2)^\kappa + (W_n^1)^\kappa} \\ &\quad \le \min\left\{\frac{(X_{\xi_n}^{[1,2]})^\rho}{(X_{\xi_n}^{[1,2]})^\rho + (X_{\xi_n}^{[1,\ell]})^\rho}, \frac{(X_{\xi_n}^{[\ell-1,\ell-2]})^\rho}{(X_{\xi_n}^{[\ell-1,\ell-2]})^\rho + (X_{\xi_n}^{[\ell-1,\ell]})^\rho}\right\}, \end{aligned} \tag{79}$$

which enables coupling with the walk on two edges, is implied by the following condition written in terms of Greek letters:

$$\left(\frac{\delta_n}{\gamma_n}\right)^\rho \ge \left(\frac{\alpha_n + \delta_n}{\beta_n + \gamma_n}\right)^\kappa,$$

$$\left(\frac{\delta_n}{\beta_n}\right)^\rho \ge \left(\frac{\alpha_n + \delta_n}{\beta_n + \gamma_n}\right)^\kappa.$$

Then it is easy to check that (79), not holding for some n, implies

$$\delta_n^{\rho} < (\beta_n + \gamma_n)^{\rho-\kappa}, \qquad \text{that is, } \delta_n < (\beta_n + \gamma_n)^{(1-1/\sqrt{\rho})}. \tag{80}$$

Define $\sigma^2 := \inf\{n \geq N_1 : (79) \text{ does not hold}\}$. Let $((J_n^1, V_n^1), n \geq N_1)$ be a reinforced random walk on two edges again with weight function $W(k) = k^\kappa = k^{\sqrt{\rho}}$, now started at the initial time N_1 with $J_{N_1}^1 = 0$, and $V_{N_1}^{1,[0,i]} = W_{N_1}^i$, $i \in \{1, 2\}$. Due to Corollary 3,

$$P\left(V_n^{1,[0,1]} \leq \frac{2}{\sqrt{M_1}} V_n^{1,[0,2]}, n \geq N_1 \middle| V_{N_1}^{1,[0,1]} \leq \frac{2}{M_1} V_{N_1}^{1,[0,2]}\right) \geq 1 - \frac{4}{\sqrt{M_1}}. \tag{81}$$

Recall σ^* defined in (72). Again there exist versions of $((I_n, X_n), n \geq N)$ and $((J_n^1, V_n^1), n \geq N_1)$ on the same probability space such that $W_{N_1+k}^1 + W_{N_1+k}^2 = V_{N_1+2k}^{1,[0,1]} + V_{N_1+2k}^{1,[0,2]}$, $0 \leq k \leq \sigma^2 \wedge \sigma^* - N_1$, and

$$W_{N_1+k}^1 \leq V_{N_1+2k}^{1,[0,1]}, \qquad 0 \leq k \leq \sigma^2 \wedge \sigma^* - N_1.$$

On event $\{\sigma^1 < \sigma^*, \sup_{n \in [N, \sigma^1]} V_n^{[0,1]} / V_n^{[0,2]} \leq 2/\sqrt{M}\}$ introduce notation $P^{*1}(\cdot) = P^*(\cdot | \mathcal{F}_{\xi_{\sigma_1}})$ and $E^{*1}(\cdot) = E^*(\cdot | \mathcal{F}_{\xi_{\sigma_1}})$. In analogy to (75) we have

$$\begin{aligned}
&P^{*1}\Big(\min_i \tau_3^{m,i} < \infty\Big) \\
&\quad \geq P^{*1}(\sigma^* < \infty, \sigma^* \leq \sigma^2) \\
&\qquad + P^{*1}\Big(\sigma^2 = \sigma^* = \infty, \sup_{n \geq N_1} V_n^{1,[0,1]} / V_n^{1,[0,2]} \leq 2/\sqrt{M_1}\Big) \\
&\qquad + E^{*1}\Big(E\big(\mathbb{1}_{\{\min_i \tau_3^{m,i} < \infty\}} | \mathcal{F}_{\xi_{\sigma^2}}\big) \\
&\qquad\qquad \times \mathbb{1}_{\{\sigma^2 < \sigma^*, \ \sup_{n \geq N_1} V_n^{1,[0,1]} / V_n^{1,[0,2]} \leq 2/\sqrt{M_1}\}}\Big).
\end{aligned} \tag{82}$$

Note that relation (80) and

$$\beta_{\sigma^2} + \gamma_{\sigma^2} \leq \frac{2}{\sqrt{M_1}} \alpha_{\sigma^2} \tag{83}$$

are valid on $\{\sigma^2 < \sigma^*, \ \sup_{n \in [N_1, \sigma^2]} V_n^{1,[0,1]} / V_n^{1,[0,2]} \leq 2/\sqrt{M_1}\}$. Assuming $\ell = 4$ (even though the case ℓ odd is interesting) and M_2 defined in (78) is such that $M_2 > (2^{1/(1-\rho)} + \varepsilon)$, one gets as in (77) that (80), (82) and (83) imply

$$P^{*1}\Big(\min_i \tau_3^{m,i} < \infty\Big) \geq 1 - 4/\sqrt{M_1}$$

on

$$\Big\{\sigma^1 < \sigma^*, \ \sup_{n \in [N, \sigma^1]} V_n^{[0,1]} / V_n^{[0,2]} \leq 2/\sqrt{M}\Big\},$$

and after combining this with calculations before (77), that

$$P^*\Big(\min_i \tau_3^{m,i} < \infty\Big) \geq \big(1 - 4/\sqrt{M}\big)\big(1 - 4/\sqrt{M_1}\big) \qquad \text{on } \{\tau_2^m < \infty\}.$$

If $\ell > 4$, the relations (80) and (83) imply (76) with M_1 instead of M and σ_2 instead of σ_1. This enables the start of the next induction step where (assuming that $X^{[1,2]}$ still corresponds to α) $N_2 = \sigma^2$, $\xi_n = \inf\{k \geq \xi_{n-1} : I_k \in \{1, \ell - 2\}\}$, $n > N_2$, $W_n^1 = X_{\xi_n}^{[1,\ell]} + X_{\xi_n}^{[\ell-2,\ell-1]}$, $W_n^2 = X_{\xi_n}^{[\ell-3,\ell-2]} + X_{\xi_n}^{[1,2]}$, $n \geq N_2$. Choose the initial $M_0 = M$ sufficiently large so that $M_j \in (0,\infty)$ from (78) are well defined for each $j = 0, \dots, \ell - 2$, so that $1 - 4/\sqrt{M_j} > 0$, $j = 0, \dots, \ell - 3$, and so that $M_{\ell-2} > 2^{1/(\rho-1)} + \varepsilon$. Then after $\ell - 2$ induction steps are completed in the above fashion, one gets

$$P^*\Big(\min_i \tau_3^{m,i} < \infty\Big) \geq \prod_{j=0}^{\ell-3}\left(1 - \frac{4}{\sqrt{M_j}}\right) \qquad \text{on } \{\tau_2^m < \infty\},$$

which completes the proof of the lemma. $\square$

The next lemma says that after $\min_i \tau_3^{m,i}$ the probability of some edge becoming the attracting edge is uniformly bounded away from 0.

LEMMA 14. *$P(\sup_k \max\{X_k^{[i+1,i+2]}, X_k^{[i,i-1]}\} < \infty \mid \mathcal{F}_{\tau_3^{m,i}}) > c^3(m, \varepsilon_1, K, \ell)$ on event $\{\tau_3^{m,i} < \infty\}$, where $\liminf_m c^3(m, \varepsilon_1, K, \ell) > 0$.*

PROOF. Without loss of generality suppose $\tau_3^{m,\ell} = \min_i \tau_3^{m,i}$, so that

$$X_{\min_i \tau_3^{m,i}}^{[1,\ell]} > (2^{1/(\rho-1)} + \varepsilon)\Big(X_{\min_i \tau_3^{m,i}}^{[\ell,\ell-1]} + X_{\min_i \tau_3^{m,i}}^{[1,2]}\Big),$$

and that $I_{\tau_3^{m,1}} \in \{1, \ell\}$. Let $N = 2X_{\tau_3^{m,1}}^{[1,\ell]} + (X_{\tau_3^{m,1}}^{[\ell,\ell-1]} + X_{\tau_3^{m,1}}^{[1,2]})$. As in the previous lemma, observe the random walk (I_n, X_n) at times $\xi_N = \tau_3^{m,1}$, $\xi_n = \inf\{k \geq \xi_{n-1} : I_k \in \{1, \ell\}\}$, $n > N$. Let $W_n^1 = 2X_{\xi_n}^{[1,\ell]}$ and $W_n^2 = X_{\xi_n}^{[\ell-1,\ell]} + X_{\xi_n}^{[1,2]}$, $n \geq N$. Again $W_n^1 - W_{n-1}^1, W_n^2 - W_{n-1}^2 \in \{0, 2\}$, $W_n^1 - W_{n-1}^1 = 2 - (W_n^2 - W_{n-1}^2)$. Let $\alpha_n = X_{\xi_n}^{[1,\ell]}/(W_n^1/2 + W_n^2)$, $\beta_n = \max\{X_{\xi_n}^{[\ell-1,\ell]}/(W_n^1/2 + W_n^2), X_{\xi_n}^{[1,2]}/(W_n^1/2 + W_n^2)\}$, and $\gamma_n = 1 - \alpha_n - \beta_n = \min\{X_{\xi_n}^{[\ell-1,\ell]}/(W_n^1/2 + W_n^2), X_{\xi_n}^{[1,2]}/(W_n^1/2 + W_n^2)\}$. A little algebra shows there exists $\kappa = \kappa(\varepsilon, \rho) > 1$ such that for any three real numbers $1 \geq \alpha > \beta \geq \gamma \geq 0$, $\alpha + \beta + \gamma = 1$, with the property $\alpha > (2^{1/(\rho-1)} + \varepsilon/2)(\beta + \gamma)$, we have

$$\left(\frac{\beta}{\alpha}\right)^\rho \leq \left(\frac{1-\alpha}{2\alpha}\right)^\kappa. \tag{84}$$

Again inequality (84) holds for $\alpha_n, \beta_n, \gamma_n$ defined above if and only if

$$\begin{aligned}(85)\qquad &\frac{(W_n^2)^\kappa}{(W_n^2)^\kappa + (W_n^1)^\kappa}\\ &\qquad \geq \max\left\{\frac{(X_{\xi_n}^{[1,2]})^\rho}{(X_{\xi_n}^{[1,2]})^\rho + (X_{\xi_n}^{[1,\ell]})^\rho}, \frac{(X_{\xi_n}^{[\ell-1,\ell]})^\rho}{(X_{\xi_n}^{[\ell-1,\ell]})^\rho + (X_{\xi_n}^{[1,\ell]})^\rho}\right\}.\end{aligned}$$

Let $((J_n, V_n), n \geq N)$ be a reinforced random walk on two edges with weight function $W(k) = k^\kappa = k^{\kappa(\varepsilon,\rho)}$, started with $J_N = 0$, and $V_N^{[0,i]} = W_N^i$, $i \in \{1, 2\}$. Let $\sigma := \inf\{n \geq N : (85) \text{ does not hold}\}$ and $\sigma^* := \inf\{n \geq N : \xi_n = \infty\}$. One can couple the steps of (I, X) and (J, V) up to time σ, so that

$$\begin{aligned}(86)\qquad & W_{N+k}^1 + W_{N+k}^2 = V_{N+2k}^{[0,1]} + V_{N+2k}^{[0,2]},\\ &\qquad W_{N+k}^2 \leq V_{N+2k}^{[0,2]}, \qquad 0 \leq k \leq \sigma \wedge \sigma^* - N,\\ & W_{N+k}^1 + W_{N+k}^2 \leq V_{N+2k}^{[0,1]} + V_{N+2k}^{[0,2]},\\ &\qquad W_{N+k}^2 \leq V_{N+2k}^{[0,2]}, \qquad \sigma \wedge \sigma^* - N \leq k \leq \sigma - N.\end{aligned}$$

Finally, let $P^*(\cdot) = P(\cdot | \tau_3^{m,\ell} < \infty, \mathcal{F}_{\tau_3^{m,\ell}})$, and note that

$$\begin{aligned}& P^*\left(\sup_n (X_n^{[1,2]} + X_n^{[\ell,\ell-1]}) < \infty\right)\\ &\quad \geq P^*\left(\sigma = \infty, \sup_n V_n^{[0,2]} < \infty\right)\\ &\quad \geq P^*\left(\sup_n V_n^{[0,2]} < \infty, \alpha_n > \left(2^{1/(\rho-1)} + \varepsilon/2\right)(\beta_n + \gamma_n), n \geq N\right)\\ &\quad \geq P\left(\sup_n \frac{V_n^{[0,2]}}{V_n^{[0,1]}} < \frac{1}{2^{\rho/(\rho-1)} + \varepsilon} \,\middle|\, \frac{V_N^{[0,2]}}{V_N^{[0,1]}} \leq \frac{1}{2^{\rho/(\rho-1)} + 2\varepsilon}\right)\\ &\quad \geq 1 - \frac{(2^{\rho/(\rho-1)} + 1 + \varepsilon)^2 (2^{\rho/(\rho-1)} + 2\varepsilon)}{(2^{\rho/(\rho-1)} + 1 + 2\varepsilon)^2 (2^{\rho/(\rho-1)} + \varepsilon)} > 0.\end{aligned}$$

The first inequality above holds due to (86), the second inequality is a consequence of the fact that $\alpha_n > (2^{1/(\rho-1)} + \varepsilon/2)(\beta_n + \gamma_n)$ enables the coupling of the two processes in the nth step, due to the choice of κ, the third inequality again follows from (86) and the definition of $\alpha_n, \beta_n, \gamma_n$, while the last inequality is an application of Corollary 3. $\square$

5. Final proofs.

PROOF OF PROPOSITION 1. Fix $\varepsilon_1 > 0$ so that $\varepsilon_1 \ell < 1/4$ and $\varepsilon_1 < \varepsilon(1/10)$. Assume that $K \in [20, \infty)$ is large so that (24) holds. Define the constant M by recursion (78), and assume it is sufficiently large so that

$$M_{\ell-2} > 2^{1/(\rho-1)} + \varepsilon \quad \text{and} \quad \prod_{j=0}^{\ell-3}\left(1 - \frac{4}{\sqrt{M_j}}\right) > 0.$$

Recall $n(\cdot,\cdot)$ and $n_1(\cdot,\cdot)$ defined in Lemmas 4 and 5. Finally, assume that the initial time m is sufficiently large so that

$$m \geq \max\{n(M, K/(2\ell)), n_1(1/10, \varepsilon_1)\},$$

and such that Lemmas 8 and 12 hold. Let $A = \{\text{there exists attracting edge}\}$. By Corollary 2 we have

$$A = \left\{\min_{1\leq i\leq \ell} \sup_k \max_{j\neq i} X_k^{[j,j+1]} < \infty\right\} = \left\{\min_{1\leq i\leq \ell} \sup_k \max\{X_k^{[i+1,i+2]}, X_k^{[i,i-1]}\} < \infty\right\}$$

almost surely.

The sequence of Lemmas 7–14 in Section 4 gives

$$\inf_m P(A|\mathcal{F}_m) \geq p^* > 0,$$

almost surely, so that Lévy's 0–1 law implies

$$p^* \leq P(A|\mathcal{F}_m) \to P(A|\mathcal{F}_\infty) = \mathbb{1}_A \qquad \text{almost surely,}$$

implying $\mathbb{1}_A = 1$ almost surely. $\square$

PROOF OF THEOREM 1. There are countably many cycles contained in $\mathcal{G}$, so it suffices to show that for each (deterministic odd) cycle $\Xi \subset \mathcal{G}$, we have $P(\Xi \subset \mathcal{G}_\infty) = 0$. We claim that otherwise a reinforced random walk (I^ℓ, X^ℓ) on $\mathcal{G}^\ell$, a cycle of length ℓ, with the same reinforcement weight function W, would satisfy (in obvious notation) $P(\mathcal{G}^\ell = \mathcal{G}^\ell_\infty) > 0$, a contradiction with Proposition 1. Indeed, assume

$$P(\Xi \subset \mathcal{G}_\infty) > 0, \tag{87}$$

for some cycle Ξ of length ℓ. Lemmas 1, 2 and Corollary 1 imply that almost surely $\{\Xi \subset \mathcal{G}_\infty\} = \{\Xi = \mathcal{G}_\infty\}$, so (87) says there exists a finite n, and a configuration $(i, x) \in V \times N^E$, such that

$$P\big(\Xi = \mathcal{G}_\infty, (I_n, X_n) = (i, x), I_{n+k} \in \Xi, k \geq 0\big) > 0. \tag{88}$$

Denote by P^ℓ the probability law of the process (I^ℓ, X^ℓ) on $\Xi = \mathcal{G}^\ell$. Note that for any $N < \infty$, and any cylinder event $A_{n,N}$ generated by positions of the particle at times $n, n+1, \ldots, n+N$, we have

$$\begin{aligned}
&P\big(A_{n,N}, I_{n+k} \in \Xi, 0 \leq k \leq N | (I_n, X_n) = (i, x)\big)\\
&\qquad \leq P^\ell\big(A_{n,N} | (I_n^\ell, X_n^\ell) = (i, x)\big),
\end{aligned}$$

since $W(x)/(W(x)+W(y)+\sum_i W(z_i)) < W(x)/(W(x)+W(y))$, $x, y, z_i \geq 1$. The above inequality holds path by path (cf. Section 6), therefore

$$\begin{aligned} &P\big(\Xi = \mathcal{G}_\infty, I_{n+k} \in \Xi, 0 \leq k \leq N | (I_n, X_n) = (i,x)\big) \\ &\qquad \leq P^\ell\big(\mathcal{G}^\ell = \mathcal{G}^\ell_\infty | (I^\ell_n, X^\ell_n) = (i,x)\big), \end{aligned}$$

and (88) would imply

$$P^\ell\big(\mathcal{G}^\ell = \mathcal{G}^\ell_\infty | (I^\ell_n, X^\ell_n) = (i,x)\big) > 0,$$

which is impossible by Proposition 1. $\square$

REMARK. The reader will soon note that the assumption $W(k) = k^\rho$ is heavily used throughout the next Section 6. In addition, various properties of power functions were used in calculations and estimates in Sections 3 and 4 as follows: proof of Lemma 3, displays before and after (20), computations (44), (63) and related results, and relations (69), (79) and (85).

6. Proofs of Lemmas 4–6. The excursion $\{(I_k, X_k) : T(n) < k \leq T(n+1)\}$ away from vertex 1 is determined by $(I_{T(n)}, X_{T(n)})$ and the random path Π of vertices $I_{T(n)+1} \to \cdots \to I_{T(n+1)-1} \to I_{T(n+1)} = 1$. Let $|\Pi| = T(n+1) - T(n)$ be the length of Π.

LEMMA 15. *For each $M < \infty$ there exist $c, C \in (0,\infty)$, $c = c(M,\ell,\rho)$, $C = C(M,\ell,\rho)$ such that on $\{\tau_2^n(M) > T(n)\}$*

$$E\big(e^{c|\Pi|}\mathbb{1}_{\{T(n+1)<\tau_2^m(2M)\}}\big|\mathcal{F}_{T(n)}\big) \leq C.$$

PROOF. Assume that ℓ is odd. If $\ell = 3$, the above inequality is a simple consequence of the fact that on the event $\{T(n+1) < \tau_2^m(2M)\}$, random variable $|\Pi| - 1$ is "stochastically dominated" by a geometric $(1/((2M)^\rho + 1))$ random variable. In symbols,

$$\begin{aligned} &P\big(|\Pi| - 1 > k, T(n+1) < \tau_2^m(2M) | \mathcal{F}_{T(n)}\big) \\ &\quad \leq P\big(T(n) + k + 1 < T(n+1) \wedge \tau_2^m(2M) | \mathcal{F}_{T(n)}\big) \\ &\quad \leq P\big(T(n) + k + 1 < T(n+1) \wedge \tau_2^m(2M) | \\ &\qquad\quad T(n) + k < T(n+1) \wedge \tau_2^m(2M), \mathcal{F}_{T(n)+k}\big) \\ &\qquad \times P\big(T(n) + k < T(n+1) \wedge \tau_2^m(2M) | \mathcal{F}_{T(n)}\big) \\ &\quad \leq \left(1 - \frac{1}{(2M)^\rho + 1}\right) P\big(T(n) + k < T(n+1) \wedge \tau_2^m(2M) | \mathcal{F}_{T(n)}\big), \end{aligned}$$

where the last inequality holds since on $\{T(n)+k<T(n+1)\wedge\tau_2^m(2M)\}$ we have $X^{[I_{T(n)+k},I_{T(n)+k}+1]}/X^{[I_{T(n)+k},I_{T(n)+k}-1]}\in(1/(2M),2M)$ and since for $\ell=3$ the particle is at each time located either at 1 or at a neighbor of 1. By induction,

$$P\big(|\Pi|-1>k,T(n+1)<\tau_2^m(2M)|\mathcal{F}_{T(n)}\big)\le\left(1-\frac{1}{(2M)^\rho+1}\right)^k.$$

Similarly, if $\ell=5$, then $|\Pi|$ is dominated by a geometric sum of independent geometric random variables, where the success probability is $1/((2M)^\rho+1)$. One proceeds by induction. The claim is proved similarly for even ℓ. $\square$

Let $\iota\in(0,1/2)$ be fixed. The previous lemma and Markov inequality imply that on $\{\tau_2^n(M)>T(n)\}$,

$$P\big(|\Pi|>T(n)^{\iota/2},T(n+1)<\tau_2^m(2M)|\mathcal{F}_{T(n)}\big)\le Ce^{-cT(n)^{\iota/2}}. \tag{89}$$

Consider a path π of vertices $i_1\to\cdots\to i_{k-1}\to i_k$ where $i_1=2,i_{k-1}=\ell,i_k=1$, and $i_j\ne1$ for all $1\le j\le k-1$. The corresponding sequence of edges along this path is $[i_1,i_2],\dots,[i_{k-1},i_k]$. The "reversed" path $\overleftarrow{\pi}$, $\ell=i_{k-1}\to\cdots\to2\to1=i_0$ has a similar encoding $[i_{k-1},i_{k-2}],\dots,[i_2,i_1],[i_1,i_0]$. For $1\le j\le k-1$, define $\delta_j^+(1)=\delta_j^-(1)=0$, $\Delta_j^+([1,2])=\Delta_j^-([1,\ell])=1$ and

$$\begin{aligned}
\delta_j^+(v)&=\#\{1\le l\le j:v=i_l\},\\
\Delta_j^+(e)&=\#\{1\le l\le j-1:e=[i_l,i_{l+1}]\},\qquad e\ne[1,2],\\
\delta_j^-(v)&=\#\{j\le l\le k-1:v=i_l\},\\
\Delta_j^-(e)&=\#\{j\le l\le k-2:e=[i_l,i_{l+1}]\},\qquad e\ne[1,\ell].
\end{aligned}$$

For $1\le j\le k-1$ we have identities

$$\begin{aligned}
\delta_j^+(i_j)&=\big(\Delta_j^+([i_j,i_j-1])+\Delta_j^+([i_j,i_j+1])+1\big)/2\\
\delta_j^-(i_j)&=\big(\Delta_j^-([i_j,i_j-1])+\Delta_j^-([i_j,i_j+1])+1\big)/2.
\end{aligned}\tag{90}$$

The conditional probability $P(\cdot|\mathcal{F}_{T(n)},\ I_{T(n)+1}=2)$ of the path π equals

$$\frac{\prod_{j=1}^{k-1}(x_{j+}^{[i_j,i_{j+1}]})^\rho}{\prod_{j=1}^{k-1}[(x_{j+}^{[i_j,i_j+1]})^\rho+(x_{j+}^{[i_j,i_j-1]})^\rho]},\tag{91}$$

where for each edge e, $x_{j+}^e=X_{T(n)}^e+\Delta_j^+(e)$. Similarly, the probability $P(\cdot|\mathcal{F}_{T(n)},$ $I_{T(n)+1}=\ell)$ of the reversed path $\overleftarrow{\pi}$ equals

$$\frac{\prod_{j=k-1}^{1}(x_{j-}^{[i_j,i_{j-1}]})^\rho}{\prod_{j=k-1}^{1}[(x_{j-}^{[i_j,i_j+1]})^\rho+(x_{j-}^{[i_j,i_j-1]})^\rho]},\tag{92}$$

where $x^e_{j-} = X^e_{T(n)} + \Delta^-_j(e)$. $\square$

PROOF OF LEMMA 4. Definition (17) implies, for $j \in \{2, \ell\}$,

$$1 - p_j(n) = P\big(I_{T(n+1)-1} = \ell + 2 - j,\ T(n+1) < \tau_2^m(2M) | \mathcal{F}_{T(n)},\ I_{T(n)+1} = j\big).$$

The proof is a consequence of "pseudo reversibility." Write

$$1 - p_j(n) = P\big(A_j^{>} | \mathcal{F}_{T(n)},\ I_{T(n)+1} = j\big) + P\big(A_j^{<} | \mathcal{F}_{T(n)},\ I_{T(n)+1} = j\big),$$

where $A_j^{>} = \{I_{T(n+1)-1} = \ell + 2 - j,\ |\Pi| > T(n)^{\iota/2}, T(n+1) < \tau_2^m(2M)\}$ and $A_j^{<} = \{I_{T(n+1)-1} = \ell + 2 - j,\ |\Pi| \le T(n)^{\iota/2}, T(n+1) < \tau_2^m(2M)\}$. Note that on $\{T(n) < \tau_2^m(M)\}$ [for $T(n) \ge m > (\ell)^{2/\iota}$],

$$P\big(A_j^{<} | \mathcal{F}_{T(n)},\ I_{T(n)+1} = j\big) \ge 1/\big((2M)^\rho + 1\big)^{\ell-1},$$

since the path $2 \to 3 \to \cdots \to \ell \to 1$ (or $\ell \to \ell - 1 \to \cdots \to 2 \to 1$) is realized with probability at least $1/((2M)^\rho + 1)^{\ell-1}$. Due to this observation and (89), it suffices to show

$$\frac{P(A_2^{<} | \mathcal{F}_{T(n)},\ I_{T(n)+1} = 2)}{P(A_\ell^{<} | \mathcal{F}_{T(n)},\ I_{T(n)+1} = \ell)} = \left(\frac{X^{[1,\ell]}_{T(n)}}{X^{[1,2]}_{T(n)}}\right)^\rho \left(1 + O\left(\frac{1}{T(n)^{1-\iota}}\right)\right). \tag{93}$$

On event $\{T(n) < \tau_2^m(M)\}$ for large enough m [and therefore $T(n)$], $A_j^{<} = \{I_{T(n+1)-1} = \ell + 2 - j,\ |\Pi| \le T(n)^{\iota/2}\}$, since each edge is traversed at most $T(n)^{\iota/2} - \ell - 1$ times during the excursion, which is not enough to achieve $\tau_2^m(2M)$. Until the end of this proof use abbreviation X^e for $X^e_{T(n)}$. The probability $P(A_2^{<} | \mathcal{F}_{T(n)},\ I_{T(n)+1} = 2)$ [resp. $P(A_\ell^{<} | \mathcal{F}_{T(n)},\ I_{T(n)+1} = \ell)$] is the sum over all paths in π (resp. $\overleftarrow{\pi}$) of their corresponding probabilities (91) [resp. (92)]. It is easily seen that the numerators in expressions (91) and (92) are proportional, with the same constant $(X^{[1,\ell]})^\rho/(X^{[1,2]})^\rho$ of proportionality over all paths. This is a consequence of the identity $\Delta^+_{|\pi|}(e) = \Delta^-_{|\overleftarrow{\pi}|}(e)$ that holds for all $e \notin \{[1,2], [1,\ell]\}$, and for all pairs of paths π and $\overleftarrow{\pi}$. Moreover, for each j, $1 \le j \le k - 1$, there exists a unique $\overline{j}$, $1 \le \overline{j} \le k - 1$, such that $i_j = i_{\overline{j}}$ and $\delta^+_j(i_j) = \delta^-_{\overline{j}}(i_{\overline{j}})$. In words, the vertex i_j is visited $\delta^+_j(i_j)$ times in the first j steps along the path π, and the same vertex is visited the same number of times in the first $\overline{j}$ steps along the reversed path $\overleftarrow{\pi}$. Now by (90),

$$x^{[i_j, i_j - 1]}_{j+} + x^{[i_j, i_j + 1]}_{j+} = x^{[i_j, i_j - 1]}_{\overline{j}-} + x^{[i_j, i_j + 1]}_{\overline{j}-}, \tag{94}$$

and the ratio of the terms in (91) and (92) equals

$$\frac{(X^{[1,\ell]})^\rho}{(X^{[1,2]})^\rho} \prod_{h=2}^{\ell} \prod_{j, i_j = h} \left(\frac{(x^{[h,h-1]}_{\overline{j}-})^\rho + (x^{[h,h+1]}_{\overline{j}-})^\rho}{(x^{[h,h-1]}_{j+})^\rho + (x^{[h,h+1]}_{j+})^\rho}\right), \tag{95}$$

where each term in the product satisfies relation (94). It is easy to check, for example by using Taylor's expansion for $s \mapsto (1+s)^\rho$ at 0, that

$$\left(\frac{(x-a)^\rho + (y+a)^\rho}{x^\rho + y^\rho}\right) = 1 + O\left(\frac{\rho\, a(x^{\rho-1} + y^{\rho-1})}{x^\rho + y^\rho}\right).$$

Then due to definition of $A_j^<$, $j = 1, 2$, we have $k = |\pi| \le T(n)^{\iota/2}$, so that each of the terms in the above product is bounded from above by $1 + \frac{\mathcal{K} T(n)^{\iota/2}}{T(n)}$, for some constant $\mathcal{K} = \mathcal{K}(M, \ell, \rho) < \infty$, and the product in (95) is bounded from above by

$$\left(\left(1 + \frac{\mathcal{K}}{T(n)^{1-\iota/2}}\right)^{T(n)^{\iota/2}}\right)^{\ell-1} \le \left(1 + \frac{\mathcal{K}'}{T(n)^{1-\iota}}\right)^{\ell-1} \le 1 + \frac{\mathcal{K}''}{T(n)^{1-\iota}},$$

where $\mathcal{K}'' < \infty$ depends only on M, ℓ and ρ. Summing over all pairs of paths π and $\overleftarrow{\pi}$, yields the estimate

$$\frac{P(A_2^< | \mathcal{F}_{T(n)},\, I_{T(n)+1} = 2)}{P(A_\ell^< | \mathcal{F}_{T(n)},\, I_{T(n)+1} = \ell)} \le \left(\frac{X^{[1,\ell]}}{X^{[1,2]}}\right)^\rho \left(1 + O\left(\frac{1}{T(n)^{1-\iota}}\right)\right), \tag{96}$$

and a symmetric argument gives

$$\frac{P(A_\ell^< | \mathcal{F}_{T(n)},\, I_{T(n)+1} = \ell)}{P(A_2^< | \mathcal{F}_{T(n)},\, I_{T(n)+1} = 2)} \le \left(\frac{X^{[1,2]}}{X^{[1,\ell]}}\right)^\rho \left(1 + O\left(\frac{1}{T(n)^{1-\iota}}\right)\right),$$

implying (93), therefore (i).

For (ii), note that it suffices to show $s < 1$, where $s = s(n) = (1 - p_2(n))/(1 - p_\ell(n))$. The last claim is true for m sufficiently large due to (i), since

$$s - 1 = \frac{(X^{[1,\ell]})^\rho - (X^{[1,2]})^\rho}{(X^{[1,2]})^\rho} + O\left(\frac{1}{T(n)^{1-\iota}}\right)$$

is negative on $\{X^{[1,2]} - X^{[1,\ell]} > \varepsilon\sqrt{T(n)}\}$, as a consequence of intermediate value theorem, and the fact $\iota < 1/2$. Namely, $1 - (X^{[1,\ell]}/X^{[1,2]})^\rho > (\rho/2)(1 - X^{[1,\ell]}/X^{[1,2]})$, gives $\frac{(X^{[1,\ell]})^\rho - (X^{[1,2]})^\rho}{(X^{[1,2]})^\rho} \le -\varepsilon\rho/(2\sqrt{T(n)})$, so it suffices to take $n(M, \varepsilon) = n(M, \varepsilon, \ell, \rho, \iota)$ such that $\varepsilon\rho/(2\sqrt{n(M,\varepsilon)}) > O(1/n(M,\varepsilon)^{1-\iota})$, for $O(\cdot)$ above. $\square$

PROOF OF LEMMA 5. Consider only $p_2(n)$, the argument for $p_\ell(n)$ is analogous. Intuitively, the claim is true since $X_{T(n)}^{[i,i-1]}/X_{T(n)}^{[i,i+1]} \approx 1$ for all i, so that the reinforced walk is very similar to the symmetric random walk on the circle, and the latter process satisfies

$$P_2 := P(\text{symmetric walk started at 2 returns to 1 via 2}) = \frac{\ell - 1}{\ell}.$$

Since $p_2(n) \geq \tilde{p}_2(n) := P(I_{T(n+1)-1} = 2, T(n+1) < \infty | \mathcal{F}_{T(n)}, I_{T(n)+1} = 2)$, it suffices to show that $\tilde{p}_2(n)$ satisfies the claim of the lemma.

Consider paths π of the form $i_1 \to \cdots \to i_{k-1} \to i_k$ where now $i_1 = i_{k-1} = 2$, $i_k = 1$, and $i_j \neq 1$ for all $1 \leq j \leq k-1$. Then

$$\tilde{p}_2(n) = \sum_{|\pi|<\infty} P(\pi),$$

where $P(\pi)$ is given by (91), and

$$P_2 = \sum_{|\pi|<\infty} \frac{1}{2^{|\pi|-1}}.$$

Take K_0 large enough so that

$$\sum_{|\pi| \leq K_0} \frac{1}{2^{|\pi|-1}} \geq \frac{\ell - 1}{\ell}(1 - \delta/2). \tag{97}$$

Take $\varepsilon(\delta) = \varepsilon(\delta, \rho, \ell) < 1/(4\ell)$ small enough so that for all $\varepsilon_1 < \varepsilon(\delta)$ we have $|(x^\rho - 1)/(x^\rho + 1)| \leq 3\rho(x-1)$ if $1 \leq x \leq \frac{1+2\varepsilon_1\ell}{1-2\varepsilon_1\ell}$, and $(1 - 24\rho\varepsilon_1\ell)^{K_0} \geq 1 - \delta/2$. Fix $\varepsilon_1 < \varepsilon(\delta)$ and let $n_1(\delta, \varepsilon_1) > K_0/\varepsilon_1$. Assume $\tau_1^m(\varepsilon_1) > T(n) \geq m \geq n_1(\delta, \varepsilon_1)$. Note that then by (26) for $j \leq k-1 \leq K_0 - 1$ we have $x_{j+}^{[i_j, i_{j+1}]} \leq (X_{T(n)}^{[i_j, i_{j+1}]} + K_0) \leq T(n)(1 + 2\varepsilon_1\ell)/\ell$. Also due to (26), $x_{j+}^{[i_j, i_{j+1}]}/x_{j+}^{[i_j, i_{j-1}]} \leq (1 + 2\varepsilon_1\ell)/(1 - \varepsilon_1\ell) < (1 + 2\varepsilon_1\ell)/(1 - 2\varepsilon_1\ell)$. Now

$$\begin{aligned}
&\prod_{j=1}^{k-1} \frac{(x_{j+}^{[i_j, i_{j+1}]})^\rho}{(x_{j+}^{[i_j, i_j+1]})^\rho + (x_{j+}^{[i_j, i_j-1]})^\rho} \\
&\qquad = \frac{1}{2^{k-1}} \prod_{j=1}^{k-1} \left(1 + \frac{(x_{j+}^{[i_j, i_{j+1}]}/x_{j+}^{[i_j, i_j-1]})^\rho - 1}{(x_{j+}^{[i_j, i_j+1]}/x_{j+}^{[i_j, i_j-1]})^\rho + 1}\right) \\
&\qquad \geq (1 - 24\rho\varepsilon_1\ell)^{k-1} \frac{1}{2^{k-1}},
\end{aligned} \tag{98}$$

so that

$$\begin{aligned}
\tilde{p}_2(n) &\geq \sum_{|\pi| \leq K_0} P(\pi) \geq (1 - 24\rho\varepsilon_1\ell)^{K_0} \sum_{|\pi| \leq K_0} \frac{1}{2^{|\pi|-1}} = \frac{\ell - 1}{\ell}\left(1 - \frac{\delta}{2}\right)^2 \\
&\geq \frac{(\ell - 1)(1 - \delta)}{\ell}.
\end{aligned}$$

□

PROOF OF LEMMA 6. (i) The proof of the lower bound is similar to the one in the previous lemma. Note that on $\{\tau_0^m(K) > T(n)\}$ we can choose $K_0(n) = \lfloor T(n)^{1/4} \rfloor$ so that the left-hand side in (97) converges to $(\ell - 1)/\ell$

as $T(n) \to \infty$, and, since $|x_{j+}^{[i_j,i_j+1]} - x_{j+}^{[i_j,i_j-1]}| \le |X_{T(n)}^{[i_j,i_j+1]} - X_{T(n)}^{[i_j,i_j-1]}| + T(n)^{1/4} \le 2K\sqrt{T(n)}$, instead of (98) we have

$$\prod_{j=1}^{k-1} \frac{(x_{j+}^{[i_j,i_{j+1}]})^\rho}{(x_{j+}^{[i_j,i_j+1]})^\rho + (x_{j+}^{[i_j,i_j-1]})^\rho} \ge \left(1 - \frac{24\rho K\ell}{\sqrt{T(n)}}\right)^{k-1} \frac{1}{2^{k-1}}.$$

Since $(1 - 24K\rho\ell/\sqrt{T(n)})^{K_0(n)} \to 1$, the lower bound holds.

For the upper bound, it suffices to to show that

$$1 - p_j(n) \ge \frac{1}{\ell}\big(1 - \tilde{c}_1(n)\big),$$

where $\tilde{c}_1(n) \to 0$. Observe that on $\{\tau_0^m(K) > T(n)\}$, in the notation of Lemma 4,

$$1 - p_j(n) \ge P\big(I_{T(n+1)-1} = \ell + 2 - j,\ |\Pi| \le T(n)^{1/4} | \mathcal{F}_{T(n)},\ I_{T(n)+1} = j\big).$$

The rest of the argument is the same as for the lower bound.

Assertion (ii) can be obtained in the same way as (26) and (27), and is also left to the reader. $\square$

Acknowledgments. This work was done while the author was a postdoctoral fellow at Cornell University. I thank Stas Volkov for introducing me to the problem on the triangle and its background, Rick Durrett for encouraging me to extend the main result to all $\rho > 1$, and Mihail Loulakis and Jose Ramirez for their careful reading of preliminary versions of this paper. I am indebted to Harry Kesten for his encouragement and for the numerous conversations.

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