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# ON THE SAMPLE PATHS OF BROWNIAN MOTIONS ON COMPACT INFINITE DIMENSIONAL GROUPS

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We study the regularity of the sample paths of certain Brownian motions on the infinite dimensional torus $\mathbb{T}^\infty$ and other compact connected groups in terms of the associated intrinsic distance. For each $\lambda \in (0, 1)$, we give examples where the intrinsic distance $d$ is continuous and defines the topology of $\mathbb{T}^\infty$ and where the sample paths satisfy

$$0 < \liminf_{t\to 0} \frac{d(X_0, X_t)}{t^{(1-\lambda)/2}} \le \limsup_{t\to 0} \frac{d(X_0, X_t)}{t^{(1-\lambda)/2}} < \infty$$

and

$$0 < \lim_{\varepsilon\to 0} \sup_{\substack{0<t<s<1\\ t-s\le\varepsilon}} \frac{d(X_s, X_t)}{(t-s)^{(1-\lambda)/2}} < \infty.$$

**1. Introduction.** Let $G$ be a compact connected metrizable group equipped with its normalized Haar measure $\nu$. A left-invariant diffusion process on $G$ is a stochastic process $X = (X_t, \mathbb{P}, \Omega)$ having the following properties:

(B0) $X_0 = e$, the identity element of $G$.
(B1) $X$ has stationary independent increments. That is, for any $0 < s < t < +\infty$, the law of $X_s^{-1}X_t$ depends only of $t - s$, and, for any $0 < t_1 < t_2 < \cdots < t_k < +\infty$, the $G$-valued random variables $X_{t_i}^{-1}X_{t_{i+1}}$, $0 < i < k$ are independent.
(B2) $X$ has continuous paths. That is, for $\mathbb{P}$ almost all $\omega \in \Omega$, $t \mapsto X_t(\omega)$ is a continuous function from $[0, +\infty)$ to $G$.

In the sequel, we will consider only processes that have the following two additional properties. For any domain $U \ni e$, let $\tau_U$ be the first exit time from $U$.

(B3) $X$ is symmetric, that is, $X_t$ and $X_t^{-1}$ have the same distribution.
(B4) $X$ is nondegenerate in the following sense. For any domain $U \ni e$ and for any open set $V$ in $U$, $(X_t, t < \tau_U)$ visits $V$ with positive probability.

For simplicity, let us call *invariant diffusion* a process having properties (B0)–(B4). Thus, according to this definition, all invariant diffusions are symmetric and nondegenerate. As we are assuming that $G$ is compact, it is natural to consider one more property, namely,

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(B5) $X$ is central, that is, for any $a \in G$, $X_t$ and $aX_ta^{-1}$ have the same distribution.

We call *Brownian motion* any stochastic process $X = (X_t, \mathbb{P}, \Omega)$ satisfying the properties (B0)–(B5). Together with (B1), (B5) implies that $X$ is by-invariant. When $G$ is Abelian, any invariant diffusion as defined above is a Brownian motion.

This work is concerned with some regularity properties of the sample paths of invariant diffusions in the case where $G$ is *infinite-dimensional*. The main novel aspect of our study is related to the fact that the underlying (metrizable) space is both *infinite-dimensional* and *compact*. The restriction to compact groups versus locally compact groups is not essential but will help us focus on the main original aspects of our results by avoiding some additional technical difficulties.

In finite-dimensional settings such as Riemannian manifolds, the regularity of the sample paths of Brownian motion is naturally studied in terms of the intrinsic (Riemannian) distance associated to the infinitesimal generator of the process. For Banach space valued processes (see, e.g., [25, 26]), sample path regularity is measured in terms of various auxiliary distances which are not canonically attached to the process, simply because there is no intrinsic distance available. In our infinite-dimensional compact setting, both situations can occur. For certain invariant diffusions on compact groups, the intrinsic "distance" equals $\infty$ almost everywhere w.r.t. Haar measure, leaving us in a situation similar to that of Banach spaces. However, any metrizable compact connected locally connected group $G$ carries many invariant diffusions which admit an intrinsic distance that is continuous and defines the topology of $G$; see [6, 9, 7]. In such cases, it is most natural to study the regularity of paths of the diffusion in terms of its intrinsic distance, as one would do on a finite-dimensional Riemannian manifold. The present paper focuses on this type of situations.

Recall that, on $\mathbb{R}^n$, any invariant, symmetric, nondegenerate diffusion process $X$ is, up to a change of coordinates, the classical Brownian motion whose distribution at time $t > 0$ has density

$$\left(\frac{1}{4\pi t}\right)^{n/2} \exp\left(-\frac{\|x\|^2}{4t}\right)$$

with respect to Lebesgue measure. Here $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^n$ and we consider that Brownian motion is driven by $\Delta = \sum_1^n \partial_i^2$ instead of $\frac{1}{2}\Delta$. With this normalization, Brownian sample paths have the following celebrated properties (see [24, 27]):

(i) The Lévy–Khinchine law of the iterated logarithm asserts that, almost surely,

$$\limsup_{t\to 0} \frac{\|X_t\|}{\sqrt{4t \log\log(1/t)}} = 1.$$

(ii) Lévy's result concerning the modulus of continuity of Brownian paths asserts that, almost surely,

$$\lim_{\varepsilon\to 0} \sup_{\substack{0<s<t<1\\ t-s<\varepsilon}} \frac{\|X_t - X_s\|}{\sqrt{4(t-s)\log(1/(t-s))}} = 1.$$

(iii) A theorem of Dvoretzky and Erdös concerning the "rate of escape" of Brownian motion asserts that, for $n \geq 3$ and any continuous increasing positive function $h$,

$$\liminf_{t\to 0} \frac{\|X_t\|}{h(t)\sqrt{t}} = \begin{cases} +\infty \\ 0 \end{cases} \quad \text{a.s. iff} \quad \sum_k [h(2^{-k})]^{n-2} \begin{cases} \text{converges,} \\ \text{diverges.} \end{cases}$$

To make the third statement more explicit, observe that, specializing to $h_\sigma(t) = |\log t|^{-\sigma}$, $\sigma > 0$, we have

$$\liminf_{t\to 0} \frac{\|X_t\| |\log t|^{\sigma}}{\sqrt{t}} = \begin{cases} +\infty \\ 0 \end{cases} \quad \text{a.s. iff} \quad \sigma \begin{cases} > \dfrac{1}{n-2}, \\ < \dfrac{1}{n-2}. \end{cases}$$

Since we restrict attention to compact groups, let us point out that these three properties of Brownian paths hold true without change on the finite-dimensional torus $\mathbb{T}^n$ and on any compact Lie group, although there seems to be no good reference for this fact. Our forthcoming paper [10] gives complete proofs of these assertions in a general finite-dimensional setting. See also the related works [18, 19]. Of course, on a compact Lie group, the Euclidean norm $\|\cdot\|$ must be replaced by the natural Riemannian distance adapted to the considered Brownian motion.

For our purpose, what is remarkable about the results quoted above is their stability as the dimension goes to infinity: (i) and (ii) are completely dimension independent whereas, in (iii), as $n$ goes to infinity, the border line gets closer and closer to the function $\sqrt{t}$. Since any compact connected group $G$ is the projective limit of compact connected Lie groups $G_\alpha$, and any Brownian motion $X$ on $G$ is, in some sense (see [1, 2]), the projective limit of Brownian motions $X_\alpha$ on $G_\alpha$, one could think that some universal forms of (i), (ii) and maybe (iii), hold true for such processes. As we shall see, this is not at all the case. For instance, on the infinite-dimensional torus, we will prove the following result.

THEOREM 1.1. *Let $X$ be a Brownian motion on the infinite-dimensional torus $\mathbb{T}^\infty$. Let $\mu_t$ denote the law of $X_t$ and assume that, for all $t > 0$, $\mu_t$ admits a continuous density $x \mapsto \mu_t(x)$ with respect to Haar measure. Assume further that,* *for some* $\lambda \in (0,1)$, *and constants* $0 < c_0 \le C_0 < \infty$,

$$\forall t \in (0,1), \qquad c_0 t^{-\lambda} \le \log \mu_t(e) \le C_0 t^{-\lambda}.$$

*Then the associated intrinsic distance* $d$ *is continuous and defines the topology of* $\mathbb{T}^\infty$. *Moreover, there are constants* $0 < c \le C < \infty$ *which depend only on* $\lambda$ *and* $c_0, C_0$ *such that*, $\mathbb{P}$-*almost surely*,

$$c \le \liminf_{t \to 0} \frac{d(e, X_t)}{t^{(1-\lambda)/2}} \le \limsup_{t \to 0} \frac{d(e, X_t)}{t^{(1-\lambda)/2}} \le C$$

*and*

$$c \le \lim_{\varepsilon \to 0} \sup_{\substack{0<s<t<1 \\ t-s<\varepsilon}} \frac{d(X_s, X_t)}{(t-s)^{(1-\lambda)/2}} \le C.$$

Although several aspects of this result are surprising, the proofs involve mostly technical adaptations of well established ideas: some estimates of the probability of being in a certain ball in a certain time interval and the use of the Borel–Cantelli lemma. Theorem 1.1 shows the existence of many different behaviors and this is the main novel aspect of this work: since many different behaviors are possible, what does one need to know about a given Brownian motion in order to predict the (intrinsic) regularity of its sample paths? This paper provide partial answers in terms of the behavior of $t \mapsto \log \mu_t(e)$ in a rather general context. The companion paper [11] gives complementary results in a very restricted setting and shows that predicting the precise path regularity of a given Brownian motion on the infinite dimensional torus is a rather subtle question.

The structure of this paper is as follows. Section 2 introduces notation and review well-known facts concerning Brownian motions on compact groups. Section 3 shows how certain Gaussian upper bounds alone suffice to imply the upper bounds stated in Theorem 1.1 in a more general context. Section 4 shows that, in certain circumstances, two-sided Gaussian bounds can be used to provide lower bounds that match the upper bounds of Section 3. Along the way, several important intermediate results are obtained. Section 4.1 relates Gaussian bounds to volume estimates in a way that already captures some of the spirit of Theorem 1.1 (see Theorem 4.7). Section 4.2 develops Green function estimates which are used in Section 4.3, together with the volume estimates of Section 4.1, to prove the most difficult result of this paper, that is, the first lower bound in Theorem 1.1. The Gaussian bounds required to apply these results and obtain concrete examples have been proved, in certain cases, in our previous works [6, 9].

The main results of this paper relating Gaussian bounds to path regularity are Theorems 3.6, 3.8 and 4.13. Theorems 1.1, 3.9, 3.11 and 4.17 provide concrete examples.

## 2. Background and notation.

2.1. *Gaussian semigroups.* Let $(\mu_t)_{t>0}$ be a weakly continuous convolution semigroup of probability measures on a group $G$. This means precisely that each $\mu_t$, $t>0$, is a probability measure on $G$ and that $(\mu_t)_{t>0}$ satisfies:

(i) $\mu_t * \mu_s = \mu_{t+s}$, $t, s > 0$;
(ii) $\mu_t \to \delta_e$ weakly as $t \to 0$.

Such a semigroup is called *Gaussian* if it also satisfies

(iii) $t^{-1}\mu_t(V^c) \to 0$ as $t \to 0$ for any neighborhood $V$ of the identity element $e \in G$.

We say that $(\mu_t)_{t>0}$ is symmetric if $\mu_t(A) = \mu_t(A^{-1})$ for all $t > 0$ and all Borel sets $A \subset G$. We say that $(\mu_t)_{t>0}$ is central if $\mu_t$ is central for all $t > 0$, that is, $\mu_t(aAa^{-1}) = \mu_t(A)$ for all Borel sets $A$ and all $a \in G$.

Given a convolution semigroup $(\mu_t)_{t>0}$, define the associated Markov semigroup $(H_t)_{t>0}$ acting on continuous functions by

$$H_t f(x) = \int_G f(xy)\, d\mu_t(y). \tag{2.1}$$

Thus $(H_t)_{t>0}$ is given by $H_t f = f * \check{\mu}_t$ where $\check{\mu}(B) = \mu(B^{-1})$ for any Borel set $B$ and any Borel measure $\mu$. If $(\mu_t)_{t>0}$ is symmetric then $(H_t)_{t>0}$ extends to $L^2(G)$ as a semigroup of self-adjoint contractions. One can then associate to $(H_t)_{t>0}$ its $L^2(G)$-infinitesimal generator $(-L, \mathrm{Dom}(L))$ and its Dirichlet form $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ so that $H_t = e^{-tL}$ on $L^2(G)$ and $\mathcal{E}(f,g) = \langle L^{1/2}f, L^{1/2}g\rangle$, $f, g \in \mathrm{Dom}(\mathcal{E}) = \mathrm{Dom}(L^{1/2})$. We will also refer to $-L$ as the infinitesimal generator of $(\mu_t)_{t>0}$.

2.2. *The projective structure.* Let $G$ be a compact connected group. Any such group is the projective limit of a family of compact connected Lie groups $G_\alpha = G/K_\alpha$, $\alpha \in \aleph$, where $(K_\alpha)_{\alpha\in\aleph}$ is a descending family of compact normal subgroups with trivial intersection. We will not need to understand explicitly what this important structural fact entails, but here is a simple and important consequence. Let $\pi_\alpha : G \to G_\alpha$ denote the canonical projection onto $G_\alpha$. Set

$$\mathcal{B}_\alpha(G) = \{f = \phi \circ \pi_\alpha : \phi \in \mathcal{C}^\infty(G_\alpha)\}.$$

That is, $\mathcal{B}_\alpha(G)$ is the set of all functions on $G$ that are obtained by lifting to $G$ any smooth function on one of the Lie groups $G_\alpha$. The space

$$\mathcal{B}(G) = \bigcup_{\alpha\in\aleph} \mathcal{B}_\alpha(G)$$

is the space of Bruhat test functions introduced in [14]. The fact that $G$ is the projective limit of the $G_\alpha$'s implies that $\mathcal{B}(G)$ is dense in $L^p(G)$, $1 \le p < +\infty$ and in $\mathcal{C}(G)$, the space of all continuous functions on $G$.

Any symmetric Gaussian semigroup $(\mu_t)_{t>0}$ on $G$ yields by projection a symmetric Gaussian semigroup $(\mu_{\alpha,t})_{t>0}$ on $G_\alpha$. By Hunt's theorem [23] and the projective structure, it follows that $\mathcal{B}(G)$ is contained in the domain of the infinitesimal generator $-L$ of $(\mu_t)_{t>0}$. In fact, $\mathcal{B}(G)$ is a core for the corresponding Dirichlet form. For any function $f = \phi \circ \pi_\alpha \in \mathcal{B}_\alpha(G)$, one has

$$Lf = [L_\alpha \phi] \circ \pi_\alpha$$

where $-L_\alpha$ is the infinitesimal generator of $(\mu_{\alpha,t})_{t>0}$. The aforementioned theorem of Hunt states in particular that each $-L_\alpha$ is a sum of squares of left-invariant vector fields on $G_\alpha$, that is $L_\alpha = -\sum_1^{k_\alpha} X_{\alpha,i}^2$. In particular, for any $\phi \in \mathcal{B}(G)$, $L\phi \in \mathcal{B}(G)$.

Call a symmetric Gaussian semigroup *sub-elliptic* if, for any $\alpha$, $L_\alpha = -\sum_1^{k_\alpha} X_{\alpha,i}^2$ satisfies Hörmander's condition, that is, the left-invariant vector fields $X_{\alpha,i}$, together with all their Lie brackets, span the Lie algebra of $G_\alpha$. This is also equivalent to saying that the projected semigroup $(\mu_{\alpha,t})_{t>0}$ on $G_\alpha$ is absolutely continuous w.r.t. Haar measure and admits a smooth positive density (this equivalence is nontrivial as it depends among other things on Hörmander subellipticity theorem, see [28]).

Call a symmetric Gaussian semigroup *elliptic* if, for any $\alpha$, the left-invariant vector fields $X_{\alpha,i}$ span the Lie algebra of $G_\alpha$.

2.3. *Invariant diffusion processes and Gaussian semigroups.* We now briefly recall the correspondence between invariant diffusions and Gaussian semigroups. The book [20] can be used as a detailed reference for this material.

Start with a process $X$ on a connected compact group $G$ satisfying (B0) and (B1). Let $\mu_t$ be the law of the process at time $t > 0$. Then $(\mu_t)_{t>0}$ is a convolution semigroup. Property (B2), that is, the continuity of the sample paths, is precisely equivalent to $(\mu_t)_{t>0}$ being Gaussian. Moreover, $(\mu_t)_{t>0}$ is symmetric if and only if (B3) holds. Assuming symmetry, the nondegeneracy property (B4) is equivalent to the fact that $(\mu_t)_{t>0}$ is *sub-elliptic* as defined in the previous section. Property (B5) is equivalent to $(\mu_t)_{t>0}$ being central [i.e., $\mu_t(aBa^{-1}) = \mu_t(B)$ for every $a \in G$ and open set $B$].

Conversely, let a Gaussian convolution semigroup $(\mu_t)_{t>0}$ be given. If we assume that $G$ is metrizable, a classical construction yields a stochastic process $X$—the associated Hunt process—whose marginal at time $t$ is $\mu_t$. This process satisfies (B0) and (B1). Because, $(\mu_t)_{t>0}$ is Gaussian, $X$ must satisfy (B2) and the previous discussion concerning properties (B3)–(B5) applies.

THEOREM 2.1. *Assume that $G$ is a connected compact metrizable group. Then there is a one to one correspondence between invariant diffusions on $G$ [i.e., processes satisfying* (B0)–(B4)] *and symmetric sub-elliptic Gaussian convolution semigroups. Moreover, an invariant diffusion is a Brownian motion [i.e., satisfies* (B5)] *if and only if the associated sub-elliptic Gaussian semigroup is central.*

2.4. *The intrinsic distance.* We are now ready to introduce the crucial notion of intrinsic distance. Given a Gaussian semigroup $(\mu_t)_{t>0}$ with infinitesimal generator $-L$, consider the field operator $\Gamma$, defined on $\mathcal{B}(G) \times \mathcal{B}(G)$ by the formula

$$\Gamma(u, v) = \tfrac{1}{2}\big(uLv + vLu - L(uv)\big).$$

DEFINITION 2.2. Let $(\mu_t)_{t>0}$ be a symmetric Gaussian semigroup on $G$. The intrinsic distance $d$ ($+\infty$ is allowed) is defined by

$$d(x, y) = \sup\big\{f(x) - f(y) : f \in \mathcal{B}(G), \Gamma(f, f) \le 1\big\}.$$

We also set $d(x) = d(e, x)$ where $e$ is the identity element in $G$.

Let us stress the fact that this "distance" can be very degenerated. For instance, it is a strong hypothesis to assume that $d$ is continuous. Set $D = \{x \in G : d(x) < +\infty\}$. Then $D$ is always a dense Borel measurable subgroup of $G$. It follows that either $D$ is equal to $G$ or it has Haar measure zero. See [6, 9] for details.

The definition of the intrinsic distance emerged in the eighties in various works, in particular, in connection with the work of Davies on Gaussian upper bounds for heat kernels; see [15], Section (3.2.9). See also [29] and the references therein.

## 3. Gaussian upper bounds and some consequences.

Concerning statements such as (i) and (ii) in the Introduction, the upper bounds are generally easier to prove than the lower bounds. This section is concern with the easier part, that is, upper bounds on $d(X_t)$ or $d(X_s^{-1}X_t)$, on general compact connected locally connected metric group. These upper bounds will be derived from Gaussian upper bounds on the density of the Gaussian semigroup $(\mu)_{t>0}$. Of course, in general, $(\mu_t)_{t>0}$ needs not have a density at all w.r.t. the Haar measure $\nu$, see [3, 4, 7].

3.1. *On-diagonal and Gaussian bounds.* Gaussian bounds can be obtained from certain on-diagonal bounds on the density of $\mu_t$, assuming that such a density exists. We will use the following notation.

DEFINITION 3.1. We say that a symmetric Gaussian convolution semigroup is:

(CK) If, for all $t > 0$, $\mu_t$ is absolutely continuous w.r.t. Haar measure and admits a continuous density.

(CK$\lambda$) For some $\lambda \in (0, \infty)$ if (CK) holds and the continuous density $\mu_t(\cdot)$ satisfies

$$\sup_{t\in(0,1)} t^{\lambda} \log \mu_t(e) < +\infty.$$

($\mathrm{CK}_\psi$) If (CK) holds and the continuous density $\mu_t(\cdot)$ satisfies

$$\sup_{t\in(0,1)} \frac{\log \mu_t(e)}{\psi(1/t)} < +\infty$$

where $\psi$ is a fixed continuous increasing function.

The following simple result from [7] and [9] is useful in the sequel.

LEMMA 3.2. *Assume that $G$ is compact, connected and not a Lie group. Assume that $(\mu_t)_{t>0}$ is a symmetric Gaussian semigroup satisfying property* (CK) *and such that*, $\forall\, t \in (0,1)$, $\mu_t(e) \le \exp(\psi(t))$. *Then*

$$\lim_{t\to\infty} \frac{\psi(t)}{\log(1+t)} = +\infty.$$

REMARK. Let $G$ be a compact connected group which admits a (CK) symmetric Gaussian semigroup. By [20] (see also [8]), the group $G$ must then be locally connected and metrizable. In this case, if $G$ is not a Lie group, $G$ must be infinite-dimensional in the sense that any sequence $G_\alpha$ of compact connected Lie groups whose projective limit equals $G$ must have $\sup_\alpha n_\alpha = +\infty$ where $n_\alpha$ is the topological dimension of the Lie group $G_\alpha$. For each Lie group $G_\alpha$, we have

$$\log \mu_{\alpha,t}(e) \sim \frac{m_\alpha}{2} \log(1+1/t)$$

where $m_\alpha \ge n_\alpha$ (see [30]). This explains the result stated in Lemma 3.2.

Let $\psi : (0,+\infty) \to (0,+\infty)$. Recall that, by definition, $\psi$ is slowly varying if

$$\forall\, x \in (0,+\infty), \qquad \lim_{t\to+\infty} \psi(tx)/\psi(t) = 1.$$

The function $\psi$ is regularly varying of index $\lambda \in (-\infty,+\infty)$ if $t^{-\lambda}\psi(t)$ is slowly varying. See [12]. Let $\mathcal{R}_\lambda$ denote the class of all $\lambda$-regularly varying functions. We will often consider property ($\mathrm{CK}_\psi$) when $\psi \in \mathcal{R}_\lambda$. Note that (CK$\lambda$) is nothing but ($\mathrm{CK}_\psi$) with $\psi(t) = t^\lambda$.

It will be useful to consider also the class $\mathcal{R}^*$ (resp. $\mathcal{R}_*$) of those positive continuous increasing functions $\psi$ such that there exists $k > 1$ satisfying

$$\limsup_{s\to\infty} \frac{\psi(ks)}{\psi(s)} < k \qquad \left(\text{resp. } \liminf_{s\to\infty} \frac{\psi(ks)}{\psi(s)} > 1\right). \tag{3.1}$$

LEMMA 3.3. *For any positive continuous increasing function $\psi$, the condition $\psi \in \mathcal{R}^*$ is equivalent to the existence of $C_0$ and $\theta^* \in (0,1)$ such that*

$$\forall\, t > 1,\ \forall\, s \in (1,t), \qquad \frac{\psi(t)}{\psi(s)} \le C_0 \left(\frac{t}{s}\right)^{\theta^*}. \tag{3.2}$$

*Similarly, the condition* $\psi \in \mathcal{R}_*$ *is equivalent to the existence of* $c_0$ *and* $\theta_* \in (0, \infty)$ *such that*

$$\forall t > 1, \forall s \in (1, t), \qquad \frac{\psi(t)}{\psi(s)} \ge c_0 \left(\frac{t}{s}\right)^{\theta_*}. \tag{3.3}$$

PROOF. Since the proofs are similar, we only prove the statement concerning $\mathcal{R}^*$. From (3.2), it follows easily that (3.1) holds true for all $k$ large enough. Conversely, (3.1) implies that there exists $\theta \in (0, 1)$ such that $\psi(ks)/\psi(s) \le k^\theta$ for all $s \ge s_0$ and some $s_0 \ge 1$. Fix $t > s \ge s_0$ and pick $n$ such that $k^{n-1} \le t/s < k^n$. Then $\psi(t) \le \psi(k^n s) \le k^{\theta n}\psi(s) \le k^\theta (t/s)^\theta \psi(s)$. The remaining case where $1 \le s \le s_0$ follows by inspection. □

LEMMA 3.4. *If* $\psi$ *is a positive continuous increasing function such that* $\psi \in \mathcal{R}_\lambda$ *with* $\lambda \in (0, 1)$ *then* $\psi \in \mathcal{R}^*$ *and* $\psi \in \mathcal{R}_*$.

PROOF. If $\psi \in \mathcal{R}_\lambda$, then for any $k > 1$,

$$\lim_{s \to \infty} \frac{\psi(ks)}{\psi(s)} = k^\lambda \in (1, k).$$

Hence, $\psi$ satisfies both properties in (3.1). □

We now recall a Gaussian upper bound taken from [6, 9]. It follows essentially from [15], Chapter 3.

THEOREM 3.5. *Let* $G$ *be a compact connected group. Fix* $\psi \in \mathcal{R}^*$. *Let* $(\mu_t)_{t>0}$ *be a symmetric Gaussian semigroup satisfying* (CK) *and such that*

$$\forall t \in (0, 1), \qquad \log \mu_t(e) \le \psi(1/t).$$

*Then, the intrinsic distance* $d$ *of Definition* 2.2 *is continuous and defines the topology of* $G$. *Moreover, for any* $\varepsilon > 0$, *there exists a constant* $C = C(\psi, \varepsilon)$ *such that*

$$\forall t \in (0, 1), \forall x \in G, \qquad \mu_t(x) \le \exp\left(C\psi(1/t) - \frac{d(x)^2}{4(1+\varepsilon)t}\right)$$

*where* $C(\psi, \varepsilon)$ *depends on* $\psi$ *only through the constants* $C_0$, $\theta^*$ *in* (3.2).

3.2. *Modulus of continuity.* Our first result deals with the analog of the law of the iterated logarithm which can be viewed as a result concerning the modulus of continuity of $t \mapsto X_t$ at $t = 0$.

THEOREM 3.6. *Let $G$ be a compact connected group which is not a Lie group. Assume that $(\mu_t)_{t>0}$ is a symmetric Gaussian semigroup satisfying property* (CK). *Assume further that there exists a continuous left-invariant distance function $\rho$ on $G$ and a continuous positive increasing function $\psi$ such that $t\psi(1/t)$ is increasing and*

$$\forall t \in (0,1),\ \forall x \in G, \qquad \mu_t(x) \le \exp\left(\psi(1/t) - \frac{\rho(x)^2}{t}\right)$$

*where $\rho(x) = \rho(e,x)$. Let $X$ be the invariant diffusion associated with $(\mu_t)_{t>0}$. Then*, *almost surely*,

$$\limsup_{t\to 0} \frac{\rho(X_t)}{\sqrt{t\psi(1/t)}} \le 4.$$

Note that we can always change a given distance $\rho$ to $c\rho$ with $c > 0$ in order to apply this result. The constant 4 in this statement is not sharp. In what follows, we denote by $\mathbb{P}_x$ the probability measure associated to $X$ started at $x$. We need the following lemma of independent interest.

LEMMA 3.7. *Under the hypothesis of Theorem* 3.6, *there exists a constant $C$ such that*, *for all $t \in (0,1)$ and $R > 0$ with $R \ge 2\sqrt{2t\psi(1/t)}$, we have*

$$\mathbb{P}_e\Big(\sup_{s\in[0,t]} \rho(X_s) \ge R\Big) \le C\exp\Big(-\frac{R^2}{8t}\Big).$$

PROOF. Observe that since $s\psi(1/s)$ increases with $s$, if $R \ge \sqrt{2t\psi(1/t)}$, then the postulated Gaussian upper bound gives, for any $0 < s < t \le 1$,

$$\begin{aligned}\mathbb{P}_e\big(\rho(X_s) \ge R\big) &= \int_{\{\rho(z)\ge R\}} \mu_s(z)\, d\nu(z)\\ &\le \exp\left(\frac{R^2}{s}\left(\frac{s\psi(1/s)}{R^2} - 1\right)\right)\\ &\le \exp\Big(-\frac{R^2}{2s}\Big).\end{aligned}$$

Hence,

$$\sup_{s\in[0,t]} \mathbb{P}_e\big(\rho(X_s) \ge R\big) \le \exp\Big(-\frac{R^2}{2t}\Big). \tag{3.4}$$

For a fixed $r$, let $\tau$ be the infimum of the time $s > 0$ such that $\rho(X_s) \ge r$. Then

$$\mathbb{P}_e\Big(\sup_{s\in[0,t]} \rho(X_s) \ge r\Big) = \mathbb{P}_e(\tau \le t).$$

By the strong Markov property, we have

$$\begin{aligned}\mathbb{P}_e\big(\rho(X_t) > r/2\big) &\geq \mathbb{P}_e\big(\rho(X_t) > r/2; \tau \leq t\big) \\ &= \mathbb{P}_e(\tau \leq t) - \mathbb{P}_e\big(\rho(X_t) \leq r/2; \tau \leq t\big) \\ &= \mathbb{P}_e(\tau \leq t) - \mathbb{E}_e\big(\mathbb{P}_{X_\tau}\big(\rho(X_{t-\tau}) \leq r/2\big)\mathbf{1}_{\tau \leq t}\big) \\ &\geq \mathbb{P}_e(\tau \leq t) - \mathbb{E}_e\big(\mathbb{P}_{X_\tau}\big(\rho(X_{t-\tau}, X_\tau) \geq r/2\big)\mathbf{1}_{\tau \leq t}\big) \\ &\geq \mathbb{P}_e(\tau \leq t)\Big(1 - \sup_{s \in [0,t]} \mathbb{P}_e\big(\rho(X_s) \geq r/2\big)\Big).\end{aligned}$$

By (3.4) with $R = r/2 \geq \sqrt{2t\psi(1/t)}$, this gives

$$\mathbb{P}_e(\tau \leq t)\big(1 - e^{-r^2/8t}\big) \leq e^{-r^2/8t}$$

from which the desired result follows. $\square$

REMARK. This type of argument has been used before by many authors (the second author learned this useful trick from Dan Stroock). It replaces André's reflection principle which is used in the classical case of Brownian motion in Euclidean space to prove the well known inequality

$$\mathbb{P}_0\Big(\sup_{0 \leq s \leq t} \|X_s\| \geq r\Big) \leq 2\mathbb{P}_0(\|X_t\| \geq r).$$

PROOF OF THEOREM 3.6. Fix $\sigma \in (0, 1)$ and consider the events

$$A_i = \Big\{\sup_{t \in [0,\sigma^i]} \rho(X_t) \geq 4\sqrt{\sigma^i \psi(\sigma^{-i})}\Big\}$$

By Lemma 3.7, $\mathbb{P}_e(A_i) \leq e^{-2\psi(\sigma^{-i})}$. By Lemma 3.2, the series $\sum \mathbb{P}_e(A_i)$ converges. Thus, by the Borel–Cantelli lemma, almost surely, for all $n$ large enough,

$$\sup_{t \in [0,\sigma^n]} \rho(X_t) \leq 4\sqrt{\sigma^n \psi(\sigma^{-n})}.$$

Since $t\psi(1/t)$ is nondecreasing and $\psi(1/t)$ nonincreasing, it follows that, almost surely for all $t$ small enough,

$$\rho(X_t) \leq 4\sqrt{(t/\sigma)\psi(\sigma/t)} \leq 4\sigma^{-1}\sqrt{t\psi(1/t)}.$$

Hence, almost surely,

$$\limsup_{t \to 0} \frac{\rho(X_t)}{\sqrt{t\psi(1/t)}} \leq 4\sigma^{-1}.$$

Since this holds for all $\sigma \in (0, 1)$, the conclusion of Theorem 3.6 follows. $\square$

The next theorem concerns the modulus of continuity. In the present context, it can be seen as an improved version of Theorem 3.6.

THEOREM 3.8. *Referring to the notation and hypotheses of Theorem* 3.6, *we have*

$$\lim_{\varepsilon\to 0} \sup_{\substack{0<s<t<1\\ t-s<\varepsilon}} \frac{\rho(X_s^{-1}X_t)}{\sqrt{(t-s)\psi(1/(t-s))}} \le 12 \qquad \textit{almost surely.}$$

The constant 12 in this statement is not sharp.

PROOF OF THEOREM 3.8. We learned the following variation on Lévy's original argument from Greg Lawler whom we thank for his remarks. Set $M(t) = \psi(1/t)$ and let $A_n$ be the event

$$A_n = \left\{ \sup_{\substack{j\in\{0,1,\dots,2^n-1\}\\ t\in[j2^{-n},(j+1)2^{-n}]}} \frac{\rho(X_{j2^{-n}}^{-1}X_t)}{\sqrt{(t-j2^{-n})M(t-j2^{-n})}} \ge 2\sqrt{2}\right\}.$$

As $t - j2^{-n} \le 2^{-n}$ and $M$ is decreasing, the Markov property and Lemma 3.7 give $\mathbb{P}_e(A_n) \le 2^n \exp(-M(2^{-n}))$. We claim that there exists $a > 0$ such that $2^n \times \exp(-M(2^{-n})) \le \exp(-an)$ for all $n$ large enough. Indeed, $2^n \exp(-M(2^{-n})) = \exp(-M(2^{-n}) + \log 2^n)$ and, by Lemma 3.2, $M(2^{-n})/\log 2^n$ tends to infinity.

By the Borel–Cantelli lemma, for almost all $\omega$, there exists $n(\omega)$ such that for all $n > n(\omega)$, $j \in \{0, \dots, 2^n - 1\}$ and $t \in [j2^{-n}, (j+1)2^{-n}]$,

$$\rho\bigl(X_{j2^{-n}}^{-1}X_t\bigr) < 2\sqrt{2(t-j2^{-n})M(t-j2^{-n})}.$$

Fix $0 \le s < t \le 1$ with $0 < t - s \le 2^{-n(\omega)}$. Let $n > n(\omega)$ be the integer such that $2^{-n-1} \le t - s < 2^{-n}$. Then we can find an integer $k \in \{1, \dots, 2^n - 1\}$ such that $(k-1)2^{-n} \le s < t \le (k+1)2^{-n}$. If $t \in (s, k2^{-n}]$, then

$$\rho(X_s^{-1}X_t) \le \rho\bigl(X_s^{-1}X_{(k-1)2^{-n}}\bigr) + \rho\bigl(X_{(k-1)2^{-n}}^{-1}X_t\bigr) < 4\sqrt{2^{-n+1}M(2^{-n})},$$

whereas, if $t \in (k2^{-n}, (k+1)2^{-n}]$, then

$$\begin{aligned}\rho(X_s^{-1}X_t) &\le \rho\bigl(X_s^{-1}X_{(k-1)2^{-n}}\bigr) + \rho\bigl(X_{(k-1)2^{-n}}^{-1}X_{k2^{-n}}\bigr) + \rho\bigl(X_{k2^{-n}}^{-1}X_t\bigr)\\ &< 6\sqrt{2^{-n+1}M(2^{-n})}.\end{aligned}$$

As $M$ is decreasing, $t \mapsto tM(t)$ increasing, and $t - s \ge 2^{-n-1}$, we obtain

$$\rho(X_s^{-1}X_t) < 12\sqrt{(t-s)M(t-s)}.$$

The desired conclusion follows. $\square$

3.3. *Applications and examples.* The results of Sections 3.1 and 3.2 yield the following statement.

THEOREM 3.9. *Let $G$ be a compact connected group. Let $(\mu_t)_{t>0}$ be a symmetric Gaussian semigroup. Let $X = (X_t, \mathbb{P}, \Omega)$ be the associated invariant diffusion. Assume that there exists a function $\psi \in \mathcal{R}^*$ such that $(\mu_t)_{t>0}$ satisfies* $(\mathrm{CK}_\psi)$. *Then, there exists a constant $C$ such that, almost surely,*

$$\lim_{\varepsilon\to 0} \sup_{\substack{0\le s<t\le 1\\ t-s\le\varepsilon}} \frac{d(X_s^{-1}X_t)}{\sqrt{(t-s)\psi(1/(t-s))}} \le C$$

*where $d$ is the intrinsic distance from Definition* 2.2.

Given an arbitrary compact connected group $G$, it is not obvious at all that it carries any Brownian motion satisfying property (CK). In fact, it is necessary that $G$ be locally connected and metrizable (see [20], Chapter 6 and the remark at the end of Section 3.1). One of the main result of [7] is that, for any compact connected locally connected metrizable group $G$, for any function $\psi$ such that

$$\lim_{t\to\infty} \frac{\psi(t)}{\log(1+t)} = +\infty, \tag{3.5}$$

there exists a (CK) Brownian motion on $G$ such that

$$\forall t \in (0,1), \qquad \log \mu_t(e) \le \psi(1/t).$$

Thus we have the following existence theorem.

THEOREM 3.10. *Let $G$ be an arbitrary compact connected locally connected metrizable group. Let $\psi$ be a slowly varying function satisfying* (3.5). *Then, there exists a Brownian motion $X = (X_t, \mathbb{P}, \Omega)$ on $G$ whose associated intrinsic distance $d$ is continuous and whose sample paths satisfy*

$$\lim_{\varepsilon\to 0} \sup_{\substack{0\le s<t\le 1\\ t-s\le\varepsilon}} \frac{d(X_s^{-1}X_t)}{\sqrt{(t-s)\psi(1/(t-s))}} \le 1.$$

*In particular, there exists a Brownian motion $X = (X_t, \mathbb{P}, \Omega)$ on $G$ whose associated intrinsic distance $d$ is continuous and whose sample paths are Hölder continuous of exponent $\sigma$ with respect to $d$, for all $\sigma \in (0, 1/2)$.*

We now describe some explicit examples, on the infinite-dimensional torus and on some noncommutative groups.

EXAMPLE 1. On the infinite-dimensional torus $\mathbb{T}^\infty = (\mathbb{R}/2\pi\mathbb{Z})^\infty$, any symmetric Gaussian semigroup $(\mu_t)_{t>0}$ is uniquely determined by an infinite symmetric matrix $A = (a_{i,j})$ such that for all $\xi = (\xi_i)$ with finitely many nonzero entries, $\langle A\xi, \xi\rangle = \sum_{i,j} a_{i,j}\xi_i\xi_j \ge 0$. Namely, the infinitesimal generator of $(\mu_t)_{t>0}$ is the second-order differential operator $\sum_{i,j} a_{i,j}\partial_i\partial_j$. Given a matrix $A$ as above, let $(\mu_t^A)_{t>0}$ be the associated symmetric Gaussian semigroup and set

$$W_A(s) = \#\{k \in \mathbb{Z}^{(\infty)} : \langle Ak, k\rangle \le s\}.$$

It turns out (see [6] for details) that

$$\begin{aligned} &\log W_A(s) = O(s^{\lambda/(1+\lambda)}) &&\text{as } s \to \infty \\ \iff \quad &\log \mu_t^A(e) = O(t^{-\lambda}) &&\text{as } t \to 0. \end{aligned}$$

Thus, if for some $\lambda \in (0, 1)$ and $C_0 > 0$,

$$\sup_{s\in(0,\infty)} \frac{\log W_A(s)}{(1+s)^{\lambda/(1+\lambda)}} \le C_0,$$

then there exists $C = C(C_0, \lambda) < \infty$ such that

$$\lim_{\varepsilon\to 0} \sup_{\substack{0\le s<t\le 1\\ t-s\le\varepsilon}} \frac{d_A(X_s^{-1}X_t)}{(t-s)^{(1-\lambda)/2}} \le C$$

where $d_A$ is the associated intrinsic distance.

EXAMPLE 2. Let us now specialize to the case where $A$ is diagonal with $a_{i,i} = a_i > 0$. Information concerning the behavior of $\log \mu_t^A(e)$ in this case can be found [2, 3]. Here, the intrinsic distance is continuous if and only if $\sum_i a_i^{-1} < \infty$ and $d_A(x)$ is given by

$$d_A(x) = \left(\sum_1^\infty a_i^{-1}|x_i|^2\right)^{1/2}$$

where $x = (x_i) \in \mathbb{T}^\infty$ with $x_i \in (-\pi, \pi]$. Together with [2] and [3], Theorem 3.10, gives the following:

1. Assume that $A$ is diagonal with $a_i \ge ci^{1/\lambda}$, for some $\lambda \in (0, 1)$. Then there exists $C = C(\lambda) < \infty$ such that

$$\lim_{\varepsilon\to 0} \sup_{\substack{0\le s<t\le 1\\ t-s\le\varepsilon}} \frac{d_A(X_s^{-1}X_t)}{(t-s)^{(1-\lambda)/2}} \le C.$$

2. Assume that $A$ is diagonal with $a_i \ge e^{i^\sigma}$, for some $\sigma \in (0, \infty)$. Then there exists $C = C(\sigma) < \infty$ such that

$$\lim_{\varepsilon\to 0} \sup_{\substack{0\le s<t\le 1\\ t-s\le\varepsilon}} \frac{d_A(X_s^{-1}X_t)}{\sqrt{(t-s)[\log(1+1/(t-s))]^{1+1/\sigma}}} \le C.$$

We will see below (Proposition 4.9 and Theorem 4.17) that the result in (1) is sharp when $a_i = i^{1/\lambda}$. The results in (2) is not sharp even when $a_i = e^{i^\sigma}$. Indeed, in [11], we use the very special structure of the present set of examples to improve upon (2) in certain cases. This however requires rather delicate arguments.

EXAMPLE 3. Let $G$ be a metrizable compact connected group. We say that $G$ is semisimple if $G' = G$ where $G' = [G, G]$ is the commutator group. See [21]. In the case of compact connected Lie groups this definition coincides with other classical ones. Structure theory (see, e.g., [21]) tells us that any metrizable compact connected semisimple group is isomorphic to the quotient

$$G \cong \Sigma/H$$

of a finite or countable direct product $\Sigma = \prod \Sigma_i$ of compact connected simple Lie groups $\Sigma_i$ by a closed central totally disconnected subgroup $H$. For details on what follows, we refer to [20, 21] and, more specifically, to [7].

On each $\Sigma_i$ there is, up to a multiplicative constant, a unique second order differential operator $\Delta_i$ which is bi-invariant and has no zero order term. This operator is the Laplace–Beltrami operator of the Killing Riemannian metric on $\Sigma_i$. Given a sequence $\mathbf{a} = (a_i)$ of positive numbers, let $(\overline{\mu}_t^{\mathbf{a}})_{t>0}$ be the Gaussian semigroup on $\Sigma$ whose infinitesimal generator is $\sum_i a_i \Delta_i$ and let $(\mu_t^{\mathbf{a}})_{t>0}$ be its projection on $G$. Denote by $X^{\mathbf{a}}$ the sochastic process associated to $(\mu_t^{\mathbf{a}})_{t>0}$. Then the Brownian motions on $G$, that is, the stochastic processes satisfying the hypotheses (B0)–(B5) of the Introduction, are exactly the processes $X^{\mathbf{a}}$, where $\mathbf{a}$ runs over all possible sequences of positive numbers. Let $d_{\mathbf{a}}$ be the associated intrinsic distance and set

$$N_{\mathbf{a}}(s) = \sum_{i:a_i \le s} n_i$$

where $n_i$ is the topological dimension of $\Sigma_i$. It is proved in [4, 5, 7] that $N_{\mathbf{a}}$ controls the behavior of the semigroup $(\mu_t^{\mathbf{a}})_{t>0}$. Using the results of [4, 5, 7] and Theorem 3.10, we obtain the following.

THEOREM 3.11. *Let $G$ be a compact connected semisimple group. Let $\mathbf{a}$ be a sequence of positive numbers. Referring to the notation introduced above, assume that for some $\lambda \in [0, 1)$ and some function $\psi \in \mathcal{R}_\lambda$, we have $N_{\mathbf{a}}(s) = O(\psi(s))$ as $s$ tend to infinity.*

1. *If $\lambda \in (0, 1)$ then*

$$\lim_{\varepsilon \to 0} \sup_{\substack{0 \le s < t \le 1 \\ t-s \le \varepsilon}} \frac{d_{\mathbf{a}}(X_s^{-1} X_t)}{\sqrt{(t-s)\psi(1/(t-s))}} \le C < \infty.$$

2. *If* $\lambda = 0$ *and* $s \to \widetilde{\psi}(s) = \psi(e^s)$ *satisfies* $\sup_s\{\widetilde{\psi}(2s)/\widetilde{\psi}(s)\} < \infty$, *then*

$$\lim_{\varepsilon\to 0} \sup_{\substack{0\le s<t\le 1\\ t-s\le\varepsilon}} \frac{d_{\mathbf{a}}(X_s^{-1}X_t)}{\sqrt{(t-s)\log(1+1/(t-s))\psi(1/(t-s))}} \le C < \infty.$$

EXAMPLE 4. Consider $G = \prod_1^\infty SO(i+2)$. The special orthogonal group $SO(i+2)$ has dimension $n_i = (i+1)(i+2)/2$. Consider $\mathbf{a} = (a_i)$ with $a_i \ge ci^\alpha$, $\alpha, c > 0$. Then $N_{\mathbf{a}}(s) \le Cs^{3/\alpha}$. Thus, for $\alpha > 3$, the associated Brownian motion $X^{\mathbf{a}} = (X_t, \mathbb{P})$ satisfies

$$\lim_{\varepsilon\to 0} \sup_{\substack{0\le s<t\le 1\\ t-s\le\varepsilon}} \frac{d(X_s^{-1}X_t)}{(t-s)^\gamma} \le C_{c,\alpha} < \infty \qquad \text{with } \gamma = \frac{1}{2}\left(1 - \frac{3}{\alpha}\right).$$

We believe this result is sharp when $a_i = ci^\alpha$ but we have no proof of this fact. The results obtained in Section 4 below could potentially yield a matching lower bound but the necessary Gaussian lower bounds have not yet been proved for groups such as the one considered here.

**4. Two-sided Gaussian bounds and some consequences.** Let $G$ be a compact connected group equipped with a continuous left-invariant distance $\rho(x, y)$ which defines the topology of $G$. Set $\rho(x) = \rho(e, x)$. Consider the associated volume growth function $V_\rho = V$ defined by (recall that $\nu$ denotes the Haar measure)

$$V(r) = \nu(\{x \in G : \rho(x) < r\}). \tag{4.1}$$

Consider a symmetric Gaussian convolution semigroup $(\mu_t)_{t>0}$ on $G$ and assume that this semigroup satisfies property (CK). As before, denote by $x \to \mu_t(x)$ continuous density of the measure $\mu_t$ with respect to Haar measure.

In this section we consider the possibility that there exists a decreasing continuous function $M : (0, +\infty) \to (0, +\infty)$ and four constants $c_1, C_1, c_2, C_2 \in (0, \infty)$ such that, for all $t \in (0, 1)$ and $x \in G$, the Gaussian semigroup $(\mu_t)_{t>0}$ satisfies

$$c_1 M(t) - C_2\frac{\rho(x)^2}{t} \le \log \mu_t(x) \le C_1 M(t) - c_2\frac{\rho(x)^2}{t}. \tag{4.2}$$

For convenience, we set $\psi(t) = M(1/t)$.

Under the hypothesis that such an inequality holds with a suitable function $M$, we give sharp volume growth estimates and sharp Green function estimates. We then use these estimates to obtain results concerning the local rate of escape of the associated diffusion, in the spirit of the classical result of Dvoretzky and Erdös stated in the Introduction.

To reassure the reader that it is not entirely foolish to assume that (4.2) holds, we quote a result from [6].

THEOREM 4.1. *On the infinite-dimensional torus $G = \mathbb{T}^\infty$, for any symmetric Gaussian semigroup $(\mu_t)_{t>0}$ satisfying* (CK), *we have*

$$\forall x \in \mathbb{T}^\infty,\ \forall t > 0, \qquad \mu_t(x) \geq \mu_t(e) \exp\left(-\frac{d(x)^2}{4t}\right) \tag{4.3}$$

*where $d$ is the associated intrinsic distance.*

Using this and Theorem 3.5, we obtain the following important result.

THEOREM 4.2. *Let $(\mu_t)_{t>0}$ be a* (CK) *symmetric Gaussian semigroup on the infinite-dimensional torus $\mathbb{T}^\infty$. Assume that*

$$c\psi(1/t) \leq \log \mu_t(e) \leq C\psi(1/t)$$

*where $\psi \in \mathcal{R}^*$ and $c, C \in (0,\infty)$. Then there exist constants $c_1, c_2, C_1, C_2 \in (0,\infty)$ such that, for all $t \in (0,1)$ and $x \in G$,*

$$c_1\psi(1/t) - C_2\frac{d(x)^2}{t} \leq \log \mu_t(x) \leq C_1\psi(1/t) - c_2\frac{d(x)^2}{t}. \tag{4.4}$$

The following technical definition will be used throughout this section.

DEFINITION 4.3. Let $M$ be a continuous decreasing function defined on the interval $(0, 1)$. Assume that $\lim_{s\to 0} M(s) = \infty$, that $s \mapsto sM(s)$ is increasing, and that $\lim_{s\to 0} sM(s) = 0$. Given $c > 0$, define the function $F_c : (0, cM(1)) \to (0, 1)$ by

$$F_c(s) = t \qquad \text{if and only if } ctM(t) = s. \tag{4.5}$$

Thus $F_c(t)$ is the inverse of $ctM(t)$. Set also $F = F_1$.

Note that $F_c$ is increasing and tends to zero at zero. Moreover, for any two constants $c, b$ with $0 < c < b$, we have $F_b < F_c$ on $(0, cM(1))$.

4.1. *Volume growth.* This subsection relates Gaussian bounds to the behavior of the volume growth function $V$ defined at (4.1). These simple results are new. Let us point out that estimating the volume function $V$ in the present setting is not an easy task. As far as we can see, even for simple examples on $\mathbb{T}^\infty$ such as those considered in Example 2, the best way to estimate the volume function $V$ is to relate it to $\log \mu_t(e)$ as done below.

LEMMA 4.4. *Let $M$ and $F_c$ be as in Definition* 4.3. *Consider a distance $\rho$ and a symmetric Gaussian semigroup $(\mu_t)_{t>0}$ satisfying* (CK).

1. *Assume that there exist* $C_1, c_2 > 0$ *such that*

$$\forall t \in (0,1),\ \forall x \in G, \qquad \mu_t(x) \le \exp\left(C_1 M(t) - c_2 \frac{\rho(x)^2}{t}\right). \tag{4.6}$$

*Then the volume growth function* $V = V_\rho$ *satisfies*

$$\inf_{r \in (0, CM(1))} \frac{\log V(r)}{M \circ F_C(r^2)} > -\infty, \tag{4.7}$$

*for* $C = 2C_1/c_2$.

2. *Assume that there exist* $c_1, C_2 > 0$ *such that*

$$\forall t \in (0,1),\ \forall x \in G, \qquad \mu_t(x) \ge \exp\left(c_1 M(t) - C_2 \frac{\rho(x)^2}{t}\right). \tag{4.8}$$

*Then the volume growth function* $V = V_\rho$ *satisfies*

$$\sup_{r \in (0, cM(1))} \frac{\log V(r)}{M \circ F_c(r^2)} < 0, \tag{4.9}$$

*for* $c = c_1/(2C_2)$.

PROOF OF 1. Set $v(s) = V(\sqrt{s})$. Then we have

$$\int_G e^{-c_2 \rho(x)^2/t}\, d\nu(x) = \int_0^\infty e^{-c_2 s/t}\, dv(s) = c_2 \int_0^\infty v(st) e^{-c_2 s}\, ds.$$

Hence, for any $A \ge 1$, the hypothesis that (4.6) holds true yields

$$\begin{aligned} e^{-C_1 M(t)} &\le c_2 \int_0^\infty v(st) e^{-c_2 s}\, ds \\ &\le c_2 \left( \int_0^A v(st) e^{-c_2 s}\, ds + \int_A^\infty v(st) e^{-c_2 s}\, ds \right) \\ &\le c_2 v(At) + e^{-c_2 A}. \end{aligned}$$

Taking $A = CM(t)$ with $C = 2C_1/c_2$, we obtain

$$\forall t \in (0,1), \qquad e^{-C_1 M(t)}\bigl(1 - e^{-C_1 M(t)}\bigr) \le c_2 v(CtM(t)).$$

As $\lim_{t \to 0} M(t) = +\infty$, for all $s$ small enough,

$$v(s) \ge \frac{1}{2c_2} \exp\bigl(-C_1 M \circ F_C(s)\bigr).$$

Returning to the volume growth function $V(s) = v(s^2)$, we obtain

$$\liminf_{r \to 0} \frac{\log V(r)}{M \circ F_C(r^2)} \ge -C_1$$

as desired. $\square$

PROOF OF 2. By the same token, assuming (4.8), we obtain

$$e^{-c_1 M(t)} \geq C_2 \int_0^\infty v(st) e^{-C_2 s}\, ds \geq v(at) e^{-aC_2},$$

for any $a > 0$ and $t \in (0, 1)$. Choosing $a = cM(t)$ with $c = c_1/(2C_2)$ yields

$$e^{-c_1/2M(t)} \geq v(ctM(t)),$$

that is,

$$V(r) \leq \exp\left(-\frac{c_1}{2} M \circ F_c(r^2)\right)$$

for all $r$ small enough. This ends the proof of Lemma 4.4. $\square$

The following simple technical lemma is useful in interpreting the estimates obtained above.

LEMMA 4.5. *Let $M$ and $F_c$ be as in Definition 4.3. Assume further that $\psi(t) = M(1/t)$ belongs to $\mathcal{R}^*$. Then, for any fixed $b, c$ with $0 < b \leq c < +\infty$, there exists a constant $A$ such that, on $(0, bM(1))$, we have $F_c \leq F_b \leq AF_c$.*

PROOF. By Lemma 3.3, $\psi(t) = M(1/t) \in \mathcal{R}^*$ is equivalent to the existence of $C_0 > 0$ and $\alpha \in (0, 1)$ such that

$$\forall s, t \text{ with } 0 < s \leq t \leq 1, \qquad \frac{sM(s)}{tM(t)} \leq C_0 \left(\frac{s}{t}\right)^\alpha. \tag{4.10}$$

The desired result follows. $\square$

THEOREM 4.6. *Assume that $(\mu_t)_{t>0}$ is (CK) and satisfies the two-sided Gaussian bound (4.2) with $\rho$ and $M$, $F_c$ as in Lemma 4.5. Then there are constants $a_0, a_1 > 0$ such that the volume growth function $V = V_\rho$ satisfies*

$$\forall r \in \big(0, M(1)\big), \qquad -a_0 M \circ F(r^2) \leq \log V(r) \leq -a_1 M \circ F(r^2), \tag{4.11}$$

*where $F = F_1$ is the inverse function of $s \mapsto sM(s)$.*

PROOF. It suffices to observe that, for any $c > 0$, by Lemma 4.5, there exist two constants $C_1, c_1 > 0$ such that

$$c_1 \frac{s}{F(s)} \leq M \circ F_c(s) = \frac{s}{cF_c(s)} \leq C_1 \frac{s}{F(s)}$$

for all $s$ small enough. The desired result then follows from $s/F(s) = M \circ F(s)$. $\square$

On $G = \mathbb{T}^\infty$, two-sided Gaussian bounds are available by (4.4). Hence we obtain Theorem 4.7. Given two positive functions $f, g$ defined in a neighborhood of $t_0 \in [0, \infty]$, we write $f \simeq g$ if there are two constants $0 < c \leq C < \infty$ such that $cf \leq g \leq Cf$ in a neighborhood of $t_0$.

THEOREM 4.7. *Let $(\mu_t)_{t>0}$ be a symmetric* (CK) *Gaussian semigroup on the infinite-dimensional torus $\mathbb{T}^\infty$. Let $V(r)$ denote the volume growth function associated with the intrinsic distance $d$ from Definition* (2.2).

1. *Assume that there exists $\lambda \in (0,1)$ such that $\log \mu_t(e) \simeq t^{-\lambda}$ at 0. Then we have*

$$\log \frac{1}{V(r)} \simeq r^{-2\lambda/(1-\lambda)} \qquad \text{at } 0.$$

2. *Assume that there exists an increasing slowly varying function $\psi$ satisfying $\psi(t\psi(t)) \sim \psi(t)$ at infinity and such that $\log \mu_t(e) \simeq \psi(1/t)$ at 0. Then*

$$\log \frac{1}{V(r)} \simeq \psi(1/r^2) \qquad \text{at } 0.$$

Note that the functions $\psi(u) = \log_1(u)\log_n(u)$ where $\log_n(u) = \log(1 + \log_{n-1}(u))$, $\log_1(u) = \log(1+u)$, or $\psi(u) = [\log_1(u)]^k$ all satisfy the condition $\psi(t\psi(t)) \sim \psi(t)$ at infinity.

To illustrate how and why volume estimates, especially upper bounds, are of interest in the study of the regularity of the sample paths of invariant diffusions, we offer the following two results.

PROPOSITION 4.8. *Referring to the setting and notation introduced above, assume that there exist $\lambda, \beta$ with $0 < \lambda < 1$ and $\beta > 0$ such that*:

(a) $\log V_\rho(r) \le -cr^{-\beta}$ *for all* $r \in (0,1)$;
(b) $(\mu_t)_{t>0}$ *satisfies* (CK) *and* (4.6) *with* $M(t) \le t^{-\lambda}$ *for all* $t \in (0,1)$.

*Then, setting $\gamma = \lambda/\beta$, we have*

$$\limsup_{t\to 0} \frac{\rho(X_t)}{t^\gamma} > 0 \qquad \text{almost surely.}$$

PROOF. First, observe that Lemma 4.4 shows that (b) implies

$$\forall r \in (0,1), \qquad \log V_\rho(r) \ge -Cr^{-2\lambda/(1-\lambda)}.$$

Hence (a) and (b) are compatible only if $\beta \le 2\lambda/(1-\lambda)$. Using (a) and (b), we have for any $r, t \in (0,1)$,

$$\mathbb{P}\big(\rho(X_t) < r\big) \le V_\rho(r)\mu_t(e) \le e^{-cr^{-\beta} + C_1 t^{-\lambda}}. \tag{4.12}$$

Let $t_n$ be a decreasing sequence tending to zero and set $A_n = \{\rho(X_{t_n}) < at_n^\gamma\}$ where $a \in (0,1)$ will be chosen later. By (4.12),

$$\mathbb{P}(A_n) \le e^{-(ca^{-\lambda} - C_1)t_n^{-\lambda}}.$$

Picking $a = [2C_1/c]^{-1/\lambda}$ and $t_n = 1/n$, we obtain that $\sum_n \mathbb{P}(A_n) < +\infty$. Hence, by the Borel–Cantelli lemma $\mathbb{P}$-almost surely, $\rho(X_{t_n}) > a^\gamma t_n^\gamma$ for all $n$ large enough. This completes the proof of Proposition 4.8. $\square$

PROPOSITION 4.9. *Let $M$ and $F$ be as in Definition* 4.3. *Assume further that $\psi(t) = M(1/t)$ belongs to both $\mathcal{R}^*$ and $\mathcal{R}_*$. Referring to the setting and notation introduced above, assume that*:

(a) $\log V_\rho(r) \le -aM \circ F(r^2)$ *for all* $r \in (0, 1)$;
(b) $(\mu_t)_{t>0}$ *satisfies* (CK) *and* (4.6).

*Then there exists a constant* $0 < c \le C < \infty$ *such that, almost surely*,

$$c \le \limsup_{t \to 0} \frac{\rho(X_t)}{\sqrt{tM(t)}} < C.$$

*The constants $c$, $C$ depend only on $a$, on the constants $C_0, c_0, \theta^*, \theta_*$ of Lemma* 3.3, *and on the constants $C_1, c_2$ in* (4.6).

PROOF. The upper bound $\limsup_{t\to 0} \frac{\rho(X_t)}{\sqrt{tM(t)}} < 4\sqrt{C_1/c_1}$ is proved in Theorem 3.6. For the lower bound, as in the proof of Proposition 4.8, we have for any $r, t \in (0, 1)$,

$$\mathbb{P}\big(\rho(X_t) < r\big) \le e^{-aM\circ F(r^2)+C_1M(t)}. \tag{4.13}$$

For any $t \in (0, 1)$ small enough, let $r = r(t)$ be defined by

$$M \circ F(r^2) = \frac{2C_1}{a} M(t).$$

Clearly, $r(t)$ is increasing. We claim that there are constants $C'c' > 0$ such that

$$c' tM(t) \le r^2(t) \le C' tM(t). \tag{4.14}$$

Indeed, we have

$$r^2 = F^{-1} \circ M^{-1}\big(2C_1M(t)/a\big) = \frac{2C_1}{a} M(t) M^{-1}\big(2C_1M(t)/a\big).$$

Now, the hypothesis that $\psi \in \mathcal{R}_*$ implies that there are constants $c_0 > 0$ and $\beta \in (0, 1)$ such that

$$\forall s, t \text{ with } 0 < s \le t < 1, \qquad \frac{M(s)}{M(t)} \ge c_0 \left(\frac{t}{s}\right)^\beta. \tag{4.15}$$

Using this and the fact that $M(t)$ is nonincreasing, we see that for any $\lambda > 0$, there exists $1 \le a_\lambda < \infty$ such that

$$a_\lambda^{-1} M^{-1}(t) \le M^{-1}(\lambda t) \le a_\lambda M^{-1}(t).$$

This proves (4.14).

Now, set $A_n = \{\rho(X_{1/n}) < r(1/n)\}$. By (4.13) and the definition of $r(t)$,

$$\mathbb{P}(A_n) \le e^{-C_1M(1/n)}.$$

By (4.15), this shows that $\sum_n \mathbb{P}(A_n) < +\infty$. Hence, the Borel–Cantelli lemma yields that $\mathbb{P}$-almost surely,

$$\rho(X_{1/n}) \geq r(1/n) \geq \sqrt{c'(1/n)M(1/n)}$$

for all $n$ large enough. Hence

$$\limsup_{t\to 0} \frac{\rho(X_t)}{\sqrt{tM(t)}} \geq \sqrt{c'} > 0$$

as desired. This completes the proof of Proposition 4.9. $\square$

4.2. *Green function estimates.* Denote by

$$g = \int_0^\infty e^{-t}\mu_t\, dt$$

the Green function associated to $(\mu_t)_{t>0}$. In general, $g$ is simply a probability measure on $G$. If this measure is absolutely continuous with respect to Haar measure and admits a continuous density on $G \setminus \{e\}$, we denote this density by $x \mapsto g(x)$. By [6], $g$ admits a continuous density on $G \setminus \{e\}$ as soon as $(\mu_t)_{t>0}$ satisfies (CK) and $\lim_{t\to 0} t \log \mu_t(e) = 0$.

THEOREM 4.10. *Let $M$ and $F_c$ be as in Definition* 4.3 . *Consider a distance $\rho$ and a symmetric Gaussian semigroup $(\mu_t)_{t>0}$ satisfying* (CK) *as above.*

1. *Assume that $(\mu_t)_{t>0}$ satisfies the Gaussian upper bound* (4.6). *Then there exists a constant $C > 0$ such that*

$$\forall x \in G, \qquad \log g(x) \leq CM \circ F_C(\rho(x)^2). \tag{4.16}$$

2. *Assume that $(\mu_t)_{t>0}$ satisfies the Gaussian lower bound* (4.8). *Then there exist two constants $C, c > 0$ such that, for all $x$ satisfying $\rho(x)^2 \leq cM(1)$,*

$$\log g(x) \geq cM \circ F_c(\rho(x)^2) + C \log F_c(\rho(x)^2) - F_c(\rho(x)^2). \tag{4.17}$$

PROOF OF 1. As $M$ is decreasing and $s \mapsto sM(s)$ increasing, for all $t \in (0, 1)$, we have

$$C_1 M(t) - c_2\rho(x)^2/t \leq C_1 M \circ F_C(\rho(x)^2)$$

with $C = \max\{C_1/c_2, \mathbf{d}^2/M(1)\}$ where $\mathbf{d} = \max_G \rho$ is the diameter of the compact metric space $(G, \rho)$. Indeed, let $t_0 \in (0, 1)$ be such that $Ct_0 M(t_0) = \rho(x)^2$ (such a $t_0$ exists because $C \geq \mathbf{d}^2/M(1)$). Then, either $0 < t \leq t_0$ and we have

$$C_1 M(t) - c_2\frac{\rho(x)^2}{t} \leq \frac{1}{t}\big(C_1 t M(t) - c_1\rho(x)^2\big) \leq \frac{1}{t}\left(\frac{C_1}{C} - c_1\right)\rho(x)^2 \leq 0,$$

or $t_0 < t < 1$ and we have

$$C_1 M(t) - c_2\rho(x)^2/t \leq C_1 M(t_0) = C_1 M \circ F_C(\rho(x)^2).$$

This implies the desired upper bound on $\log g(x)$. $\square$

PROOF OF 2. Observe that, for any $c > 0$, $cM \circ F_c(s) = s/F_c(s)$. Set $\rho = \rho(x)$ and assume that $\rho^2 \le cM(1)$. Choose $c = c_1/(3C_2)$. As $M$ is decreasing and $F_c$ increasing, we have

$$\begin{aligned} g(x) &\ge \int_{F_c(\rho^2)/2}^{F_c(\rho^2)} \exp\left(c_1 M(t) - C_2 \frac{\rho^2}{t} - t\right) dt \\ &\ge \frac{F_c(\rho^2)}{2} \exp\left(c_1 M \circ F_c(\rho^2) - 2C_2 \frac{\rho^2}{F_c(\rho^2)} - F_c(\rho^2)\right) \\ &= \frac{F_c(\rho^2)}{2} \exp\left(\left(\frac{c_1}{c} - C_2\right) \frac{\rho^2}{F_c(\rho^2)} - F_c(\rho^2)\right) \\ &= \frac{F_c(\rho^2)}{2} \exp\left(C_2 \frac{\rho^2}{F_c(\rho^2)} - F_c(\rho^2)\right) \\ &\ge \frac{F_c(\rho^2)}{2} \exp\big((C_1/3) M \circ F_c(\rho^2) - F_c(\rho^2)\big). \end{aligned}$$ $\square$

REMARK 1. If $G$ is not a Lie group, then Lemma 3.2 shows that $M(t)$ grows to infinity strictly faster than $\log(1 + 1/t)$ as $t$ tends to 0. Thus, in this case, Theorem 4.10(2) yields the lower bound

$$\log g(x) \ge cM \circ F_c\big(\rho(x)^2\big)$$

for all $x$ such that $\rho(x)$ is small enough.

THEOREM 4.11. *Let $M$ and $F_c$ be as in Definition* 4.3. *Consider a distance $\rho$ and a symmetric* (CK) *Gaussian semigroup $(\mu_t)_{t>0}$ as above.*

1. *Assume that $(\mu_t)_{t>0}$ satisfies the two-sided Gaussian bound* (4.2). *Then there exist two constants $0 < c \le C < \infty$ such that, for all $x$ with $\rho(x)$ small enough,*

$$\frac{c\rho(x)^2}{F_c(\rho(x)^2)} \le \log g(x) \le \frac{C\rho(x)^2}{F_C(\rho(x)^2)}. \tag{4.18}$$

2. *Assume in addition that $\psi(t) = M(1/t)$ belongs to $\mathcal{R}^*$* [*i.e., $M$ satisfies* (4.10)]. *Then for all $x$ with $\rho(x)$ small enough,*

$$\log g(x) \simeq \log \frac{1}{V(\rho(x))}. \tag{4.19}$$

By (4.4), in the case of the infinite-dimensional torus, we have the following result.

THEOREM 4.12. *Let $(\mu_t)_{t>0}$ be a symmetric* (CK) *Gaussian semigroup on the infinite-dimensional torus $\mathbb{T}^\infty$.*

1. *Assume that there exists* $\lambda \in (0, 1)$ *such that* $\log \mu_t(e) \simeq t^{-\lambda}$ *at* $0$. *Then, for all $x$ in a small enough neighborhood of the identity element, the Green function $g$ satisfies*

$$\log g(x) \simeq d(x)^{-2\lambda/(1-\lambda)}$$

*where $d$ is the intrinsic distance associated to* $(\mu_t)_{t>0}$.

2. *Assume that there exists an increasing slowly varying function* $\psi$ *satisfying* $\psi(t\psi(t)) \sim \psi(t)$ *at infinity and such that* $\log \mu_t(e) \simeq \psi(1/t)$ *at* $0$. *Then, for all $x$ in a small enough neighborhood of the identity element*,

$$\log g(x) \simeq \psi\bigl(d(x)^{-2}\bigr).$$

4.3. *Local rate of escape.* The aim of this section is to prove a result analogous to the classic result of Dvoretzky and Erdös. In the present setting it takes a slightly different form. Indeed, call a positive increasing function $h$ an upper radius for $X$ if, almost surely, $\rho(X_t) \le h(t)$ for all $t$ small enough. Call it a lower radius if, almost surely $\rho(X_t) \ge h(t)$ for all $t$ small enough. For Brownian motion on $\mathbb{R}^n$, or on any compact Lie group of dimension $n$, the law of the iterated logarithm describes an almost optimal upper radius whereas Dvoretzky–Erdös result describes almost optimal lower radii. In this classic case, these almost optimal upper and lower radii are significantly different, the former being $\sqrt{2t \log\log 1/t}$, the latter $\sqrt{t}[\log 1/t]^{-(1+\varepsilon)/(n-2)}$. This section shows that, under some natural hypotheses, in the present infinite-dimensional setting, optimal upper and lower radii are comparable.

THEOREM 4.13. *Let* $(\mu_t)_{t>0}$ *be a* (CK) *symmetric Gaussian semigroup. Let $M$ and $F_c$ be as in Definition* 4.3. *Assume that* $\psi(t) = M(1/t)$ *belongs to both* $\mathcal{R}^*$ *and* $\mathcal{R}_*$. *Assume further that the two-sided Gaussian bound* (4.2) *holds true. Then there are constants* $0 < c \le C < \infty$ *such that*

$$c \le \liminf_{t\to 0} \frac{\rho(X_t)}{\sqrt{tM(t)}} \le \limsup_{t\to 0} \frac{\rho(X_t)}{\sqrt{tM(t)}} \le C.$$

*The constants $c, C$ depend only on the constants $c_1, C_1, c_2, C_2$ from* (4.2) *and on the constants $c_0, C_0, \theta^*, \theta_*$ corresponding to $\psi$ by Lemma* 3.3.

PROOF. The upper bound is proved in Theorem 3.6 under weaker hypotheses. Thus we now focus on the lower bound

$$0 < \liminf_{t\to 0} \frac{\rho(X_t)}{\sqrt{tM(t)}}. \tag{4.20}$$

Let us observe that all the hypotheses will be needed for our proof of the lower bound (4.20) itself. Set

$$A_n = \bigl\{\rho(X_t) < \kappa\sqrt{a^n M(a^n)} \text{ for some } t \in [a^{n+1}, a^n]\bigr\}$$

where $a \in (0,1)$ is an arbitrary fixed constant and $\kappa > 0$ is a constant which will be chosen later. Our aim is to show that $\kappa > 0$ can be chosen so that $\sum \mathbb{P}(A_n) < \infty$. If this is the case, then the Borel–Cantelli lemma shows that, almost surely,

$$\liminf_{t\to 0} \frac{\rho(X_t)}{\sqrt{tM(t)}} \geq \kappa > 0.$$

Consider the process $X^1 = (X_t^1)$ associated with the convolution semigroup $e^{-t}\mu_t$. This process takes value in $G \cup \{\infty\}$ where $\infty$ is an isolated point added to $G$. Set $\rho(x,\infty) = +\infty$ for any $x \in G$. The process $X^1 = (X_t^1)$ can be obtained from $X$ in the following way. Let $\xi$ be a real random variable, independent of the process $X$ and with $\mathbb{P}(\xi > t) = e^{-t}$. Then

$$X_t^1 = \begin{cases} X_t, & \text{if } t < \xi, \\ \infty, & \text{if } t \geq \xi. \end{cases}$$

Thus $X_t^1$ is $X_t$ killed at an exponential time. For any $0 < s < s' \leq 1$, $0 < r \leq 1$, we have

$$\begin{aligned} &\mathbb{P}\big(\rho(X_t) < r \text{ for some } t \in [s,s']\big) \\ &\quad = \mathbb{P}\big(\rho(X_t) < r \text{ for some } t \in [s,s']; t \geq \xi\big) \\ &\qquad + \mathbb{P}\big(\rho(X_t) < r \text{ for some } t \in [s,s']; t < \xi\big) \\ &\quad \leq \mathbb{P}(s' \geq \xi) + \mathbb{P}\big(\rho(X_t^1) < r \text{ for some } t \geq s\big) \\ &\quad = 1 - e^{-s'} + \mathbb{P}\big(\rho(X_t^1) < r \text{ for some } t \geq s\big). \end{aligned} \tag{4.21}$$

Since $1 - e^{-s'} \sim s'$ at 0, (4.21) shows that the series $\sum \mathbb{P}(A_n)$ converges if and only if the series $\sum \mathbb{P}(B_n)$ converges where

$$B_n = \big\{\rho(X_t^1) < \kappa\sqrt{a^n M(a^n)} \text{ for some } t \geq a^{n+1}\big\}. \tag{4.22}$$

Thus it suffices to show that we can choose $\kappa$ small enough so that $\sum \mathbb{P}(B_n)$ converges. $\square$

In what follows, we will need some properties of the functions $F_c$ defined at (4.5) and of $t \mapsto t/F_c(t)$. By Lemma 4.5, when $\psi \in \mathcal{R}^*$, the functions $F_c$ are all comparable to $F = F_1$. We will use this fact several times in what follows. Moreover, we have the following simple result.

LEMMA 4.14. *Under the hypothesis that $\psi(t) = M(1/t)$ belongs to $\mathcal{R}^* \cap \mathcal{R}_*$, there exist $K_0$ and $\sigma > 0$ such that*

$$\forall t \in \big(0, M(1)\big),\ \forall s \in (0,t), \qquad \frac{t}{F(t)} \leq K_0\left(\frac{s}{t}\right)^{\sigma} \frac{s}{F(s)}. \tag{4.23}$$

PROOF. Since $F$ is the inverse of $s \mapsto sM(s)$, (4.23) is equivalent to

$$\forall t \in (0,1), \forall s \in (0,t), \qquad \frac{M(t)}{M(s)} \le \left(K_0 \frac{s}{t}\right)^{\sigma/(1+\sigma)}. \tag{4.24}$$

The hypothesis that $\psi(t) = M(1/t)$ belongs to $\mathcal{R}^* \cap \mathcal{R}_*$, implies that (4.15) holds true with $\beta \in (0,1)$ and this proves that (4.24) holds true for some $K_0$ and $\sigma > 0$. $\square$

PROPOSITION 4.15. *Consider the function*

$$w_r(z) = \mathbb{P}_z\big(\rho(X_t^1) < r \textit{ for some } t > 0\big).$$

*Under the hypotheses of Theorem* 4.13, *there exist two constants* $0 < c \le C < \infty$ *such that, for all* $r$ *small enough and all* $z$ *with* $\rho(z)^2 \le M(1)$, *we have*

$$w_r(z) \le \exp\left(\frac{C\rho(z)^2}{F(\rho(z)^2)} - \frac{cr^2}{F(r^2)}\right). \tag{4.25}$$

PROOF. To prove this, note that $w = w_r$ is a superharmonic function for the process $X^1$. That means that $H_s^1 w \le w$ for all $s > 0$ where $H_s^1 f = e^{-s} f * \mu_s$. Since $\lim_{s\to\infty} H_s^1 w = 0$ point-wise, $w$ is a potential [w.r.t. $(H_t^1)_{t>0}$]. Since the Gaussian semigroup $(\mu_t)_{t>0}$ is symmetric and satisfies (CK), the Hunt duality theory (see [13], Chapter 6) shows that there exists a Borel measure $m$ such that

$$w = (I + L)^{-1} m = g * m$$

where $-L$ denote the infinitesimal generator of $(\mu_t)_{t>0}$ and $g$ is the Green function defined by $g(x) = \int_0^\infty e^{-s}\mu_s(x)\,ds$. Since $w$ is $X^1$-harmonic outside the ball $\overline{B(e,r)}$, $m$ is supported by $\overline{B(e,r)}$. Moreover, since $w \equiv 1$ on $B(e,r)$, the measure $m$ restricted to $B(e,r)$ coincides with the Haar measure (in the sense of distributions, we have $(I + L)w = m$ and since $w$ is constant equal to 1 in $B(e,r)$, we get that $m = 1$ in $B(e,r)$). Thus

$$dm = \mathbf{1}_{B(e,r)}\,d\nu + \mathbf{1}_{\partial B(e,r)}\,dm$$

where $\partial B(e,r) = \{y : \rho(y) = r\}$ is the boundary of $B(e,r)$. For any $z$ with $\rho(z) \ge 2r$ and any $y$ in $B(e,r)$, the upper bound in (4.18), Lemma 4.5 and (4.23) together give

$$g(z^{-1}y) \le \exp\left(\frac{C\rho(z)^2}{F(\rho(z)^2)}\right).$$

Hence

$$w(z) = \int_G g(z^{-1}y)\,dm(y) \le \left(\int_{\partial B(e,r)} dm + V(r)\right) \exp\left(\frac{C\rho(z)^2}{F(\rho(z)^2)}\right).$$

By the lower bound in (4.18), we also have

$$1 = w(e) = \int_G g(y)\,dm(y) \geq \int_{\partial B(e,r)} g(y)\,dm(y) \geq \left( \int_{\partial B(e,r)} dm \right) \exp\left( \frac{cr^2}{F(r^2)} \right)$$

for all $r$ small enough. Hence

$$\int_{\partial B(e,r)} dm \leq \exp\left( -\frac{cr^2}{F(r^2)} \right).$$

By (4.11) we also have

$$V(r) \leq \exp\left( -\frac{cr^2}{F(r^2)} \right).$$

Thus, we obtain

$$w(z) \leq 2 \exp\left( \frac{C\rho(z)^2}{F(\rho(z)^2)} - \frac{cr^2}{F(r^2)} \right)$$

for all $z$ and $r$ with $\rho(z) \geq 2r$ and $r$ small enough. Since $w_r$ is bounded above by 1 and with the help of (4.23), one easily extends this inequality to all $z$ with $\rho(z)^2 \leq M(1)$. This ends the proof of Proposition 4.15. □

Now, to complete the proof of Theorem 4.13 it suffices to prove the following result.

LEMMA 4.16. *Under the hypotheses of Theorem* 4.13, *if* $\kappa$ *is small enough*, *then* $\sum \mathbb{P}(B_n) < \infty$ *where* $B_n$ *is defined at* (4.22).

PROOF. By Proposition 4.15 and the strong Markov property, we have

$$\begin{aligned} &\mathbb{P}_x\big(\rho(X_t^1) \leq r \text{ for some } t > s\big) \\ &\quad = e^{-s} \int w_r(z)\mu_s(z^{-1}x)\,d\nu(z) \\ &\quad = e^{-s} \left( \int_{\rho(z) > Kr} w_r(z)\mu_s(z^{-1}x)\,d\nu(z) \right. \\ &\qquad\qquad \left. + \int_{\rho(z) \leq Kr} w_r(z)\mu_s(z^{-1}x)\,d\nu(z) \right) \end{aligned} \tag{4.26}$$

for any $K > 0$. If we pick $K = \max\{1, 2K_0C/c)^{1/(2\sigma)}\}$ where $C, c$ are the constants given by Proposition 4.15 and $K_0, \sigma$ are as in (4.23), Proposition 4.15 gives

$$\int_{\rho(z) > Kr} w_r(z)\mu_s(z^{-1}x)\,d\nu(z) \leq \exp\left( -\frac{cr^2}{2F(r^2)} \right).$$

Moreover, since $\mu_s(z) \le \exp(M(s))$ for any $s \in (0,1)$ and $z \in G$, Theorem 4.6 gives

$$\int_{\rho(z)\le Kr} w_r(z)\mu_s(z^{-1}x)\,d\nu(z) \le V(Kr)\exp(M(s)) \le \exp\left(M(s) - \frac{a_1(Kr)^2}{F((Kr)^2)}\right).$$

To obtain an estimate of $\mathbb{P}(B_n)$, we pick $r = \kappa\sqrt{a^n M(a^n)}$, $s = a^{n+1}$ and obtain

$$\mathbb{P}(B_n) \le \exp\left(-\frac{c}{2}\frac{\kappa^2 a^n M(a^n)}{F(\kappa^2 a^n M(a^n))}\right) + \exp\left(M(a^{n+1}) - \frac{a_1\kappa^2 K^2 a^n M(a^n)}{F(\kappa^2 K^2 a^n M(a^n))}\right).$$

Using Lemma 4.14, the fact that $F(tM(t)) = t$, and (4.10), we get

$$\mathbb{P}(B_n) \le \exp\left(-\frac{cM(a^n)}{2K_0\kappa^{2\sigma}}\right) + \exp\left(C_0 a^{\alpha-1} M(a^n) - \frac{a_1 M(a^n)}{K_0(\kappa K)^{2\sigma}}\right).$$

Thus, for $\kappa$ small enough, we have $\mathbb{P}(B_n) \le 2\exp(-c' M(a^n))$. By (4.15), this shows that $\sum \mathbb{P}(B_n) < \infty$ as desired. This completes the proof of 4.13. □

Theorem 4.2 shows that 4.13 applies nicely when $G = \mathbb{T}^\infty$. In this case, it yields the following result which contains the first part of Theorem 1.1.

THEOREM 4.17. *Let $X$ be a Brownian motion on the infinite dimensional torus $\mathbb{T}^\infty$ with associated Gaussian semigroup $(\mu_t)_{t>0}$. Assume that $(\mu_t)_{t>0}$ satisfies* (CK) *and that there exist an increasing function $\psi \in \mathcal{R}^* \cap \mathcal{R}_*$ and two constants $0 < c_1 \le C_1 < \infty$ such that*

$$\forall t \in (0,1), \qquad c_1\psi(1/t) \le \mu_t(e) \le C_1\psi(1/t).$$

*Then there are constants $0 < c \le C < \infty$ such that*

$$c \le \liminf_{t\to 0} \frac{d(X_t)}{\sqrt{t\psi(1/t)}} \le \limsup_{t\to 0} \frac{d(X_t)}{\sqrt{t\psi(1/t)}} \le C$$

*where $d$ is the associated intrinsic distance. The constants $c, C$ depend only on $c_1, C_1$ and on the constants $c_0, C_0, \theta^*, \theta_*$ corresponding to $\psi$ by* Lemma 3.3.

For a very explicit example, take $(\mu_t)_{t>0}$ as in Example 2 of Section 3.3 with $a_i \simeq i^{1/\lambda}$ for some $\lambda \in (0,1)$. Then (see [2, 3]) $\log \mu_t(e) \simeq t^{-\lambda}$ and

$$0 < \liminf_{t\to 0} \frac{d_A(X_t)}{t^{(1-\lambda)/2}} \le \limsup_{t\to 0} \frac{d_A(X_t)}{t^{(1-\lambda)/2}} < +\infty.$$

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