*The Annals of Probability*
2003, Vol. 31, No. 3, 1377–1412


# ON TRANSITION SEMIGROUPS OF $(A, \Psi)$-SUPERPROCESSES WITH IMMIGRATION

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We study the global properties of transition semigroups $(p_t^{\nu,\Psi,A})$ of $(A, \Psi)$-superprocesses over compact type spaces with possibly nonzero immigration $\nu$ in various function spaces. In particular, we compare the different rates of convergence of $(p_t^{\nu,\Psi,A})$ to equilibrium. Our analysis is based on an explicit formula for the Gateaux derivative of $p_t^{\nu,\Psi,A}F$.

**0. Introduction.** Let us start with four observations concerning the transition semigroup of one of the most elementary superprocesses. More precisely, assume that the type space consists of only one type (so that the state space can be identified with $\mathbb{R}_+ := [0, \infty)$). Assume that the branching mechanism is given by $\Psi(\lambda) = \lambda^2 - \theta\lambda$, $\theta > 0$. Then the generator of the corresponding superprocess with nonzero immigration $q > 0$ is given by

$$L_{\theta,q} f(x) = x\ddot{f}(x) + (q - \theta x)\dot{f}(x), \qquad x \in \mathbb{R}_+ := [0, \infty),\ f \in C_b^2(\mathbb{R}_+).$$

It is easy to see that the Gamma measure

$$\Gamma_{\theta,q}(dx) = \frac{\theta^q}{\Gamma(q)} x^{q-1} e^{-\theta x}\, dx$$

is a symmetrizing measure and that $(L_{\theta,q}, C_b^2(\mathbb{R}_+))$ is essentially self-adjoint. Moreover, the unique self-adjoint extension has a discrete spectrum with eigenvalues $-\theta n$, $n \geq 0$ (independent of $q$), and corresponding eigenvectors

$$e_n^{\theta,q}(x) := \sum_{k=0}^{n} \frac{\Gamma(n+q)}{\Gamma(k+q)} \frac{\theta^k(-x)^k}{k!(n-k)!}, \qquad n \geq 0.$$

The $e_n^{\theta,q}$ are nothing but the classical Laguerre polynomials (cf. [2], Chapter 6, and [13], Chapter 5). A classical recurrence relation for the Laguerre polynomials states that $\frac{d}{dx} e_n^{\theta,q} = -\theta e_{n-1}^{\theta,q+1}$, $n \geq 1$. Hence, if we denote by $p_t^{\theta,q} := e^{tL_{\theta,q}}$, $t \geq 0$, the semigroup generated by the self-adjoint extension of $L_{\theta,q}$, this relation implies

Received February 2002; revised August 2002.
*AMS 2000 subject classifications.* Primary 60J35; secondary 47D07, 60F10, 60G57, 60J80, 60K35, 92D10.
*Key words and phrases.* Superprocesses, ergodic theorems, gradient estimates, Gamma processes, short-time asymptotics.

that

$$\frac{d}{dx} p_t^{\theta,q} e_n^{\theta,q} = \frac{d}{dx} e^{-\theta n t} e_n^{\theta,q} = -e^{-\theta n t} \big(\theta e_{n-1}^{\theta,q+1}\big)$$
$$= -e^{-\theta t} p_t^{\theta,q+1} \big(\theta e_{n-1}^{\theta,q+1}\big) = e^{-\theta t} p_t^{\theta,q+1} \bigg(\frac{d}{dx} e_n^{\theta,q}\bigg).$$

Hence,

$$\frac{d}{dx} p_t^{\theta,q} f = e^{-\theta t} p_t^{\theta,q+1} \bigg(\frac{d}{dx} f\bigg) \tag{0.1}$$

for all $f \in \operatorname{span}\{e_n^{\theta,q} : n \geq 0\}$. It is quite easy to see that (0.1) can be generalized even further to all $f \in C_{\mathrm{b}}^1(\mathbb{R}_+)$ and easily implies that

$$\|p_t^{\theta,q} f\|_{\mathrm{Lip}} \leq e^{-\theta t} \|f\|_{\mathrm{Lip}} \tag{0.2}$$

for all bounded Lipschitz-continuous $f$. But, even more, $p_t^{\theta,q}$, $t > 0$, maps bounded measurable functions into Lipschitz-continuous ones. More precisely, we have

$$\|p_t^{\theta,q} f\|_{\mathrm{Lip}} \leq \frac{\theta e^{-\theta t}}{1 - e^{-\theta t}} \|f\|_\infty \tag{0.3}$$

for all $f \in \mathscr{B}_{\mathrm{b}}(\mathbb{R}_+)$. Note that, again, the constant is independent of $q$, but only depends on $\theta$. For our last observation, define $\Gamma(f)(x) := x \dot{f}^2(x)$ for all $f \in C_{\mathrm{b}}^1(\mathbb{R}_+)$. Then

$$\Gamma\big(p_t^{\theta,q} f\big) \leq e^{-\theta t} p_t^{\theta,q}\big(\Gamma(f)\big), \qquad f \in C_{\mathrm{b}}^1(\mathbb{R}_+),\ q \geq \tfrac{1}{2}. \tag{0.4}$$

The main purpose of this paper now is to study generalizations of (0.1)–(0.4) to arbitrary superprocesses over compact type spaces. More precisely, the generalization of (0.1) is contained in Theorem 2.1 and Corollaries 2.3 and 2.4, the generalization of (0.2) can be found in Section 2.1; Section 2.2 contains generalizations of (0.4) and Section 3 contains generalizations of (0.3).

In Section 4, we study the particular case of random Gamma processes in more detail. For this particular class of process, it is possible to obtain explicit formulas for the transition semigroup (cf. Section 4.1 for a series representation (which generalizes the corresponding series representation obtained by Ethier and Griffiths [7], Theorem 1.1, in the particular case of a constant branching mechanism) and Section 4.2 for an integral representation). Since the process is reversible, the analysis of the process simplifies considerably [cf., e.g., the formula for the Gateaux derivative (2.7)] and some results can be strengthened (cf. Section 4.3). Moreover, using a general result on the small-time asymptotics of the heat kernel in symmetric Dirichlet spaces in [10] we study in Section 4.4 the short-time asymptotics of heat kernels of random Gamma processes.

In Section 1, we introduce our framework and provide some basic facts about superprocesses with immigration and corresponding invariant measures. In particular, as a generalization of a previous result obtained by Ethier and Griffiths for random Gamma processes with type-independent branching mechanism ([7], Corollary 1.2), we will derive in Theorem 1.7 explicit rates on the convergence of superprocesses with immigration toward equilibrium in total variation norm. These estimates are nonuniform, in contrast to the estimates obtained in the Lipschitz norm and the $L^2$-norms.

**1. Superprocesses with immigration.** Let us first introduce our framework, which will be kept throughout the whole paper. Let $S$ be a compact metric type space and $E := \mathcal{M}_+(S)$ the set of all finite positive Borel measures on $S$. Since $S$ is compact, it follows that $E$ is locally compact w.r.t. the weak topology. Let $A$ be a Feller generator on $C(S)$ and

$$\begin{aligned}\Psi(x, \lambda) :=& -a(x)\lambda^2 \\ &+ \int_0^\infty (1 - e^{-\lambda s} - \lambda s)\, n(x, ds) - b(x)\lambda, \qquad x \in S, \lambda \geq 0,\end{aligned} \tag{1.1}$$

where $a, b \in C(S)$, $a \geq 0$ and $n$ is a kernel of positive measures $n(x, \cdot)$ on $\mathbb{R}_+$ such that

$$\begin{gathered}\sup_{x \in S} \int s \wedge s^2 n(x, ds) < +\infty, \\ (x, \lambda) \mapsto \int_0^\infty (1 - e^{-\lambda s} - \lambda s)\, n(x, ds) \in C(S \times \mathbb{R}_+).\end{gathered} \tag{1.2}$$

For $f \in C(S)_+$ (the set of all strictly positive continuous functions on $S$), let $\psi_t : \mathbb{R}_+ \to C(S)_+$ be the unique mild solution to the semilinear equation

$$\frac{\partial \psi_t(f)}{\partial t} = A\psi_t(f) + \Psi\big(\cdot, \psi_t(f)\big), \qquad \psi_0(f) = f. \tag{1.3}$$

Given $\nu \in E$, we denote by $(p_t^{\nu,\Psi,A})$ the transition semigroup of the corresponding $(A, \Psi)$-superprocess $\mathbb{M}_\nu^{\Psi,A}$ with immigration $\nu$. Note that $(p_t^{\nu,\Psi,A})$ is uniquely determined by

$$\begin{aligned}&p_t^{\nu,\Psi,A} \varphi_f(\mu) \\ &\quad = \exp\biggl(-\int_0^t \langle \psi_s(f), \nu\rangle\, ds\biggr) \varphi_{\psi_t(f)}(\mu), \qquad f \in C(S)_+, \mu \in E, t \geq 0.\end{aligned}$$

Here we used the notation $\varphi_f(\mu) = \exp(-\langle f, \mu\rangle)$ and $\langle f, \mu\rangle := \langle \mu, f\rangle := \int f\, d\mu$. It is well known that $(p_t^{\nu,\Psi,A})$ induces a $C_0$-semigroup on $C_\infty(E)$ (the space of all continuous functions vanishing at $\infty$). Moreover, $p_t^{\nu,\Psi,A}(C_\mathrm{b}(E)) \subset C_\mathrm{b}(E)$ for all $t \geq 0$. Its infinitesimal generator $L_\nu^{\Psi,A}$ can be obtained as the closure of

$$
\begin{aligned}
L_\nu^{\Psi,A} F(\mu) := & \int \mu(dx) a(x) \frac{\partial^2 F}{\partial \delta_x \partial \delta_x}(\mu) \\
& + \int \mu(dx) \int_0^\infty n(x, ds) \Big( F(\mu + s\delta_x) - F(\mu) - s \frac{\partial F}{\partial \delta_x}(\mu) \Big) \\
& + \int \big(\nu(dx) - \mu(dx) b(x)\big) \frac{\partial F}{\partial \delta_x}(\mu) + \int \mu(dx) \Big( A \frac{\partial F}{\partial \delta_\cdot}(\mu) \Big)(x),
\end{aligned}
\tag{1.4}
$$

where

$$
F \in \mathcal{F}C_0^2(D(A)_+) := \big\{ F(\mu) = \varphi(\langle f_1, \mu\rangle, \ldots, \langle f_n, \mu\rangle) : \\
n \geq 1,\ f_i \in D(A) \cap C(S)_+, \varphi \in C_0^2(\mathbb{R}^n) \big\}.
$$

Here

$$
\frac{\partial F}{\partial \delta_x}(\mu) := \frac{dF}{ds}(\mu + s\delta_x)\Big|_{s=0}
$$

denotes the Gateaux derivative of $F$ at $\mu$ in direction $\delta_x$. For general $\overline{\mu} \in E$, let $(\partial F / \partial \overline{\mu})(\mu)$ be the Gateaux derivative of $F$ at $\mu$ in direction $\overline{\mu}$.

If $b \in C(S)_+$, it follows that $f \mapsto \int_0^\infty \langle \psi_t(f), \nu\rangle\, dt$ is the log-Laplace functional of a probability measure $m_\nu^{\Psi,A}$ that is invariant for $\mathbb{M}_\nu^{\Psi,A}$ and hence for $L_\nu^{\Psi,A}$. Note that in the particular case $A = n = 0$, hence $\Psi(x, \lambda) = -a(x)\lambda^2 - b(x)\lambda$, thus

$$
\psi_t(f)(x) = \frac{e^{-b(x)t} f(x)}{1 + a(x)/b(x)(1 - e^{-b(x)t}) f(x)},
$$

it follows that $\int_0^\infty \langle \psi_t(f), \nu\rangle\, dt = \int \log(1 + (a/b) f) a^{-1}\, d\nu$ is the log-Laplace functional of a random Gamma measure. For this reason we will call the associated superprocess $\mathbb{M}_\nu^{\Psi,0}$ a random Gamma process (cf. Section 4).

Using the method of a priori estimates, we will prove the following result on the existence of exponential moments, which will be needed later. We will use the following notation $J_\varepsilon(x) := \varepsilon^{-1} \int n(x, ds)(e^{\varepsilon s} - 1 - \varepsilon s)$. Note that $\lim_{\varepsilon \downarrow 0} \|J_\varepsilon\|_\infty = 0$ by Dini's theorem, since $J_\varepsilon(x) \downarrow 0$ as $\varepsilon \downarrow 0$ and $J_\varepsilon$ is continuous.

PROPOSITION 1.1. *Assume that* $b_0 := \inf_{x \in S} b(x) > 0$. *Then*

$$
\int e^{\varepsilon |\mu|} m_\nu^{\Psi,A}(d\mu) < +\infty
$$

*for all* $\varepsilon < \|a\|_\infty^{-1}(b_0 - \|J_\varepsilon\|_\infty)$.

PROOF. Let $\varepsilon$ such that $\delta := (b_0 - \|J_\varepsilon\|_\infty) - \|a\|_\infty \varepsilon > 0$ and $V(\mu) := e^{\varepsilon|\mu|}$. Formally applying the generator $L_\nu^{\Psi,A}$ to $V$ yields

$$\begin{aligned} L_\nu^{\Psi,A} V(\mu) &= \varepsilon^2 \langle a, \mu\rangle V(\mu) + \varepsilon\langle J_\varepsilon, \mu\rangle V(\mu) + \varepsilon(|\nu| - \langle b, \mu\rangle)V(\mu) \\ &\le \varepsilon\bigl(|\nu| + (\varepsilon\|a\|_\infty + \|J_\varepsilon\|_\infty - b_0)|\mu|\bigr)V(\mu) \\ &\le \varepsilon(|\nu| - \delta|\mu|)V(\mu). \end{aligned}$$

Let $\varphi_n \in C_b^\infty(\mathbb{R}_+)$ satisfying $\varphi_n(t) = t$ if $t \le n$, $\varphi_n(t) = n+1$ if $t \ge n+2$, $0 \le \dot\varphi_n \le 1$ and $\ddot\varphi_n \le 0$. By approximation, it is then easy to see that $V_n(\mu) := \varphi_n(V(\mu))$ is in the domain of the closure of $(L_\nu^{\Psi,A}, \mathcal{F}C_0^2(D(A)_+))$ in $L^2(m_\nu^{\Psi,A})$ and

$$\begin{aligned} L_\nu^{\Psi,A} V_n(\mu) &= \dot\varphi_n\bigl(V(\mu)\bigr) L_\nu^{\Psi,A} V(\mu) + \ddot\varphi_n\bigl(V(\mu)\bigr)\Gamma(V,V)(\mu) \\ &\le \dot\varphi_n\bigl(V(\mu)\bigr)\varepsilon(|\nu| - \delta|\mu|)V(\mu). \end{aligned}$$

Here

$$\begin{aligned} \Gamma(F,G)(\mu) &= \left\langle \mu, a\frac{\partial F}{\partial \delta_\cdot}(\mu)\frac{\partial G}{\partial \delta_\cdot}(\mu)\right\rangle \\ &\quad + \left\langle \mu, \int_0^\infty n(\cdot, ds)\bigl(F(\mu + s\delta_\cdot) - F(\mu)\bigr)\bigl(G(\mu+s\delta_\cdot) - G(\mu)\bigr)\right\rangle. \end{aligned}$$

In particular, $\Gamma(F,F) \ge 0$. The invariance of $m_\nu^{\Psi,A}$ now implies that

$$\begin{aligned} &\delta \int \dot\varphi_n\bigl(V(\mu)\bigr)|\mu| V(\mu) m_\nu^{\Psi,A}(d\mu) \\ &\quad \le |\nu| \int \dot\varphi_n\bigl(V(\mu)\bigr) V(\mu) m_\nu^{\Psi,A}(d\mu) \\ &\quad \le |\nu| \int_{\{|\mu| \le 2|\nu|/\delta\}} V(\mu) m_\nu^{\Psi,A}(d\mu) \\ &\qquad + \frac{\delta}{2} \int_{\{|\mu| > 2|\nu|/\delta\}} \dot\varphi_n\bigl(V(\mu)\bigr)|\mu| V(\mu) m_\nu^{\Psi,A}(d\mu) \end{aligned}$$

or, equivalently,

$$\int \dot\varphi_n\bigl(V(\mu)\bigr)|\mu| V(\mu) m_\nu^{\Psi,A}(d\mu) \le \frac{2}{\delta}|\nu| \exp\left(\varepsilon\frac{2}{\delta}|\nu|\right).$$

Taking the limit $n \to \infty$, we obtain

$$\int |\mu| \exp(\varepsilon|\mu|) m_\nu^{\Psi,A}(d\mu) \le \frac{2}{\delta}|\nu| \exp\left(\varepsilon\frac{2}{\delta}|\nu|\right),$$

and thus the assertion. $\square$

*Convergence to equilibrium in total variation norm.* Recall that for any finite signed measure $\mu$ on $S$ its total variation norm is defined by

$$\|\mu\|_{\mathrm{var}} = \sup\{\langle f, \mu\rangle \mid f \text{ measurable and } \|f\|_\infty \le 1\}.$$

In particular, if $|S| = d$ contains only $d$ points, the norm coincides with one-half of the usual $\ell^1$-norm on $\mathbb{R}^d$ (identifying the space of finite signed measures with $\mathbb{R}^d$).

In [7], Corollary 1.2, Ethier and Griffiths have proved in the particular case of zero mutation and type-independent branching mechanism $\Psi(x, \lambda) = -\frac{1}{2}\lambda^2 - b\lambda$ the following estimate:

$$\begin{aligned}
&\|p_t^{\nu,\Psi,0}(\mu,\cdot) - m_\nu^{\Psi,0}\|_{\mathrm{var}} \\
&\quad \le \left(1 - \exp\left(-2b\frac{e^{-bt}}{1-e^{-bt}}|\mu|\right)\right) + \left(1 - (1 - e^{-bt})^{2|\nu|}\right)
\end{aligned}$$

on the rate of convergence of $(p_t^{\nu,\Psi,0})$ in total variation norm. The purpose of this section is to generalize their result to arbitrary nonzero mutation and possibly type-dependent branching mechanism.

To this end, fix $\nu \in E$ and let $m_{t,\nu}^{\Psi,A}$ and $n_{t,\nu}^{\Psi,A}$ be the probability measures with Laplace transform

$$\int \exp(-\langle f, \mu\rangle) m_{t,\nu}^{\Psi,A}(d\mu) = \exp\left(-\int_0^t \langle \psi_s(f), \nu\rangle\, ds\right)$$

and

$$\int \exp(-\langle f, \mu\rangle) n_{t,\nu}^{\Psi,A}(d\mu) = \exp\left(-\int_t^\infty \langle \psi_s(f), \nu\rangle\, ds\right).$$

Then

$$p_t^{\nu,\Psi,A}(\mu,\cdot) = m_{t,\nu}^{\Psi,A} * p_t^{0,\Psi,A}(\mu,\cdot) \quad \text{and} \quad m_\nu^{\Psi,A} = m_{t,\nu}^{\Psi,A} * n_{t,\nu}^{\Psi,A}.$$

PROPOSITION 1.2.

$$\begin{aligned}
\|p_t^{\nu,\Psi,A}(\mu,\cdot) - m_\nu^{\Psi,A}\|_{\mathrm{var}} &\le \|p_t^{0,\Psi,A}(\mu,\cdot) - n_{t,\nu}^{\Psi,A}\|_{\mathrm{var}} \\
&\le p_t^{0,\Psi,A}(\mu, E\setminus\{0\}) + n_{t,\nu}^{\Psi,A}(E\setminus\{0\}).
\end{aligned}$$

PROOF. For the proof of the first inequality, observe that

$$\begin{aligned}
&\int F\, dp_t^{\nu,\Psi,A}(\mu,\cdot) - \int F\, dm_\nu^{\Psi,A} \\
&\qquad = \int m_{t,\nu}^{\Psi,A}(d\mu_1)\left(\int p_t^{0,\Psi,A}(\mu, d\mu_2) F(\mu_1+\mu_2)\right. \\
&\qquad\qquad\qquad \left. - \int n_{t,\nu}^{\Psi,A}(d\mu_2) F(\mu_1+\mu_2)\right) \\
&\qquad \le \int m_{t,\nu}^{\Psi,A}(d\mu_1) \|p_t^{0,\Psi,A}(\mu,\cdot) - n_{t,\nu}^{\Psi,A}\|_{\mathrm{var}} \|F\|_\infty.
\end{aligned}$$

The second inequality follows from the fact that, for any probability measure $m$,

$$\|m - \delta_0\|_{\mathrm{var}} = m(E \setminus \{0\}). \qquad \square$$

In the next step, we will estimate the right-hand side in the proposition using the simple observation contained in the following lemma.

LEMMA 1.3. *Let $m$ be a finite nonnegative measure on $E$ with log-Laplace functional $L$. Let $f \in C(S)_+$. Then*

$$m\left(\left\{\mu \in E : \int f \, d\mu = 0\right\}\right) = \lim_{\lambda \to \infty} \exp\bigl(-L(\lambda f)\bigr).$$

*In particular*,

$$m(\{0\}) = \lim_{\lambda \to \infty} \exp\bigl(-L(\lambda \mathbb{1}_S)\bigr).$$

PROOF. Clearly,

$$\lim_{\lambda \to \infty} \exp(-\lambda \langle f, \mu \rangle) = \mathbb{1}_{\{\tilde{\mu} \in E : \int f \, d\tilde{\mu} = 0\}}(\mu)$$

for all $\mu \in E$. Consequently, by Lebesgue's theorem,

$$\begin{aligned} &\lim_{\lambda \to \infty} \exp\bigl(-L(\lambda f)\bigr) \\ &\quad = \lim_{\lambda \to \infty} \int \exp(-\lambda \langle f, \mu \rangle) m(d\mu) = m\left(\left\{\mu \in E : \int f \, d\mu = 0\right\}\right). \qquad \square \end{aligned}$$

Combining the last two results, we obtain

$$\begin{aligned} \| p_t^{\nu, \Psi, A}(\mu, \cdot) - m_\nu^{\Psi, A} \|_{\mathrm{var}} \le \lim_{\lambda \to \infty} \bigl(1 - \exp\bigl(-\langle \psi_t(\lambda \mathbb{1}_S), \mu \rangle\bigr)\bigr) \\ + \left(1 - \exp\left(-\int_t^\infty \langle \psi_s(\lambda \mathbb{1}_S), \nu \rangle \, ds\right)\right). \end{aligned} \tag{1.5}$$

REMARK 1.4. In the case of a type-independent branching mechanism and $a$, $b > 0$, the two quantities on the right-hand side of (1.5) can be easily estimated from above. Indeed, since $\int_0^\infty e^{-\lambda s} - 1 + \lambda s n(ds) \ge 0$, it follows that $\Psi(\lambda) \le -a\lambda^2 - b\lambda$ and, consequently,

$$\psi_t(\lambda) \le \frac{e^{-bt}\lambda}{1 + (a/b)(1 - e^{-bt})\lambda}.$$

Thus,

$$p_t^{0, \Psi, A}(\mu, \{0\}) = \lim_{\lambda \to \infty} \exp\bigl(-\langle \psi_t(\lambda \mathbb{1}_S), \mu \rangle\bigr) \ge \exp\left(-\frac{b}{a} \frac{e^{-bt}}{1 - e^{-bt}} |\mu|\right).$$

Similarly,

$$n_{t,\nu}^{\Psi,A}(\{0\}) \geq (1 - e^{-bt})^{|\nu|/a},$$

and therefore

$$\begin{aligned} &\| p_t^{\nu,\Psi,A}(\mu, \cdot) - m_\nu^{\Psi,A} \|_{\mathrm{var}} \\ &\quad \leq \biggl(1 - \exp\biggl(-\frac{b}{a}\frac{e^{-bt}}{1 - e^{-bt}}|\mu|\biggr)\biggr) + \bigl(1 - (1 - e^{-bt})^{|\nu|/a}\bigr). \end{aligned} \tag{1.6}$$

In the particular case $a = \frac{1}{2}$, $n = 0$ and $A = 0$, we recover the result of Ethier and Griffiths.

We will show next that the same estimate (1.6) is true in the nonconstant case if we replace $a$ and $b$ by

$$a_0 := \inf_{x \in S} a(x) \quad \text{and} \quad b_0 := \inf_{x \in S} b(x), \tag{1.7}$$

respectively. To this end, we need the following lemma containing an upper bound of $\psi_t(f)$. Since we could not find a reference in the literature, a proof is included here for the reader's convenience. Note that we cannot expect that $\psi_t(f)$ is a strong solution, so that the standard comparison principle does not work.

LEMMA 1.5. *Assume that* $b_0 > 0$. *Let* $f \in C(S)_+$ *and let* $\psi_t(f)$, $t \geq 0$, *be the unique mild solution of* (1.3). *Then*

$$\psi_t(f) \leq \frac{\exp(-b_0 t) p_t f}{1 + (a_0/b_0)(1 - \exp(-b_0 t)) p_t f}, \qquad t \geq 0.$$

*Here* $a_0$ *and* $b_0$ *are defined by* (1.7) *and* $(p_t)$ *is the semigroup generated by* $A$.

PROOF.

*Step* 1.

$$p_t f \geq \exp(b_0 t)\psi_t(f) + a_0 \int_0^t \exp(b_0 s) p_{t-s}\psi_s(f)^2\, ds, \qquad t \geq 0.$$

*Proof of Step* 1. Clearly, it suffices to prove by induction that, for all $n \geq 0$,

$$\begin{aligned} p_t f \geq{}& \sum_{k=0}^n \frac{(b_0 t)^k}{k!}\psi_t(f) + a_0 \sum_{k=0}^n \int_0^t \frac{(b_0 s)^k}{k!} p_{t-s}\psi_s(f)^2\, ds \\ &+ \frac{b_0^{n+1}}{n!}\int_0^t s^n p_{t-s}\psi_s(f)\, ds. \end{aligned} \tag{1.8}$$

If $n=0$, the assertion follows from the fact that $\Psi(x,\lambda)\le -a_0\lambda^2-b_0\lambda$ and thus

$$\begin{aligned}p_t f &= \psi_t(f)-\int_0^t p_{t-s}\Psi\big(\cdot,\psi_s(f)\big)\,ds\\ &\ge \psi_t(f)+a_0\int_0^t p_{t-s}\psi_s(f)^2\,ds+b_0\int_0^t p_{t-s}\psi_s(f)\,ds, \qquad t\ge 0.\end{aligned}$$

Suppose now that (1.8) is proved for $n$. Then, using (1.3) again, we obtain

$$\begin{aligned}&\int_0^t s^n p_{t-s}\psi_s(f)\,ds\\ &\quad\ge \int_0^t s^n\psi_{t-s}\big(\psi_s(f)\big)\,ds+a_0\int_0^t s^n\int_0^{t-s} p_{t-s-r}\psi_r\big(\psi_s(f)\big)^2\,dr\,ds\\ &\qquad+b_0\int_0^t s^n\int_0^{t-s} p_{t-s-r}\psi_r\big(\psi_s(f)\big)\,dr\,ds\\ &\quad=\int_0^t s^n\psi_t(f)\,ds+\frac{a_0}{n+1}\int_0^t s^{n+1}p_{t-s}\psi_s(f)^2\,ds\\ &\qquad+\frac{b_0}{n+1}\int_0^t s^{n+1}p_{t-s}\psi_s(f)\,ds\end{aligned}$$

and, consequently,

$$\begin{aligned}p_t f &\ge \sum_{k=0}^{n+1}\frac{(b_0t)^k}{k!}\psi_t(f)+a_0\sum_{k=0}^{n+1}\int_0^t\frac{(b_0s)^k}{k!}p_{t-s}\psi_s(f)^2\,ds\\ &\quad+\frac{b_0^{n+2}}{(n+1)!}\int_0^t s^{n+1}p_{t-s}\psi_s(f)\,ds.\end{aligned}$$

*Step* 2.

$$\begin{aligned}p_t f &\ge \sum_{k=0}^{\infty}\left(\frac{a_0}{b_0}\right)^k\exp(b_0t)\big(\exp(b_0t)-1\big)^k\psi_t(f)^{k+1}\\ &=\frac{\exp(b_0t)\psi_t(f)}{1-(a_0/b_0)(\exp(b_0t)-1)\psi_t(f)}.\end{aligned}$$

*Proof of Step* 2. This time, it suffices to show that

$$p_t f \ge \sum_{k=0}^{n}\left(\frac{a_0}{b_0}\right)^k\exp(b_0t)\big(\exp(b_0t)-1\big)^k\psi_t(f)^{k+1} \tag{1.9}$$

for all $n\in\mathbb{N}_0$. If $n=0$, this follows from Step 1 since $p_t f\ge e^{b_0t}\psi_t(f)$. Suppose now that (1.9) is proved for $n$ and for all $f\in C(S)_+$ and $t\ge 0$. Step 1 and Jensen's inequality then imply that

$$p_t f \geq \exp(b_0 t)\psi_t(f) + a_0 \int_0^t \exp(b_0 s) p_{t-s}\psi_s(f)^2\, ds$$
$$\geq \exp(b_0 t)\psi_t(f) + a_0 \int_0^t \exp(b_0 s)\big(p_{t-s}\psi_s(f)\big)^2\, ds.$$

Using the assumption, the second term on the right-hand side can be estimated from below by

$$\begin{aligned} a_0 \int_0^t \exp(b_0 s) \Bigg( \sum_{k=0}^n \left(\frac{a_0}{b_0}\right)^k \exp(b_0(t-s)) \\ \times \big(\exp(b_0(t-s)) - 1\big)^k \psi_{t-s}\big(\psi_s(f)\big)^{k+1} \Bigg)^2 ds \\ = a_0 \sum_{k,l=0}^n \left(\frac{a_0}{b_0}\right)^{k+l} \exp(b_0 t) \\ \times \int_0^t \exp(b_0(t-s))\big(\exp(b_0(t-s)) - 1\big)^{k+l}\, ds\, \psi_t(f)^{k+l+2} \\ \geq \sum_{k=0}^n \left(\frac{a_0}{b_0}\right)^{k+1} \exp(b_0 t)\big(\exp(b_0 t) - 1\big)^{k+1} \psi_t(f)^{k+2}. \end{aligned}$$

Combining the last two inequalities now implies

$$p_t f \geq \sum_{k=0}^{n+1} \left(\frac{a_0}{b_0}\right)^k \exp(b_0 t)\big(\exp(b_0 t) - 1\big)^k \psi_t(f)^{k+1}.$$

Clearly, Step 2 is equivalent to the assertion; hence, the lemma is proved. □

REMARK 1.6. In the same manner as above, one can show that if $b_0 = 0$, then

$$\psi_t(f) \leq \frac{p_t f}{1 + a_0 t p_t f}, \qquad t \geq 0.$$

To this end, note that Step 1 is trivial and in Step 2 the assertion has to be replaced by

$$p_t f \geq \sum_{k=0}^{\infty} a_0^k t^k \psi_t(f)^{k+1},$$

which can be shown exactly in the same way as above.

THEOREM 1.7. *Assume that $a_0 > 0$ and $b_0 > 0$. Then, for each $t > 0$ and $\mu \in E$,*

$$\begin{aligned}
&\| p_t^{\nu,\Psi,A}(\mu, \cdot) - m_\nu^{\Psi,A} \|_{\mathrm{var}} \\
&\quad \le \left(1 - \exp\left(-\frac{b_0}{a_0}\frac{\exp(-b_0 t)}{1-\exp(-b_0 t)}|\mu|\right)\right) \\
&\qquad + \left(1 - \left(1 - \exp(-b_0 t)\right)^{|\nu|/a_0}\right).
\end{aligned}$$

PROOF. Lemma 1.5 implies that

$$\begin{aligned}
p_t^{0,\Psi,A}(\mu, \{0\}) &= \lim_{\lambda\to\infty} \exp(-\langle \psi_t(\lambda \mathbb{1}_S), \mu\rangle) \\
&\ge \exp\left(-\frac{b_0}{a_0}\frac{\exp(-b_0 t)}{1-\exp(-b_0 t)}|\mu|\right).
\end{aligned}$$

Similarly,

$$n_{t,\nu}^{\Psi,A}(\{0\}) \ge \left(1 - \exp(-b_0 t)\right)^{|\nu|/a_0},$$

and therefore

$$\begin{aligned}
&\| p_t^{\nu,\Psi,A}(\mu, \cdot) - m_\nu^{\Psi,A} \|_{\mathrm{var}} \\
&\quad \le \left(1 - \exp\left(-\frac{b_0}{a_0}\frac{\exp(-b_0 t)}{1-\exp(-b_0 t)}|\mu|\right)\right) + \left(1 - \left(1 - \exp(-b_0 t)\right)^{|\nu|/a_0}\right).
\end{aligned}$$

□

Note that all the estimates obtained above are nonuniform in $\mu$. Uniform estimates on the exponential rate of convergence will be obtained below in spaces of Lipschitz-continuous functions and $L^2$-spaces.

## 2. The derivative of the transition semigroup and applications.

Assume from now on that $a_0 > 0$ and $b_0 > 0$. Since, for any $\mu \in E$, $f \mapsto \langle \mu, \psi_t(f)\rangle$ is the log-Laplace functional of an infinitely divisible probability measure, it follows from the canonical representation theorem that there exist uniquely determined $\nu_{t,\mu} \in E$ and $m_{t,\mu} \in \mathcal{M}_1(E)$ with $m_{t,\mu}(\{0\}) = 0$ such that

$$\langle \mu, \psi_t(f)\rangle = \langle \nu_{t,\mu}, f\rangle + \int \left(1 - \exp(-\langle f, \overline{\mu}\rangle)\right) m_{t,\mu}(d\overline{\mu})$$

(cf. [5], Theorem 3.3.1, or [6], Theorem 1.28). We call $m_{t,\mu}$ the canonical measure. Lemma 1.5 now implies that, for $a_0 > 0$ and $b_0 > 0$,

$$\lim_{\lambda\to\infty} \frac{1}{\lambda}\psi_t(\lambda \mathbb{1}_S) \le \lim_{\lambda\to\infty} \frac{1}{\lambda} \frac{\exp(-b_0 t)\lambda}{1 + (a_0/b_0)(1-\exp(-b_0 t))\lambda} = 0,$$

and therefore $\nu_{t,\mu} = 0$.

THEOREM 2.1. *Let $F \in C_b(E)$, $\mu, \overline{\mu} \in E$ and $t > 0$. Then $p_t^{\nu,\Psi,A}F$ is Gateaux differentiable at $\mu$ in direction $\overline{\mu}$ and*

$$\frac{\partial p_t^{\nu,\Psi,A}F}{\partial \overline{\mu}}(\mu) = \int m_{t,\overline{\mu}}(d\mu_1) \int p_t^{\nu,\Psi,A}(\mu, d\mu_2)\big(F(\mu_1+\mu_2) - F(\mu_2)\big).$$

*Here $m_{t,\overline{\mu}}$ is the canonical measure corresponding to $f \mapsto \langle \overline{\mu}, \psi_t(f)\rangle$. In particular*,

$$\text{(i)} \qquad \mu \mapsto \frac{\partial p_t^{\nu,\Psi,A}F}{\partial \overline{\mu}}(\mu) \in C_b(E),$$

$$\text{(ii)} \qquad \frac{\partial p_t^{\nu,\Psi,A}F}{\partial \overline{\mu}}(\mu) = \int \overline{\mu}(dx) \frac{\partial p_t^{\nu,\Psi,A}F}{\partial \delta_x}(\mu).$$

REMARK. A similar representation for the Gateaux derivative has been obtained independently by Jacka and Tribe in the particular case of binary branching with constant rate (cf. [9]).

PROOF OF THEOREM 2.1. Clearly, for all $F \in \mathcal{B}_b(E)$ and $h > 0$,

$$p_t^{\nu,\Psi,A}F(\mu + h\overline{\mu}) = \int p_t^{\nu,\Psi,A}(\mu, d\mu_1) \int p_t^{0,\Psi,A}(h\overline{\mu}, d\mu_2) F(\mu_1 + \mu_2),$$

since

$$\begin{aligned} p_t^{\nu,\Psi,A}\varphi_f(\mu + h\overline{\mu}) &= \exp\left(-\int_0^t \langle \psi_s(f), \nu\rangle\, ds\right) \exp\big(-\langle \psi_t(f), \mu + h\overline{\mu}\rangle\big) \\ &= p_t^{\nu,\Psi,A}\varphi_f(\mu) p_t^{0,\Psi,A}\varphi_f(h\overline{\mu}). \end{aligned}$$

It follows that

$$\begin{aligned} &\frac{1}{h}\big(p_t^{\nu,\Psi,A}F(\mu + h\overline{\mu}) - p_t^{\nu,\Psi,A}F(\mu)\big) \\ &\qquad = \frac{1}{h}\int p_t^{\nu,\Psi,A}(\mu, d\mu_1) \int p_t^{0,\Psi,A}(h\overline{\mu}, d\mu_2)\big(F(\mu_1+\mu_2) - F(\mu_1)\big). \end{aligned} \tag{2.1}$$

We will show next that, for all $F \in C_b(E)$ with $F(0) = 0$,

$$\lim_{h \downarrow 0} \frac{1}{h} \int p_t^{0,\Psi,A}(h\overline{\mu}, d\mu) F(\mu) = \int F\, dm_{t,\overline{\mu}}. \tag{2.2}$$

To this end, define for $h > 0$ the measure

$$\Pi_h = \frac{1}{h}\big(p_t^{0,\Psi,A}(h\overline{\mu}, \cdot) - p_t^{0,\Psi,A}(h\overline{\mu}, \{0\})\delta_0\big)$$

and note that

$$\begin{aligned}
&\lim_{h\downarrow 0}\int \exp(-\langle f, \mu\rangle)\Pi_h(d\mu)\\
&\quad= \lim_{h\downarrow 0}\frac{1}{h}\Big(\exp\big(-h\langle\overline{\mu}, \psi_t(f)\rangle\big) - p_t^{0,\Psi,A}(h\overline{\mu}, \{0\})\Big) \qquad (2.3)\\
&\quad= \lim_{h\downarrow 0}\frac{1}{h}\big(\exp\big(-h\langle\overline{\mu}, \psi_t(f)\rangle\big) - 1\big) + \lim_{h\downarrow 0}\frac{1}{h}p_t^{0,\Psi,A}(h\overline{\mu}, E\setminus\{0\}).
\end{aligned}$$

We claim that

$$\lim_{h\downarrow 0}\frac{1}{h}p_t^{0,\Psi,A}(h\overline{\mu}, E\setminus\{0\}) = m_{t,\overline{\mu}}(E\setminus\{0\}) \qquad \big(= m_{t,\overline{\mu}}(E)\big). \tag{2.4}$$

Indeed, note that

$$\begin{aligned}
\frac{1}{h}p_t^{0,\Psi,A}(h\overline{\mu}, E\setminus\{0\}) &= \lim_{\lambda\to\infty}\frac{1}{h}\int 1 - \exp(-\lambda|\mu|)p_t^{0,\Psi,A}(h\overline{\mu}, d\mu)\\
&= \lim_{\lambda\to\infty}\frac{1}{h}\big(1 - \exp\big(-h\langle\overline{\mu}, \psi_t(\lambda\mathbb{1}_S)\rangle\big)\big)\\
&= \lim_{\lambda\to\infty}\int_0^{\langle\overline{\mu}, \psi_t(\lambda\mathbb{1}_S)\rangle}\exp(-hr)\,dr.
\end{aligned}$$

Since $\int_0^{\langle\overline{\mu}, \psi_t(\lambda\mathbb{1}_S)\rangle} e^{-hr}\,dr$ is increasing for increasing $\lambda$ and decreasing $h$, it follows that

$$\begin{aligned}
\lim_{h\downarrow 0}\frac{1}{h}p_t^{0,\Psi,A}(h\overline{\mu}, E\setminus\{0\}) &= \lim_{h\downarrow 0}\lim_{\lambda\to\infty}\int_0^{\langle\overline{\mu}, \psi_t(\lambda\mathbb{1}_S)\rangle} e^{-hr}\,dr\\
&= \lim_{\lambda\to\infty}\lim_{h\downarrow 0}\int_0^{\langle\overline{\mu}, \psi_t(\lambda\mathbb{1}_S)\rangle} e^{-hr}\,dr\\
&= \lim_{\lambda\to\infty}\langle\overline{\mu}, \psi_t(\lambda\mathbb{1}_S)\rangle \qquad (2.5)\\
&= \lim_{\lambda\to\infty}\int 1 - e^{-\lambda|\mu|}m_{t,\overline{\mu}}(d\mu)\\
&= m_{t,\overline{\mu}}(E\setminus\{0\}).
\end{aligned}$$

Hence, (2.4) is proved. Inserting (2.4) into (2.3), we obtain

$$\begin{aligned}
\lim_{h\downarrow 0}\int e^{-\langle f,\mu\rangle}\Pi_h(d\mu) &= -\langle\overline{\mu}, \psi_t(f)\rangle + m_{t,\overline{\mu}}(E\setminus\{0\})\\
&= \int e^{-\langle f,\mu\rangle}m_{t,\overline{\mu}}(d\mu).
\end{aligned}$$

Dawson [5], Theorem 3.2.6, now implies that $\lim_{h\downarrow 0}\Pi_h = m_{t,\overline{\mu}}$ weakly. In particular, for $F \in C_{\mathrm{b}}(E)$ with $F(0) = 0$,

$$\lim_{h\downarrow 0}\frac{1}{h}\int p_t^{0,\Psi,A}(h\overline{\mu}, d\mu)F(\mu) = \lim_{h\downarrow 0}\int F\,d\Pi_h = \int F\,dm_{t,\overline{\mu}},$$

hence, (2.2) is proved.

Fix $F \in C_b(E)$. Inserting (2.2) into (2.1), we obtain

$$\lim_{h\downarrow 0} \frac{1}{h} \int p_t^{0,\Psi,A}(h\overline{\mu}, d\mu_2) F(\mu_1+\mu_2) - F(\mu_1)$$
$$= \int m_{t,\overline{\mu}}(d\mu_2) F(\mu_1+\mu_2) - F(\mu_1)$$

for all $\mu_1 \in E$. Since, by (2.5),

$$\left| \frac{1}{h} \int p_t^{0,\Psi,A}(h\overline{\mu}, d\mu_2) F(\mu_1+\mu_2) - F(\mu_1) \right|$$
$$\le \frac{2}{h} p_t^{0,\Psi,A}(h\overline{\mu}, E \setminus \{0\}) \|F\|_\infty$$

the dominated convergence theorem now implies that

$$\lim_{h\downarrow 0} \frac{1}{h} \big( p_t^{\nu,\Psi,A} F(\mu + h\overline{\mu}) - p_t^{\nu,\Psi,A} F(\mu) \big)$$
$$= \int p_t^{\nu,\Psi,A}(\mu, d\mu_1) \int m_{t,\overline{\mu}}(d\mu_2) \big( F(\mu_1+\mu_2) - F(\mu_1) \big).$$

For the proof of (i), it suffices now to note that

$$\mu \mapsto \int m_{t,\overline{\mu}}(d\mu_1) \int p_t^{\nu,\Psi,A}(\mu, d\mu_2) \big( F(\mu_1+\mu_2) - F(\mu_2) \big)$$

is clearly continuous on $E$ and bounded. Part (ii) follows from the fact that

$$\int 1 - e^{-\langle f,\mu\rangle} m_{t,\overline{\mu}}(d\mu) = \langle \overline{\mu}, \psi_t(f) \rangle = \int \overline{\mu}(dx) \int 1 - e^{-\langle f,\mu\rangle} m_{t,\delta_x}(d\mu)$$

and therefore $m_{t,\overline{\mu}} = \int \overline{\mu}(dx) m_{t,\delta_x}$ by uniqueness of the canonical measure. □

For the next corollary, let us assume that $A = 0$. Then $\psi_t(f)(x) = \psi_t(x, f(x))$, where $\psi_t(x,\lambda)$, $t \ge 0$, is the solution of the ordinary differential equation $\dot{\psi}_t(x,\lambda) = \Psi(x, \psi_t(x,\lambda))$, $t \ge 0$, $\psi_0(x,\lambda) = \lambda$. It follows, in particular, that the support of the canonical measure $m_{t,\delta_x}$ corresponding to $f \mapsto \psi_t(f)(x)$ is contained in $\{h\delta_x \mid h \in \mathbb{R}_+\}$ and thus $m_{t,\delta_x}$ can be identified with the canonical measure $m_{t,x}$ corresponding to the Bernstein function $\lambda \mapsto \psi_t(x,\lambda)$, $\lambda > 0$.

We will need the following notion of differentiability.

DEFINITION 2.2. A function $F \in C_b(E)$ belongs to the class $C_b^1(E)$ if, for any differentiable curve $\gamma : [0,1] \to E$, $t \mapsto F(\gamma(t))$ is differentiable with derivative

$$\frac{dF \circ \gamma}{dt}(t) = \big\langle \dot{\gamma}(t), DF\big(\cdot, \gamma(t)\big) \big\rangle$$

with a bounded measurable function $DF : S \times E \to \mathbb{R}$.

Clearly, $F \in C_{\mathrm{b}}^1(E)$ implies that $F$ is Gateaux differentiable in direction $\delta_x$ with $(\partial F/\partial\delta_x)(\mu) = DF(x,\mu)$. Note that our definition differs from the definition given in [11] to the extent that we consider the weak topology instead of the strong topology on $E$. However, we will need the stronger notion in Section 4.4.

COROLLARY 2.3. *Assume in addition to the last theorem that* $A = 0$ *and* $F \in C_{\mathrm{b}}^1(E)$. *Let* $m_{t,x}$ *be the canonical measure corresponding to the Bernstein function* $\lambda \mapsto \psi_t(x,\lambda)$. *Then*

$$\frac{\partial p_t^{\nu,\Psi,0} F}{\partial \delta_x}(\mu) = \int_0^\infty m_{t,x}\big([h,\infty)\big)\,dh \int p_t^{\nu,\Psi,0}(\mu, d\overline{\mu})\left(\frac{\partial F}{\partial \delta_x}\right)(\overline{\mu} + h\delta_x). \tag{2.6}$$

*In the case of a random Gamma process* [*i.e.*, $\Psi(x,\lambda) = -a(x)\lambda^2 - b(x)\lambda$], (2.6) *reduces to*

$$\frac{\partial p_t^{\nu,\Psi,0} F}{\partial \delta_x}(\mu) = e^{-b(x)t} p_t^{\nu + a\delta_x,\Psi,0}\left(\frac{\partial F}{\partial \delta_x}\right)(\mu). \tag{2.7}$$

PROOF. Theorem 2.1 implies that

$$\begin{aligned}
\frac{\partial p_t^{\nu,\Psi,0} F}{\partial \delta_x}(\mu) &= \int p_t^{\nu,\Psi,0}(\mu, d\mu_1) \int_0^\infty m_{t,x}(ds)\big(F(\mu_1 + s\delta_x) - F(\mu_1)\big) \\
&= \int p_t^{\nu,\Psi,0}(\mu, d\mu_1) \int_0^\infty m_{t,x}(ds) \int_0^s \frac{\partial F}{\partial \delta_x}(\mu_1 + h\delta_x)\,dh \\
&= \int p_t^{\nu,\Psi,0}(\mu, d\mu_1) \int_0^\infty m_{t,x}\big([h,\infty)\big) \frac{\partial F}{\partial \delta_x}(\mu_1 + h\delta_x)\,dh,
\end{aligned}$$

which gives (2.6).

In the case of a random Gamma process,

$$\psi_t(x,\lambda) = \frac{e^{-b(x)t}\lambda}{1 + (a(x)/b(x))(1 - e^{-b(x)t})\lambda}.$$

Hence, if we let $c_t(x) = (a(x)/b(x))(1 - e^{-b(x)t})$, then $m_{t,x} = (e^{-b(x)t}/c_t(x)) \times \Gamma_{c_t^{-1}(x),1}$. Note that

$$\begin{aligned}
&\int_0^\infty m_{t,x}\big([h,\infty)\big)\,dh \int p_t^{\nu,\Psi,0}(\mu, d\overline{\mu})\varphi_f(\overline{\mu} + h\delta_x) \\
&\qquad = \frac{e^{-b(x)t}}{1 + c_t(x) f(x)} (p_t^{\nu,\Psi,0}\varphi_f)(\mu) \\
&\qquad = e^{-b(x)t} p_t^{\nu + a\delta_x,\Psi,0}\varphi_f(\mu).
\end{aligned}$$

Hence,

$$m_{t,x}\big([h,\infty)\big)\,dh * p_t^{\nu,\Psi,0}(\mu,\cdot) = e^{-b(x)t} p_t^{\nu + a\delta_x,\Psi,0}(\mu,\cdot)$$

by uniqueness of the Laplace transform. Consequently, (2.6) implies (2.7) in this case. □

In the finite-dimensional case $S = \{1, \dots, d\}$, the strong topology and the weak topology on $E \cong \mathbb{R}_+^d$ coincide with the usual topology. Hence, the last corollary implies the following result.

COROLLARY 2.4. *Let $S = \{1, \dots, d\}$ be finite, so that we can identify $E$ with $\mathbb{R}_+^d$. Let $F \in C_{\mathrm{b}}^1(\mathbb{R}_+^d)$. Then*

$$\begin{aligned}\partial_i p_t^{\nu,\Psi,0} F(x) \\ &= \int_0^\infty m_{t,i}\big([h,\infty)\big)\,dh \int p_t^{\nu,\Psi,0}(x, dy)\,\partial_i F(y + he_i), \qquad 1 \le i \le d.\end{aligned}$$

*In particular,*

$$\partial_i p_t^{\nu,\Psi,0} F(x) = \exp(-b_i t)\, p_t^{\nu+a_i e_i,\Psi,0}(\partial_i F)(x), \qquad 1 \le i \le d,$$

*in the case of a random Gamma process.*

2.1. *Convergence in spaces of Lipschitz-continuous functions.*

NOTATION. Let $d$ be a metric on $E$. For any function $F : E \to \mathbb{R}$, let

$$\|F\|_{\mathrm{Lip(d)}} := \sup_{\mu \ne \overline{\mu}} \frac{|F(\mu) - F(\overline{\mu})|}{d(\mu, \overline{\mu})}$$

be its Lipschitz norm w.r.t. $d$. Finally, denote by $\mathrm{Lip}(d)$ the space of all bounded $d$-Lipschitz-continuous functions.

We will study in the following Lipschitz constants of $p_t^{\nu,\Psi,A} F$ w.r.t. the distance $d_{\mathrm{var}}$ induced by the total variation norm $\|\cdot\|_{\mathrm{var}}$.

THEOREM 2.5. *Let $(p_t^{\nu,\Psi,A})$ be the transition semigroup of the $(A, \Psi)$-superprocess with immigration $\nu$. Let $b_0 := \inf_{x\in S} b(x)$. Then*

$$\|p_t^{\nu,\Psi,A} F\|_{\mathrm{Lip}(d_{\mathrm{var}})} \le \exp(-b_0 t)\|F\|_{\mathrm{Lip}(d_{\mathrm{var}})}$$

*for all $F \in C_{\mathrm{b}}(E)$.*

We need the following result.

LEMMA 2.6. *Let $F \in C_{\mathrm{b}}(E)$, $t > 0$ and $\mu, \overline{\mu} \in E$. Then*

$$p_t^{\nu,\Psi,A} F(\mu) - p_t^{\nu,\Psi,A} F(\overline{\mu}) = \int_0^1 \left\langle \mu - \overline{\mu}, \frac{\partial p_t^{\nu,\Psi,A} F}{\partial \delta.}(\overline{\mu} + s(\mu - \overline{\mu}))\right\rangle ds.$$

PROOF. To simplify the notation, let $G := p_t^{\nu,\Psi,A} F$. Theorem 2.1 clearly implies that for arbitrary $\mu_1 \in E$ the mapping $s \mapsto G(\overline{\mu} + s\mu_1)$, $s \in (0, 1)$, is continuously differentiable and

$$\frac{dG}{ds}(\overline{\mu} + s\mu_1) = \left\langle \mu_1, \frac{\partial G}{\partial \delta_\cdot}(\overline{\mu} + s\mu_1) \right\rangle.$$

To prove the assertion, it is now sufficient to generalize the last equality to $\mu_1 = s(\mu - \overline{\mu})$ for $s \in (0, 1)$. To this end, note that for small $h$ with $s + h \in (0, 1)$, we have

$$\begin{aligned}
&\frac{1}{h}\Big(G\big(\overline{\mu} + (s+h)(\mu - \overline{\mu})\big) - G\big(\overline{\mu} + s(\mu - \overline{\mu})\big)\Big) \\
&\quad = \frac{1}{h}\Big(G\big((1-(s+h))\overline{\mu} + (s+h)\mu\big) \\
&\qquad\qquad - G\big((1-(s+h))\overline{\mu} + s\mu\big)\Big) \\
&\qquad - \frac{1}{h}\Big(G\big((1-s)\overline{\mu} + s\mu\big) - G\big((1-s)\overline{\mu} + s\mu - h\overline{\mu}\big)\Big) \\
&\quad = \int_0^1 \left\langle \mu, \frac{\partial G}{\partial \delta_\cdot}\big((1-(s+h))\overline{\mu} + (s+rh)\mu\big)\right\rangle dr \\
&\qquad - \int_0^1 \left\langle \overline{\mu}, \frac{\partial G}{\partial \delta_\cdot}\big((1-s)\overline{\mu} + s\mu + (1-r)h\overline{\mu}\big)\right\rangle dr \\
&\quad \to \left\langle \mu - \overline{\mu}, \frac{\partial G}{\partial \delta_\cdot}\big(\overline{\mu} + s(\mu - \overline{\mu})\big)\right\rangle, \qquad h \to 0,
\end{aligned}$$

by the dominated convergence theorem, since $\mu \mapsto (\partial G/\partial \delta_x)(\mu) \in C_{\mathrm{b}}(E)$. $\square$

PROOF OF THEOREM 2.5. Let $m_{t,\delta_x}$ be the canonical measure corresponding to $f \mapsto \psi_t(f)(x)$. Theorem 2.1 now implies that, for $F \in C_{\mathrm{b}}(E)$,

$$\begin{aligned}
\frac{\partial p_t^{\nu,\Psi,A} F}{\partial \delta_x}(\mu) &= \int p_t^{\nu,\Psi,A}(\mu, d\mu_1) \int m_{t,\delta_x}(d\mu_2)\big(F(\mu_1 + \mu_2) - F(\mu_1)\big) \\
&\le \int p_t^{\nu,\Psi,A}(\mu, d\mu_1) \int m_{t,\delta_x}(d\mu_2)|\mu_2| \|F\|_{\mathrm{Lip}(d_{\mathrm{var}})}.
\end{aligned}$$

Since $\psi_t(h\mathbb{1}_S) \le e^{-b_0 t} h$, by Lemma 1.5 it follows that

$$\begin{aligned}
\int m_{t,\delta_x}(d\mu_2)|\mu_2| &= \lim_{h \downarrow 0} \frac{1}{h} \int 1 - \exp(-h|\mu_2|) m_{t,\delta_x}(d\mu_2) \\
&= \lim_{h \downarrow 0} \frac{1}{h} \psi_t(h\mathbb{1}_S)(x) \le \exp(-b_0 t).
\end{aligned}$$

Hence,

$$\left|\frac{\partial p_t^{\nu,\Psi,A}F}{\partial\delta_x}(\mu)\right| \le \exp(-b_0 t)\|F\|_{\mathrm{Lip}(d_{\mathrm{var}})}.$$

Since the last inequality holds for all $\mu \in E$ and all $x \in S$, the assertion now follows from Lemma 2.6 since

$$\begin{aligned} p_t^{\nu,\Psi,0}F(\mu) - p_t^{\nu,\Psi,0}F(\overline{\mu}) &= \int_0^1 \left\langle \mu - \overline{\mu}, \frac{\partial p_t^{\nu,\Psi,0}F}{\partial\delta.}(\overline{\mu} + s(\mu - \overline{\mu}))\right\rangle ds \\ &\le \exp(-b_0 t)\|\mu - \overline{\mu}\|_{\mathrm{var}}\|F\|_{\mathrm{Lip}(d_{\mathrm{var}})}. \qquad \square \end{aligned} \tag{2.8}$$

Since in the finite-dimensional case the total variation norm can be identified with one-half of the $\ell_1$-norm and since the set $C_{\mathrm{b}}^1(\mathbb{R}_+^d)$ is dense in the space of bounded Lipschitz-continuous functions, we obtain the following result:

COROLLARY 2.7. *Assume in addition to Theorem* 2.4 *that* $S = \{1, \dots, d\}$ *is finite*, *so that we can identify* $E$ *with* $\mathbb{R}_+^d$. *Then*

$$\|p_t^{\nu,\Psi,A}F\|_{\mathrm{Lip}(\ell_1)} \le \exp(-b_0 t)\|F\|_{\mathrm{Lip}(\ell_1)}$$

*for all bounded* $\ell_1$*-Lipschitz-continuous* $F$.

REMARK 2.8. Theorem 2.5 implies, in particular, convergence of $p_t^{\nu,\Psi,A}F$ with exponential rate $b_0$ in $L^p(m_\nu^{\Psi,A})$, $p \ge 1$, for $F \in C_{\mathrm{b}}(E)$ with $F$ $d_{\mathrm{var}}$-Lipschitz continuous, since

$$\begin{aligned} &\|p_t^{\nu,\Psi,A}F - \langle F\rangle\|_{L^p(m_\nu^{\Psi,A})}^p \\ &\quad \le \iint |p_t^{\nu,\Psi,A}F(\mu_1) - p_t^{\nu,\Psi,A}F(\mu_2)|^p m_\nu^{\Psi,A}(d\mu_1)\, m_\nu^{\Psi,A}(d\mu_2) \\ &\quad \le \exp(-pb_0 t)\|F\|_{\mathrm{Lip}(d_{\mathrm{var}})}^p \iint \|\mu_1 - \mu_2\|_{\mathrm{var}}^p\, m_\nu^{\Psi,A}(d\mu_1)\, m_\nu^{\Psi,A}(d\mu_2) \\ &\quad \le 2^p \int |\mu|^p\, m_\nu^{\Psi,A}(d\mu) \exp(-pb_0 t)\|F\|_{\mathrm{Lip}(d_{\mathrm{var}})}^p. \end{aligned}$$

Since $m_\nu^{\Psi,A}$ has finite exponential moments (cf. Proposition 1.1), the assertion now follows.

2.2. *Uniform pointwise gradient estimates and* $L^2$*-convergence.* Using an explicit representation of the heat kernel in the Gamma case which will be proved in Section 4, we will obtain next pointwise gradient estimates on the transition semigroup of the $(A, \Psi)$-superprocess in the finite-dimensional case under additional assumptions on the immigration.

THEOREM 2.9. *Assume that* $n=0$, $S=\{1,\dots,d\}$ *and* $\min_{1\le i\le d}(\nu(i)/a(i))\ge\frac{1}{2}$. *Let* $P_i:\mathbb{R}_+^d\to\mathbb{R}_+$, $x\mapsto x_i$. *Then*:

$$\sum_{i=1}^d x_i\big(\partial_i p_t^{\nu,\Psi,A} f\big)^2(x)\le \exp(-b_0 t)p_t^{\nu,\Psi,A}\left(\sum_{i=1}^d P_i(\partial_i f)^2\right)(x)$$

*for all* $f\in C_{\mathrm{b}}^1(\mathbb{R}_+^d)$.

PROOF. To simplify the notation, let $p_t:=p_t^{\nu,\Psi,A}$, $a_i:=a(i)$, $b_i:=b(i)$ and $\nu_i:=\nu(i)$. Let us first assume that $A=0$ and $q_i/a_i=\frac{1}{2}$, $1\le i\le d$. Proposition 4.4 (cf. also Remark 4.5) implies that $p_t(x,\cdot)$ is absolutely continuous and that its density $p_t(x,y)$ admits the representation

$$\begin{aligned}p_t(x,y)&=\prod_{i=1}^d\sqrt{\frac{b_i}{a_i}\frac{1}{1-\exp(-b_i t)}}\frac{1}{\sqrt{\pi}\sqrt{y_i}}\\&\qquad\times E\left[\exp\left(-\sum_{i=1}^d\frac{b_i}{a_i}\frac{(\exp(-b_i t/2)\sqrt{x_i}-Z_i\sqrt{y_i})^2}{1-\exp(-b_i t)}\right)\right],\end{aligned}$$

where $(Z_i)_{1\le i\le d}$ are i.i.d. with $P[Z_i=+1]=P[Z_i=-1]=\frac{1}{2}$. To further simplify the notation, let

$$c:=\prod_{i=1}^d\sqrt{\frac{b_i}{a_i}\frac{1}{1-\exp(-b_i t)}}\frac{1}{\sqrt{\pi}}.$$

Then

$$\begin{aligned}&\partial_i p_t f(x)\\&\quad=-cE\left[\int\frac{\exp(-b_i t/2)}{\sqrt{x_i}}\frac{b_i}{a_i}\frac{\exp(-b_i t/2)\sqrt{x_i}-Z_i\sqrt{y_i}}{1-\exp(-b_i t)}f(y)\prod_{j=1}^d\frac{1}{\sqrt{y_j}}\right.\\&\qquad\qquad\left.\times\exp\left(-\sum_{j=1}^d\frac{b_j}{a_j}\frac{(\exp(-b_j t/2)\sqrt{x_j}-Z_j\sqrt{y_j})^2}{1-\exp(-b_j t)}\right)dy\right]\\&\quad=-\frac{\exp(-b_i t/2)}{\sqrt{x_i}}\\&\qquad\times cE\left[Z_i\int f(y)\prod_{j\ne i}\frac{1}{\sqrt{y_j}}\,\partial_{y_i}\right.\\&\qquad\qquad\left.\times\exp\left(-\sum_{j=1}^d\frac{b_j}{a_j}\frac{(\exp(-b_j t/2)\sqrt{x_j}-Z_j\sqrt{y_j})^2}{1-\exp(-b_j t)}\right)dy\right]\end{aligned}$$

$$
\begin{aligned}
&= \frac{\exp(-b_i t/2)}{\sqrt{x_i}} \\
&\quad \times cE\Bigg[Z_i \int \sqrt{y_i}\,\partial_i f(y) \prod_{j=1}^d \frac{1}{\sqrt{y_j}} \\
&\qquad\qquad \times \exp\Bigg(-\sum_{j=1}^d \frac{b_j}{a_j} \frac{(\exp(-b_j t/2)\sqrt{x_j} - Z_j\sqrt{y_j})^2}{1-\exp(-b_j t)}\Bigg)\,dy\Bigg] \\
&\le \frac{\exp(-b_0 t/2)}{\sqrt{x_i}} p_t\big(\sqrt{P_i}|\partial_i f|\big)(x).
\end{aligned}
$$

Here the third equality follows from integration by parts, using the fact that

$$
E\bigg[Z_i \int_{\mathbb{R}_+^{d-1}} g(x_1, \dots, x_{i-1}, 0, x_i, \dots, x_{d-1})\,dx\bigg] = 0
$$

for all integrable $g$. Hence, Jensen's inequality yields

$$
x_i(\partial_i p_t f)^2(x) \le \exp(-b_0 t)\, p_t(P_i(\partial_i f)^2)(x).
$$

Summation over $1, \dots, d$ proves the assertion in this case.

In the next step, assume again that $\nu_i/a_i = \frac{1}{2}$, $1 \le i \le d$, and denote by $(p_t^0)$ the semigroup of the corresponding superprocess with zero mutation. Let $(q_t)$ be the Markovian semigroup generated by the mutation $A$ and let $R_t f(x) := f(\hat{q}_t x)$. Clearly, $(R_t)$ induces a $C_0$-semigroup of contractions on $C_\infty(\mathbb{R}_+^d)$ and, for any $g \in C_b^1(\mathbb{R}_+^d)$,

$$
\begin{aligned}
\sum_{i=1}^d x_i(\partial_i R_t g)^2(x) &= \sum_{i=1}^d x_i \Bigg(\sum_{j=1}^d q_t(i,j)(\partial_j g)(\hat{q}_t x)\Bigg)^2 \\
&\le \sum_{i=1}^d x_i \sum_{j=1}^d q_t(i,j)(\partial_j g)^2(\hat{q}_t x) \\
&= R_t\Bigg(\sum_{j=1}^d P_j(\partial_j g)^2\Bigg)(x).
\end{aligned}
$$

Since $(L_\nu^{\Psi,A}, C_0^2(\mathbb{R}_+^d))$ is maximal (cf. [12]), the Trotter–Kato product formula implies that

$$
p_t = \lim_{n\to\infty} \big(p_{t/n}^0 \circ R_{t/n}\big)^n
$$

in the strong operator topology (cf. [4]). Fix $f \in C_0^\infty(\mathbb{R}_+^d)$, $x \in \mathbb{R}_+^d$ and $g \in \mathbb{R}^d$.

Then

$$
\begin{aligned}
&p_t f\big(\exp(hg)x\big) - p_t f(x)\\
&\quad = \lim_{n\to\infty} \big(p^0_{t/n}\circ R_{t/n}\big)^n f\big(\exp(hg)x\big) - \big(p^0_{t/n}\circ R_{t/n}\big)^n f(x)\\
&\quad = \lim_{n\to\infty} \int_0^h \big\langle g\exp(sg)x, \nabla\big(p^0_{t/n}\circ R_{t/n}\big)^n f\big(\exp(sg)x\big)\big\rangle\, ds\\
&\quad \le \lim_{n\to\infty} \int_0^h \left(\sum_{i=1}^d g_i^2 \exp(sg_i)x_i\right)^{1/2}\\
&\qquad\qquad \times \left(\sum_{i=1}^d \exp(sg_i)x_i\big(\partial_i\big(p^0_{t/n}\circ R_{t/n}\big)^n f\big)^2\big(\exp(sg)x\big)\right)^{1/2} ds\\
&\quad \le \exp(-b_0 t/2)\lim_{n\to\infty} \int_0^h \left(\sum_{i=1}^d g_i^2 \exp(sg_i)x_i\right)^{1/2}\\
&\qquad\qquad \times \left(\big(p^0_{t/n}\circ R_{t/n}\big)^n \left(\sum_{i=1}^d P_i(\partial_i f)^2\right)\big(\exp(sg)x\big)\right)^{1/2} ds\\
&\quad = \exp(-b_0 t/2)\int_0^h \left(\sum_{i=1}^d g_i^2 \exp(sg_i)x_i\right)^{1/2}\\
&\qquad\qquad \times \left(p_t \left(\sum_{i=1}^d P_i(\partial_i f)^2\right)\big(\exp(sg)x\big)\right)^{1/2} ds.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\langle gx, \nabla p_t f(x)\rangle &= \lim_{h\to 0}\frac{1}{h}\big(p_t f\big(\exp(hg)x\big) - p_t f(x)\big)\\
&\le \exp(-b_0 t/2)\left(\sum_{i=1}^d g_i^2 x_i\right)^{1/2}\left(p_t\left(\sum_{i=1}^d P_i(\partial_i f)^2\right)\right)^{1/2}(x),
\end{aligned}
$$

which implies the assertion for $f\in C_0^\infty(\mathbb{R}_+^d)$ if we take $g=\nabla p_t f(x)$.

The general case $\min_{1\le i\le d}\nu_i/a_i \ge \frac{1}{2}$ can be deduced from this case in a similar way. Let $(p_t^0)$ now denote the semigroup of the superprocess with immigration $\tilde{\nu}_i = \frac{1}{2}a_i$, $1\le i\le d$, and mutation $A$. Let $R_t f(x) = f(x + t(\nu-\tilde{\nu}))$, $t\ge 0$. Then $(R_t)$ induces a $C_0$-semigroup of contractions on $C_\infty(\mathbb{R}_+^d)$ (here we use $\nu_i/a_i \geq \frac{1}{2}$) and

$$\sum_{i=1}^d x_i(\partial_i R_t g)^2(x) \leq R_t\left(\sum_{i=1}^d P_i(\partial_i g)^2\right)(x)$$

for any $g \in C_b^1(\mathbb{R}_+^d)$. Using again the Trotter–Kato product formula in the same way as above, we obtain the assertion for $f \in C_0^\infty(\mathbb{R}_+^d)$. The general case now follows by approximation. □

REMARK 2.10. It seems that pointwise gradient estimates no longer hold with constants independent of $\nu(i)/a(i)$ as soon as $\min_{1\leq i\leq d}\nu(i)/a(i) < \frac{1}{2}$. This might be related to the fact that the second iterated gradient $\Gamma_2(f, f) := \frac{1}{2}\{L_\nu^{\Psi,A}\Gamma(f, f) - 2\Gamma(L_\nu^{\Psi,A} f, f)\}$ associated with the generator of the $(A, \Psi)$-superprocess with immigration $\nu$, at least in the case $A = 0$, is neither positive definite nor bounded from below in this case. Here $\Gamma(f, g)(x) := \sum_{i=1}^d a_i x_i\, \partial_i f(x) \times \partial_i g(x)$. Indeed, a direct calculation yields for $f \in C_b^2(\mathbb{R}_+^d)$

$$\begin{aligned}\Gamma_2(f, f)(x) = \sum_{i=1}^d a_i^2 x_i\, \partial_{ii} f(x)\, \partial_i f(x) + \sum_{i,j=1}^d a_i a_j x_i x_j\big(\partial_{ij} f(x)\big)^2 \\ + \sum_{i=1}^d \tfrac{1}{2} a_i b_i x_i\big(\partial_i f(x)\big)^2 + \sum_{i=1}^d \tfrac{1}{2}\nu_i a_i\big(\partial_i f(x)\big)^2.\end{aligned}$$

Now it is easily verified that $\min_{1\leq i\leq d}\nu_i/a_i \geq \frac{1}{2}$ implies

$$\Gamma_2(f, f)(x) \geq \min_{1\leq i\leq d}\frac{b_i}{2}\Gamma(f, f)(x).$$

Note that the constant can be obtained independent of $\nu$. For consequences [e.g., logarithmic Sobolev inequalities, hypercontractivity of $(p_t^{\nu,\Psi,A})$], see [1]. On the other hand, $\min_{1\leq i\leq d}\nu_i/a_i < \frac{1}{2}$ implies that there is no constant $K > -\infty$ such that the inequality

$$\Gamma_2(f, f)(x) \geq K\Gamma(f, f)(x) \tag{2.9}$$

holds for all $f \in C_b^2(\mathbb{R}_+^d)$. Indeed, by independence, it is enough to consider the case $d = 1$ only. Let $f_\varepsilon(x) := e^{-\varepsilon x}$, $\varepsilon > 0$. Then

$$\Gamma_2(f_\varepsilon, f_\varepsilon)(x) = \big(-a^2\varepsilon^3 x + a^2\varepsilon^4 x^2 + \tfrac{1}{2}a\varepsilon^2(\nu + bx)\big)f_\varepsilon(x)^2,$$

whereas, on the other hand,

$$\Gamma(f_\varepsilon, f_\varepsilon)(x) = ax\varepsilon^2 f_\varepsilon^2(x).$$

Consequently, inequality (2.9) would imply, in particular, the following inequality:

$$-ax\varepsilon + a\varepsilon^2 x^2 + \tfrac{1}{2}(\nu + bx) \geq Kx, \qquad x \in \mathbb{R}_+.$$

Choosing $x_\varepsilon := 1/2\varepsilon$, we obtain that $\frac{1}{2}(\nu - a/2) + b/4\varepsilon \ge K(1/2\varepsilon)$ for all $\varepsilon > 0$, which is clearly impossible if we let $\varepsilon$ tend to $\infty$ since $\nu - a/2 < 0$.

The last theorem also implies convergence of the semigroup toward its invariant measure with exponential rate $b_0/2$.

COROLLARY 2.11. *Let the assumptions be as in Theorem* 2.9. *Then the bilinear form*

$$\mathcal{E}(f) := \sum_{i=1}^{d} \int x_i (\partial_i f)^2(x) m_\nu^{\Psi,A}(dx), \qquad f \in C_{\mathrm{b}}^1(\mathbb{R}_+^d),$$

*determines a Poincaré inequality with constant less than* $2(\|a\|_\infty/b_0)$. *Moreover*,

$$\|p_t^{\nu,\Psi,A} f - \langle m_\nu^{\Psi,A}, f\rangle\|_{L^2(m_\nu^{\Psi,A})} \le \exp\left(-\frac{b_0}{\|a\|_\infty}\frac{t}{2}\right)\|f\|_{L^2(m_\nu^{\Psi,A})}.$$

PROOF. Let $f \in C_{\mathrm{b}}^1(\mathbb{R}_+^d)$. Since $\lim_{t\to\infty} p_t^{\nu,\Psi,0} f = \langle m_\nu^{\Psi,A}, f\rangle$, it follows from the invariance of $m_\nu^{\Psi,A}$ that

$$\begin{aligned}
\int f^2\, dm_\nu^{\Psi,A} - \langle m_\nu^{\Psi,A}, f\rangle^2 &= -\int_0^\infty \frac{d}{dt}\langle m_\nu^{\Psi,A}, p_t^{\nu,\Psi,A} f\rangle^2\, dt \\
&= -2\int_0^\infty \int L_\nu^{\Psi,A} p_t^{\nu,\Psi,A} f p_t^{\nu,\Psi,A} f\, dm_\nu^{\Psi,A}\, dt \\
&\le 2\|a\|_\infty \int_0^\infty \mathcal{E}\big(p_t^{\nu,\Psi,A} f\big)\, dt \\
&\le 2\|a\|_\infty \int_0^\infty \exp(-b_0 t)\mathcal{E}(f)\, dt = 2\frac{\|a\|_\infty}{b_0}\mathcal{E}(f).
\end{aligned}$$

In particular, for $f \in C_{\mathrm{b}}^2(\mathbb{R}_+^d)$, the last inequality implies that

$$\int f^2\, dm_\nu^{\Psi,A} - \langle m_\nu^{\Psi,A}, f\rangle^2 \le -2\frac{\|a\|_\infty}{b_0}\int L_\nu^{\Psi,A} f\, f\, dm_\nu^{\Psi,A}$$

and by $L^2$-uniqueness of $(L_\nu^{\Psi,A}, C_{\mathrm{b}}^2(\mathbb{R}_+^d))$ the last inequality extends to all $f \in D(L_\nu^{\Psi,A})$. The rest of the assertion now follows from standard semigroup theory. □

The result of the last corollary is not optimal. In the particular case of zero mutation, explicit rates of convergence of $(p_t^{\nu,\Psi,0})$ have been obtained in [12] for arbitrary compact metric type spaces and arbitrary $\nu$. In particular, it was shown in [12], Theorem 3.1, that the semigroup of the random Gamma process converges to equilibrium with exponential rate $b_0$ (independent of $a$ and $\nu$) (cf. also Section 4.2).

## 3. Regularity of the transition semigroup.

### 3.1. *Strong Feller property.*

THEOREM 3.1. *Let*

$$s(t) := \sup_{x\in S} m_{t,\delta_x}(E\setminus\{0\}) \qquad \left(= \sup_{x\in S}\lim_{\lambda\to\infty}\psi_t(\lambda \mathbb{1}_S)(x) \le \frac{b_0}{a_0}\frac{e^{-b_0 t}}{1-e^{-b_0 t}}\right).$$

*Let* $F \in \mathcal{B}_b(E)$. *Then*

$$\|p_t^{\nu,\Psi,A}F\|_{\mathrm{Lip}(d_{\mathrm{var}})} \le 2s(t)\|F\|_\infty.$$

*In particular*, $p_t^{\nu,\Psi,A} : \mathcal{B}_b(E) \to \mathrm{Lip}(d_{\mathrm{var}})$ *is continuous for* $t > 0$.

PROOF. Theorem 2.1 implies, for $F \in C_b(E)$,

$$\begin{aligned}\left|\frac{\partial p_t^{\nu,\Psi,A}F}{\partial\delta_x}(\mu)\right| &= \left|\int p_t^{\nu,\Psi,A}(\mu, d\mu_1)\int m_{t,\delta_x}(d\mu_2)\big(F(\mu_1+\mu_2) - F(\mu_1)\big)\right| \\ &\le 2m_{t,\delta_x}(E\setminus\{0\})\|F\|_\infty \le 2s(t)\|F\|_\infty.\end{aligned}$$

Thus, for all $\mu, \overline{\mu} \in E$, by Lemma 2.6,

$$\begin{aligned}\left|p_t^{\nu,\Psi,A}F(\mu) - p_t^{\nu,\Psi,A}F(\overline{\mu})\right| &= \left|\int_0^1 \left\langle \mu - \overline{\mu}, \frac{\partial p_t^{\nu,\Psi,A}F}{\partial\delta.}\big(\overline{\mu} + s(\mu - \overline{\mu})\big)\right\rangle ds\right| \\ &\le 2s(t)\|\mu - \overline{\mu}\|_{\mathrm{var}}\|F\|_\infty.\end{aligned}$$

For arbitrary $F \in \mathcal{B}_b(E)$, fix $\mu$ and $\overline{\mu} \in E$. Since $C_b(E) \subset L^1(p_t^{\nu,\Psi,A}(\mu,\cdot) + p_t^{\nu,\Psi,A}(\overline{\mu},\cdot))$ dense, we can find a sequence $(F_n) \subset C_b(E)$ converging to $F$ both $p_t^{\nu,\Psi,A}(\mu,\cdot)$-a.e. and $p_t^{\nu,\Psi,A}(\overline{\mu},\cdot)$-a.e. We may assume that $\|F_n\|_\infty \le \|F\|_\infty$. Then, by the dominated convergence theorem,

$$\begin{aligned}\left|p_t^{\nu,\Psi,A}F(\mu) - p_t^{\nu,\Psi,A}F(\overline{\mu})\right| &= \lim_{n\to\infty}\left|p_t^{\nu,\Psi,A}F_n(\mu) - p_t^{\nu,\Psi,A}F_n(\overline{\mu})\right| \\ &\le \lim_{n\to\infty} 2s(t)\|\mu - \overline{\mu}\|_{\mathrm{var}}\|F_n\|_\infty \\ &\le 2s(t)\|\mu - \overline{\mu}\|_{\mathrm{var}}\|F\|_\infty. \qquad \square\end{aligned}$$

As a corollary to the last theorem, we now obtain the following result:

COROLLARY 3.2. *The transition semigroup* $(p_t^{\nu,\Psi,A})$ *of the* $(A,\Psi)$*-superprocess with immigration* $\nu$ *is strong Feller w.r.t. the strong topology. In particular, if* $S$ *is finite, then* $(p_t^{\nu,\Psi,A})$ *is strong Feller w.r.t. the usual topology.*

REMARK 3.3. (i) Theorem 3.1 becomes wrong in the linear case, that is, in the case $a=n=0$. Indeed, in this case, $f\mapsto \psi_t(f)$, $t\geq 0$, is the linear semigroup generated by $A-b$. It follows in the case of zero immigration that $p_t^{0,\Psi,A}F(\mu)=F(\delta_{\hat{\psi}_t\mu})$. In particular, $p_t^{0,\Psi,A}F$ is not Lipschitz for general $F\in\mathscr{B}_{\mathrm{b}}(E)$.

(ii) Although the theorem looks like a strong regularity result on the transition semigroup, it does not imply any additional compactness properties of $(p_t^{\nu,\Psi,A})$, since $E$ equipped with the strong topology is no longer locally compact as soon as $S$ contains infinitely many points. What would be needed instead (e.g., for the construction of invariant measures for models with interactive selection) is the strong Feller property w.r.t. the weak topology or additional compactness properties of the semigroup (e.g., in $L^p$-spaces induced by the invariant measures). However, this cannot be expected unless additional assumptions on the mutation are imposed.

3.2. *Compactness.* Recall that the Wasserstein metric $d_w$ on $E$ is given by

$$d_w(\mu,\overline{\mu}):=\sup_{\|f\|_{\mathrm{Lip}(d)}\leq 1,\|f\|_\infty\leq 1}\int f\,d\mu-\int f\,d\overline{\mu}$$

and metrizes the weak topology on $E$.

THEOREM 3.4. *Let $(p_t)$ be the semigroup generated by $A$. Assume that there exists $d:(0,\infty)\to\mathbb{R}_+$ such that*

$$\|p_tf\|_{\mathrm{Lip}(d)}\leq d(t)\|f\|_\infty,\qquad f\in\mathscr{B}_{\mathrm{b}}(E).$$

*For $s>0$, let $\overline{B_s(0)}=\{\mu\in E\mid|\mu|\leq s\}$ and denote by $r_s:C_\infty(E)\to C(\overline{B_s(0)})$, $F\mapsto F|_{\overline{B_s(0)}}$, the natural restriction. Then $r_s\circ p_t^{\nu,\Psi,A}:C_\infty(E)\to C(\overline{B_s(0)})$ is compact for all $s>0$. Moreover, $r_s(D(L_\nu^{\Psi,A}))\subset C(\overline{B_s(0)})$ is compact, too.*

PROOF. First, note that the assumption on $(p_t)$ clearly implies that

$$d_{\mathrm{var}}(\hat{p}_t\mu,\hat{p}_t\overline{\mu})\leq d(t)d_w(\mu,\overline{\mu}),\qquad \mu,\overline{\mu}\in E. \tag{3.1}$$

Let $R_tF(\mu):=F(\hat{p}_t\mu)$. Then $(R_t)$ defines a $C_0$-semigroup of contractions on $C_\infty(E)$ and

$$S_t^n:=\big(p_{t/n}^{\nu,\Psi,0}\circ R_{t/n}\big)^n\to p_t^{\nu,\Psi,A}$$

in the strong operator topology on bounded linear operators on $C_\infty(E)$. Using Theorem 3.1 and (3.1), we now obtain, for $F\in\mathscr{B}_{\mathrm{b}}(E)$,

$$\begin{aligned}&\big|p_{t/n}^{\nu,\Psi,0}\circ R_{t/n}F(\mu)-p_{t/n}^{\nu,\Psi,0}\circ R_{t/n}F(\overline{\mu})\big|\\&\quad\leq 2s(t/n)\|F\|_\infty\|\hat{p}_{t/n}(\mu-\overline{\mu})\|_{\mathrm{var}}\\&\quad\leq 2s(t/n)\,d(t/n)\|F\|_\infty\,d_w(\mu,\overline{\mu}).\end{aligned}$$

Consequently, $|S_t^n F(\mu) - S_t^n F(\overline{\mu})| \le 2s(t/n)\, d(t/n) \|F\|_\infty d_w(\mu, \overline{\mu})$ and thus $r_s \circ S_t^n(C_\infty(E))$ is compactly embedded in $C(\overline{B_s(0)})$ by Arzela-Ascoli. Since $\lim_{n\to\infty} S_t^n = p_t^{n,\Psi,A}$ in the strong operator topology, we conclude that $r_s \circ p_t^{\nu,\Psi,A}(C_\infty(E))$ is compactly embedded in $C(\overline{B_s(0)})$, too.

Finally, let $G_\alpha := \int_0^\infty e^{-\alpha t} p_t^{\nu,\Psi,A}\, dt$ be the $\alpha$-resolvent of $L_\nu^{\Psi,A}$, $\alpha > 0$. Since $r_s \circ p_t^{\nu,\Psi,A}$ is compact for all $t > 0$, it follows that $r_s \circ G_\alpha$ is compact, too. Hence, $r_s(D(L_\nu^{\Psi,A})) = r_s \circ G_\alpha(C_\infty(E))$ is compactly embedded in $C(\overline{B_s(0)})$. $\square$

## 4. Gamma processes.

NOTATION. For arbitrary $\theta \in \mathcal{B}_b(S)_+$ and $\nu \in \mathcal{M}_+(S)$, denote by $\Gamma_{\theta,\nu}$ the random Gamma measure with log-Laplace functional $\int \log(1 + \theta^{-1} f)\, d\nu$. In analogy to the classical Gamma measure, we call $\theta$ the scale parameter and $\nu$ the shape parameter. Note that for $\theta \in C(S)_+$ the mapping $\nu \mapsto \Gamma_{\theta,\nu}$ is weakly continuous. Indeed, for $f \in C(S)_+$ and $\lim_{n\to\infty} \nu_n = \nu$ weakly, it follows that $\lim_{n\to\infty} \int \log(1+\theta^{-1} f)\, d\nu_n = \int \log(1+\theta^{-1} f)\, d\nu$, so that $\lim_{n\to\infty} \Gamma_{\theta,\nu_n} = \Gamma_{\theta,\nu}$ weakly, by [5], Theorem 3.2.6.

Throughout this section, we assume that $n = 0$ [so that $\Psi(x,\lambda) = -a(x)\lambda^2 - b(x)\lambda$] and that $A = 0$. We have already seen in Section 1 that in the particular case $n = 0$ the invariant measure $m_\nu^{\Psi,0}$ of the $(0,\Psi)$-superprocess with immigration $\nu$ is the random Gamma measure $\Gamma_{a^{-1}b, a^{-1}\nu}$.

4.1. *A series representation of* $p_t^{\nu,\Psi,0}$. The following theorem was first proved by Ethier and Griffiths in the case of a constant branching mechanism (cf. [7], Theorem 1.1).

THEOREM 4.1. *Let*

$$c_t := \frac{b}{a} \frac{e^{-bt}}{1 - e^{-bt}}.$$

*Fix* $\nu \in \mathcal{M}_+(S)$ *and denote by* $(p_t^{\nu,\Psi,0})$ *the semigroup of the* $(0,\Psi)$*-superprocess with immigration* $\nu$. *Then*

$$\begin{aligned} p_t^{\nu,\Psi,0}&(\mu, d\overline{\mu}) \\ &= \exp(-\langle c_t, \mu\rangle) \Gamma_{e^{bt}c_t, a^{-1}\nu}(d\overline{\mu}) \\ &\quad + \sum_{n=1}^\infty \exp(-\langle c_t, \mu\rangle) \frac{1}{n!} \\ &\qquad \times \int_{S^n} \mu^n(dx_1,\ldots,dx_n) c_t(x_1)\cdots c_t(x_n) \Gamma_{e^{bt}c_t, a^{-1}\nu + \sum_{k=1}^n \delta_{x_k}}(d\overline{\mu}). \end{aligned} \tag{4.1}$$

PROOF. Denote by $q_t(\mu, d\overline{\mu})$ the right-hand side of (4.1). Since

$$
\begin{aligned}
p_t^{\nu,\Psi,0}&\big(\exp(-\langle f,\cdot\rangle)\big)(\mu)\\
&= \exp\biggl(-\int_0^t \langle \psi_s(f),\nu\rangle\,ds - \langle \psi_t(f),\mu\rangle\biggr)\\
&= \exp\biggl(-\int_0^t \biggl\langle \frac{e^{-bs}f}{1+(a/b)(1-e^{-bs})f},\nu\biggr\rangle ds\\
&\qquad\qquad - \biggl\langle \frac{e^{-bt}f}{1+(a/b)(1-e^{-bt})f},\mu\biggr\rangle\biggr) \qquad (4.2)\\
&= \exp\biggl(-\biggl\langle \frac{1}{a}\log\biggl(1+\frac{a}{b}(1-e^{-bt})f\biggr),\nu\biggr\rangle\\
&\qquad\qquad - \biggl\langle \frac{e^{-bt}f}{1+(a/b)(1-e^{-bt})f},\mu\biggr\rangle\biggr),
\end{aligned}
$$

it suffices to verify that the right-hand side of (4.2) coincides with the Laplace transform of $q_t(\mu,\cdot)$. To this end, note that

$$
\begin{aligned}
q_t&\big(\exp(-\langle f,\cdot\rangle)\big)(\mu)\\
&= \exp(-\langle c_t,\mu\rangle)\int \exp(-\langle f,\overline{\mu}\rangle)\Gamma_{e^{bt}c_t,a^{-1}\nu}(d\overline{\mu})\\
&\quad + \sum_{n=1}^{\infty}\exp(-\langle c_t,\mu\rangle)\frac{1}{n!}\int_{S^n}\mu^n(dx_1,\dots,dx_n)c_t(x_1)\cdots c_t(x_n)\\
&\qquad\quad \times \int \exp(-\langle f,\overline{\mu}\rangle)\Gamma_{e^{bt}c_t,a^{-1}\nu+\sum_{k=1}^n\delta_{x_k}}(d\overline{\mu})\\
&= \exp(-\langle c_t,\mu\rangle)\exp\biggl(-\biggl\langle a^{-1}\log\biggl(1+\frac{e^{-bt}}{c_t}f\biggr),\nu\biggr\rangle\biggr)\\
&\quad + \sum_{n=1}^{\infty}\exp(-\langle c_t,\mu\rangle)\frac{1}{n!}\int_{S^n}\mu^n(dx_1,\dots,dx_n)c_t(x_1)\cdots c_t(x_n)\\
&\qquad\quad \times \exp\biggl(-\sum_{k=1}^n \log\biggl(1+\frac{e^{-b(x_k)t}}{c_t(x_k)}f(x_k)\biggr)\biggr)\\
&\qquad\quad \times \exp\biggl(-\biggl\langle a^{-1}\log\biggl(1+\frac{e^{-bt}}{c_t}f\biggr),\nu\biggr\rangle\biggr)\\
&= \exp\biggl(-\biggl\langle a^{-1}\log\biggl(1+\frac{e^{-bt}}{c_t}f\biggr),\nu\biggr\rangle\biggr)\\
&\quad \times \exp\biggl(-\langle c_t,\mu\rangle + \biggl\langle c_t\frac{1}{1+(e^{-bt}/c_t)f},\mu\biggr\rangle\biggr)
\end{aligned}
$$

$$= \exp\left(-\left\langle a^{-1}\log\left(1+\frac{e^{-bt}}{c_t}f\right), \nu\right\rangle\right)$$
$$\times \exp\left(-\left\langle \frac{e^{-bt}f}{1+(a/b)(1-e^{-bt})f}, \mu\right\rangle\right),$$

which implies the assertion. $\square$

In the finite-dimensional case, (4.1) reduces to the following series representation already contained in [8] in the case $d = 1$.

PROPOSITION 4.2. *Assume that $|S| = d$, so that we can identify $\mathcal{M}_+(S)$ with $\mathbb{R}_+^d$, and that $\nu_i := \nu(i) > 0$, $1 \le i \le d$. Then $p_t^{\nu,\Psi,0}(x,\cdot)$ has a density $p_t^{\nu,\Psi,0}(x,y)$ w.r.t. the $d$-dimensional Lebesgue measure given by*

$$\begin{aligned} p_t^{\nu,\Psi,0}(x,y) &= \prod_{i=1}^d \frac{b_i}{a_i}\frac{\exp((b_i t/2)(\nu_i/a_i - 1))}{1-\exp(-b_i t)} \exp\left(-\frac{b_i}{a_i}\frac{\exp(-b_i t)x_i + y_i}{1-\exp(-b_i t)}\right) \\ &\quad \times \left(\frac{y_i}{x_i}\right)^{\nu_i/2a_i - 1/2} I_{\nu_i/a_i - 1}\left(2\frac{b_i}{a_i}\frac{\exp(-(b_i/2)t)}{1-\exp(-b_i t)}\sqrt{x_i y_i}\right). \end{aligned} \tag{4.3}$$

*Here*

$$I_q(x) := \sum_{n=0}^\infty \frac{(x/2)^{2n+q}}{n!\Gamma(n+q+1)}, \qquad q > -1,$$

*denotes the modified Bessel function.*

PROOF. The assertion follows from a straightforward calculation of the Laplace transform of the right-hand side of (4.3). $\square$

REMARK 4.3. Note that in the multidimensional case $p_t^{\nu,\Psi,0}(x,y)$ may be rewritten as

$$\begin{aligned} p_t^{\nu,\Psi,0}(x,y) &= \prod_{i=1}^d \left(\frac{b_i}{a_i}\frac{1}{1-\exp(-b_i t)}\right)^{\nu_i/a_i} y_i^{\nu_i/a_i - 1} \exp\left(-\frac{b_i}{a_i}\frac{x_i\exp(-b_i t) + y_i}{1-\exp(-b_i t)}\right) \\ &\quad \times \sum_{n=0}^\infty \sum_{|k|=n} \prod_{i=1}^d \left(\frac{b_i}{a_i}\frac{\exp(-b_i t/2)}{1-\exp(-b_i t)}\right)^{2k_i} \frac{(x_i y_i)^{k_i}}{k_i!\Gamma(k_i + \nu_i/a_i)}. \end{aligned}$$

4.2. *Integral representation in the finite-dimensional case.* The series representation of $p_t^{\nu,\Psi,0}$ obtained in Theorem 4.1 clearly indicates the long-time behavior of $p_t^{\nu,\Psi,0}$ since $c_t \to 0$ exponentially, but it does not easily show the short-time asymptotics. To this end, we will give below in the finite-dimensional case an alternative integral representation. This problem is equivalent to the classical problem of finding integral representations for products of (modified) Bessel functions. So the main difficulty arises in the case $\min_{1\le i\le d} \nu_i/a_i < \frac{1}{2}$.

For $\alpha, \beta > 0$, denote by $\nu_{\alpha,\beta}$ the Beta distribution on [0, 1] with parameters $\alpha$ and $\beta$. Recall that

$$\nu_{\alpha,\beta}(du) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha-1}(1-u)^{\beta-1}\,du.$$

PROPOSITION 4.4. *Assume that* $\min_{1\le i\le d} \nu_i/a_i \ge \frac{1}{2}$. *Then*

$$\begin{aligned} p_t^{\nu,\Psi,0}(x,y) &= \prod_{i=1}^d \left(\frac{b_i}{a_i}\frac{1}{1-\exp(-b_i t)}\right)^{\nu_i/a_i} \frac{y_i^{\nu_i/a_i-1}}{\Gamma(\nu_i/a_i)} \\ &\quad \times E\left[\exp\left(-\sum_{i=1}^d \frac{b_i}{a_i}\frac{(\exp(-b_i t/2)\sqrt{x_i} - Z_i\sqrt{U_i}\sqrt{y_i})^2}{1-\exp(-b_i t)}\right)\right. \\ &\qquad\qquad \left.\times \exp\left(-\sum_{i=1}^d \frac{b_i}{a_i}\frac{y_i(1-U_i)}{1-\exp(-b_i t)}\right)\right], \end{aligned} \tag{4.4}$$

*where* $Z_i, U_i$ *are independent*, $P[Z_i = +1] = P[Z_i = -1] = \frac{1}{2}$ *and* $U_i$ $\nu_{1/2,\nu_i/a_i-1/2}$ *distributed*, $1 \le i \le d$. *Here we let* $\nu_{1/2,0} = \delta_1$.

PROOF. It suffices to consider the case $d = 1$ only. Hence,

$$p_t^{\nu,\Psi,0}(x,y) = \left(\frac{b}{a}\frac{1}{1-e^{-bt}}\right)^{\nu/a} \frac{y^{\nu/a-1}}{\Gamma(\nu/a)} h(x,y),$$

where

$$h(x,y) = \exp\left(-\frac{b}{a}\frac{e^{-bt}x+y}{1-e^{-bt}}\right)\sum_{n=0}^\infty \left(\frac{b}{a}\frac{e^{-bt/2}}{1-e^{-bt}}\right)^{2n} \frac{(xy)^n\Gamma(\nu/a)}{n!\Gamma(n+\nu/a)}.$$

Using the fact that, for $p > \frac{1}{2}$,

$$\frac{\Gamma(p)}{n!\Gamma(n+p)} = \frac{1}{n!\Gamma(n+\frac{1}{2})}\frac{\Gamma(p)}{\Gamma(p-\frac{1}{2})}\int_0^1 u^{n-1/2}(1-u)^{p-3/2}\,du\ ,$$

Legendre's formula $n!\Gamma(n+\frac{1}{2}) = (\sqrt{\pi}/2^{2n})(2n)!$ implies that

$$\frac{\Gamma(p)}{n!\Gamma(n+p)} = \frac{2^{2n}}{(2n)!}\int_0^1 u^n \nu_{1/2,p-1/2}(du).$$

The last formula also holds for $p = \frac{1}{2}$. Consequently,

$$
\begin{aligned}
h(x,y) &= \exp\biggl(-\frac{b}{a}\frac{e^{-bt}x+y}{1-e^{-bt}}\biggr) \\
&\quad \times \sum_{n=0}^{\infty}\biggl(\frac{b}{a}\frac{e^{-bt/2}}{1-e^{-bt}}\biggr)^{2n}\int_0^1 \frac{2^{2n}(xyu)^n}{(2n)!}\,\nu_{1/2,\nu/a-1/2}(du) \\
&= \frac{1}{2}\exp\biggl(-\frac{b}{a}\frac{e^{-bt}x+y}{1-e^{-bt}}\biggr) \\
&\quad \times \int_0^1 \exp\biggl(2\frac{b}{a}\frac{e^{-bt/2}}{1-e^{-bt}}\sqrt{xyu}\biggr) \\
&\qquad\qquad + \exp\biggl(-2\frac{b}{a}\frac{e^{-bt/2}}{1-e^{-bt}}\sqrt{xyu}\biggr)\nu_{1/2,\nu/a-1/2}(du) \\
&= \frac{1}{2}\int_0^1 \biggl(\exp\biggl(-\frac{b}{a}\frac{(e^{-bt/2}\sqrt{x}-\sqrt{yu}\,)^2}{1-e^{-bt}}\biggr) \\
&\qquad\qquad + \exp\biggl(-\frac{b}{a}\frac{(e^{-bt/2}\sqrt{x}+\sqrt{yu}\,)^2}{1-e^{-bt}}\biggr)\biggr) \\
&\qquad\quad \times \exp\biggl(-\frac{b}{a}\frac{y(1-u)}{1-e^{-bt}}\biggr)\nu_{1/2,\nu/a-1/2}(du). \qquad \square
\end{aligned}
$$

REMARK 4.5. Let us mention as a particular example the case $\nu_i/a_i = \frac{1}{2}$, $1 \le i \le d$. In this case, (4.4) reduces to the simple formula

$$
\begin{aligned}
p_t^{\nu,\Psi,0}(x,y) &= \prod_{i=1}^{d}\sqrt{\frac{b_i}{a_i}\frac{1}{1-\exp(-b_i t)}}\frac{1}{\sqrt{\pi}\sqrt{y_i}} \\
&\qquad \times E\biggl[\exp\biggl(-\sum_{i=1}^{d}\frac{b_i}{a_i}\frac{(\exp(-b_i t/2)\sqrt{x_i}-Z_i\sqrt{y_i}\,)^2}{1-\exp(-b_i t)}\biggr)\biggr] \\
&= \prod_{i=1}^{d}\sqrt{\frac{b_i}{a_i}\frac{1}{1-\exp(-b_i t)}}\frac{1}{\sqrt{\pi}\sqrt{y_i}}\frac{1}{2} \\
&\qquad \times \biggl(\exp\biggl(-\frac{b_i}{a_i}\frac{(\exp(-b_i t/2)\sqrt{x_i}-\sqrt{y_i}\,)^2}{1-\exp(-b_i t)}\biggr) \\
&\qquad\qquad + \exp\biggl(-\frac{b_i}{a_i}\frac{(\exp(-b_i t/2)\sqrt{x_i}+\sqrt{y_i}\,)^2}{1-\exp(-b_i t)}\biggr)\biggr).
\end{aligned}
$$

This representation of the heat kernel very much reminds one of the heat kernel of the Ornstein–Uhlenbeck process on $\mathbb{R}_+^d$ with reflecting boundary conditions. To understand this connection, let $Lf(x) := \sum_{i=1}^d a_i\, \partial_{ii}^2 f(x) - \sum_{i=1}^d b_i x_i\, \partial_i f(x)$ be the generator of the $d$-dimensional Ornstein–Uhlenbeck operator with diffusion matrix $\operatorname{diag}(a_1, \dots, a_d)$ and linear drift $-\operatorname{diag}(b_1, \dots, b_d)$ and consider the transformation $T : \mathbb{R}^d \to \mathbb{R}_+^d$, $x \mapsto (x_1^2, \dots, x_d^2)$. Then

$$L(f \circ T)(x) = \sum_{i=1}^d 4a_i T_i(x)\, \partial_{ii}^2 f\bigl(T(x)\bigr) + \sum_{i=1}^d \bigl(2a_i - 2b_i T_i(x)\bigr)\, \partial_i f\bigl(T(x)\bigr),$$

which implies that the image of $L$ under the transformation $T$ is precisely the generator of the superprocess on $\mathbb{R}_+^d$ with branching mechanism $\Psi(i, \lambda) = -4a_i\lambda^2 - 2b_i\lambda$, $\lambda \in \mathbb{R}_+$, and immigration $\nu = 2(a_1, \dots, a_d)$. Similarly, the image of the corresponding invariant measure of $L$ [the Gaussian distribution with mean 0 and covariance matrix $\operatorname{diag}(a_1 b_1^{-1}, \dots, a_d b_d^{-1})$] is precisely the Gamma distribution $\Gamma_{b/2a, 1/2}$.

The integral representation of $p_t^{\nu,\Psi,0}(x, y)$ immediately reveals the short-time asymptotics. For simplicity, let us only consider the particular case $\nu_i/a_i = \frac{1}{2}$, $1 \le i \le d$ (cf. Section 4.4 for the general case). In this case, it is easy to see that

$$\lim_{t \downarrow 0} t \log p_t^{\nu,\Psi,0}(x, y) = -\sum_{i=1}^d \frac{1}{a_i}\bigl(\sqrt{x_i} - \sqrt{y_i}\,\bigr)^2. \tag{4.5}$$

Indeed, we may assume that $d = 1$. Then

$$\begin{aligned}
&\lim_{t \downarrow 0} t \log p_t^{\nu,\Psi,0}(x, y) \\
&\quad = \lim_{t \downarrow 0} \frac{t}{2} \log\biggl(\frac{b}{a}\frac{1}{1 - e^{-bt}}\frac{1}{4\pi y}\biggr) \\
&\qquad - t\frac{b}{a}\biggl(\frac{(e^{-bt/2}\sqrt{x} - \sqrt{y}\,)^2}{1 - e^{-bt}}\biggr) \\
&\qquad + t \log\biggl(\exp\biggl(-\frac{b}{a}\frac{4e^{-bt/2}\sqrt{xy}}{1 - e^{-bt}}\biggr) + 1\biggr),
\end{aligned}$$

where we used $\log(r + s) = \log(r) + \log(s/r + 1)$, and, consequently,

$$\lim_{t \downarrow 0} t \log p_t^{\nu,\Psi,0}(x, y) = -\frac{1}{a}\bigl(\sqrt{x} - \sqrt{y}\,\bigr)^2.$$

Equation (4.5) will be generalized in Section 4.4, replacing $(\sum_{i=1}^d 1/a_i(\sqrt{x_i} - \sqrt{y_i}\,)^2)^{1/2}$ by a weighted Kakutani–Hellinger distance.

REMARK 4.6. One can use Proposition 4.4 to obtain integral representations of $p_t^{\nu,\Psi,0}(x, y)$ also for small $\nu_i/a_i$. To this end, let us consider for simplicity the one-dimensional case only. Using the fact that, for $p > 0$,

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\frac{b}{a}\frac{e^{-bt/2}}{1-e^{-bt}}\right)^{2n}\frac{(xy)^n\Gamma(p)}{n!\Gamma(n+p)}\\
&\quad=\sum_{n=0}^{\infty}\left(\frac{b}{a}\frac{e^{-bt/2}}{1-e^{-bt}}\right)^{2n}\frac{(xy)^n\Gamma(p+1)}{n!\Gamma(n+p+1)}\\
&\qquad+\left(\frac{b}{a}\frac{e^{-bt/2}}{1-e^{-bt}}\right)^{2}\frac{xy}{p(p+1)}\\
&\qquad\quad\times\sum_{n=0}^{\infty}\left(\frac{b}{a}\frac{e^{-bt/2}}{1-e^{-bt}}\right)^{2n}\frac{(xy)^n\Gamma(p+2)}{n!\Gamma(n+p+2)},
\end{aligned}$$

Proposition 4.4 now implies that

$$\begin{aligned}
p_t^{\nu,\Psi,0}(x,y)&=\left(\frac{b}{a}\frac{1}{1-e^{-bt}}\right)^{\nu/a}\frac{y^{\nu/a-1}}{\Gamma(\nu/a)}\\
&\quad\times\frac{1}{2}\int_0^1\left(\exp\left(-\frac{b}{a}\frac{(e^{-bt/2}\sqrt{x}-\sqrt{yu})^2}{1-e^{-bt}}\right)\right.\\
&\qquad\qquad\left.+\exp\left(-\frac{b}{a}\frac{(e^{-bt/2}\sqrt{x}+\sqrt{yu})^2}{1-e^{-bt}}\right)\right)\\
&\qquad\times\exp\left(-\frac{b}{a}\frac{y(1-u)}{1-e^{-bt}}\right)\\
&\qquad\times\left(\nu_{1/2,\nu/a+1/2}\right.\\
&\qquad\qquad\left.+\left(\frac{b}{a}\frac{e^{-bt/2}}{1-e^{-bt}}\right)^2\frac{xy}{\nu/a(\nu/a+1)}\nu_{1/2,\nu/a+3/2}\right)(du).
\end{aligned}$$

4.3. *Convergence in* $L^2(\Gamma_{\theta,\nu})$. It is shown in [12], Theorem 3.1, that $\Gamma_{a^{-1}b,a^{-1}\nu}$ is a symmetrizing measure for $(p_t^{\nu,\Psi,0})$ and $(L_\nu^{\Psi,0},\mathcal{F}C_0^2(C(S)_+))$ is essentially self-adjoint in $L^2(\Gamma_{a^{-1}b,a^{-1}\nu})$. The quadratic form associated to $L_\nu^{\Psi,0}$ is given by the closure $(\mathcal{E},D(\mathcal{E}))$ of

$$\mathcal{E}(F):=\int\left\langle\mu,a\left(\frac{\partial F}{\partial\delta_\cdot}\right)^2(\mu)\right\rangle\Gamma_{a^{-1}b,a^{-1}\nu}(d\mu),\qquad F\in\mathcal{F}C_0^2\big(C(S)_+\big),$$

in $L^2(\Gamma_{a^{-1}b,a^{-1}\nu})$. Clearly, $\mathcal{E}$ is a symmetric Dirichlet form, that is, $F^+\wedge 1\in D(\mathcal{E})$ for $F\in D(\mathcal{E})$ and $\mathcal{E}(F^+\wedge 1)\le\mathcal{E}(F)$ (cf. [3]). The following result has already been obtained in [12], Theorem 3.1. Using the explicit representation of $(\partial p_t^{\nu,\Psi,A} F/\partial\delta_x)(\mu)$ obtained in Corollary 2.3, we give an alternative proof here.

PROPOSITION 4.7. *Let $b_0 := \inf_{x\in S} b(x)$. Then $\mathcal{E}$ determines a Poincaré inequality with constant less than $b_0^{-1}$. Moreover, the transition semigroup of the $(0, \Psi)$-superprocess with immigration $\nu$ converges to equilibrium in $L^2(\Gamma_{a^{-1}b,a^{-1}\nu})$ with exponential rate $b_0$.*

For the proof, we need the following remarkable feature of random Gamma measures.

LEMMA 4.8. *Let $\theta \in C(S)_+$ and $\nu \in E$. Then*

$$\int_E \int_S \mu(dx) F(x,\mu)\Gamma_{\theta,\nu}(d\mu) = \int_S \nu(dx)\theta^{-1}(x)\biggl(\int_E F(x,\mu)\Gamma_{\theta,\nu+\delta_x}(d\mu)\biggr)$$

*for all $F : S \times E \to \mathbb{R}$, $F$ bounded and $\mathcal{B}(S\times E)$-measurable.*

PROOF. By monotone class theorems, it suffices to consider $F(x,\mu) = g(x)e^{-\langle f,\mu\rangle}$, $g, f \in C(S)_+$. Then

$$\begin{aligned}
&\int_E \int_S \mu(dx) F(x,\mu)\Gamma_{\theta,\nu}(d\mu) \\
&\quad = -\frac{d}{d\varepsilon}\int_E \exp(-\varepsilon\langle g,\mu\rangle - \langle f,\mu\rangle)\Gamma_{\theta,\nu}(d\mu)\bigg|_{\varepsilon=0} \\
&\quad = -\frac{d}{d\varepsilon}\exp\biggl(-\int \log\bigl(1+\theta^{-1}(\varepsilon g + f)\bigr)\,d\nu\biggr)\bigg|_{\varepsilon=0} \\
&\quad = \biggl\langle \nu, \frac{g}{\theta(1+\theta^{-1}f)}\biggr\rangle \exp\biggl(-\int \log(1+\theta^{-1}f)\,d\nu\biggr) \\
&\quad = \int_S \nu(dx)\theta^{-1}(x)\biggl(\int_E F(x,\mu)\Gamma_{\theta,\nu+\delta_x}(d\mu)\biggr). \qquad \square
\end{aligned}$$

PROOF OF PROPOSITION 4.7. Fix $F \in \mathcal{F}C_0^2(C(S)_+)$. Corollary 2.3 and Jensen's inequality imply that

$$\begin{aligned}
\mathcal{E}(p_t^{\nu,\Psi,0}F) &= \int \biggl\langle \mu, a\biggl(\frac{\partial p_t^{\nu,\Psi,0}F}{\partial\delta.}\biggr)^2(\mu)\biggr\rangle \Gamma_{a^{-1}b,a^{-1}\nu}(d\mu) \\
&\le \int_E \int_S \mu(dx) a(x) e^{-2b(x)t} p_t^{\nu+a\delta_x,\Psi,0} \\
&\qquad\qquad \times \biggl(\frac{\partial F}{\partial\delta_x}\biggr)^2(\mu)\Gamma_{a^{-1}b,a^{-1}\nu}(d\mu).
\end{aligned}$$

Using the last lemma and the $\Gamma_{a^{-1}b,a^{-1}(\nu+a\delta_x)}$-invariance of $p_t^{\nu+a\delta_x,\Psi,0}$, we obtain

$$\begin{aligned}
\mathcal{E}(p_t^{\nu,\Psi,0}F) &\le \exp(-2b_0t)\int_S \nu(dx)\frac{a^2(x)}{b(x)} \\
&\qquad\times\left(\int_E p_t^{\nu+a\delta_x,\Psi,0}\left(\frac{\partial F}{\partial\delta_x}\right)^2(\mu)\,\Gamma_{a^{-1}b,a^{-1}(\nu+a\delta_x)}(d\mu)\right) \\
&= \exp(-2b_0t)\int_S \nu(dx)\frac{a^2(x)}{b(x)}\left(\int_E \left(\frac{\partial F}{\partial\delta_x}\right)^2(\mu)\,\Gamma_{a^{-1}b,a^{-1}(\nu+a\delta_x)}(d\mu)\right) \\
&= \exp(-2b_0t)\int\left\langle\mu, a\left(\frac{\partial F}{\partial\delta_\cdot}\right)^2(\mu)\right\rangle\Gamma_{a^{-1}b,a^{-1}\nu}(d\mu) \\
&= \exp(-2b_0t)\mathcal{E}(F).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\int F^2\,d\Gamma_{a^{-1}b,a^{-1}\nu} &- \langle\Gamma_{a^{-1}b,a^{-1}\nu}, F\rangle^2 \\
&= -\int_0^\infty \frac{d}{dt}\|p_t^{\nu,\Psi,0}F\|^2_{L^2(\Gamma_{a^{-1}b,a^{-1}\nu})}\,dt \\
&= 2\int_0^\infty \mathcal{E}(p_t^{\nu,\Psi,0}F)\,dt \\
&\le 2\int_0^\infty \exp(-2b_0t)\,dt\,\mathcal{E}(F) \\
&= b_0^{-1}\mathcal{E}(F).
\end{aligned}$$

By density, the last inequality extends to all $F$ in the domain of the closure of $\mathcal{E}$, in particular, to all $F \in D(L_\nu^{\Psi,0})$. The convergence to equilibrium with an exponential rate $b_0$ of $(p_t^{\nu,\Psi,0})$ in $L^2(\Gamma_{a^{-1}b,a^{-1}\nu})$ now follows from standard semigroup theory. □

4.4. *Short-time asymptotics.* Since $L_\nu^{\Psi,0}$ is associated with a symmetric Dirichlet form, we can apply the general result obtained in [10] to study the short-time asymptotics of $p_t^{\nu,\Psi,0}(A, B)$ for any measurable subsets $A, B$ of $E$. To this end, note that $\mathcal{E}$ admits a carré du champ

$$\Gamma_a := D(\mathcal{E}) \times D(\mathcal{E}) \to L^1(\Gamma_{a^{-1}b,a^{-1}\nu}),$$

given by

$$\Gamma_a(F, G)(\mu) = 2\left\langle\mu, a\frac{\partial F}{\partial\delta_\cdot}(\mu)\frac{\partial G}{\partial\delta_\cdot}(\mu)\right\rangle.$$

For the terminology in this section, we refer to [3]. Clearly, $\Gamma_a$ is local. The corresponding intrinsic metric $d_a(A, B)$ associated with $\Gamma_a$ is defined by

$$d_a(A, B) = \sup_{F \in D(\mathcal{E}), \Gamma_a(F,F) \le 1} \inf_{\mu \in A, \overline{\mu} \in B} F(\mu) - F(\overline{\mu}).$$

Here "sup" and "inf" are meant in the essential sense. Using Proposition 4.7 and [10], Theorem 1.1 now imply that for any $A$ and $B$ with $\Gamma_{a^{-1}b, a^{-1}\nu}(A) > 0$ and $\Gamma_{a^{-1}b, a^{-1}\nu}(B) > 0$ it follows that

$$\lim_{t \downarrow 0} 2t \log p_t^{\nu, \Psi, 0}(A, B) = -d_a^2(A, B). \tag{4.6}$$

Here $p_t^{\nu, \Psi, 0}(A, B) := \int_B p_t^{\nu, \Psi, 0} \mathbb{1}_A \, d\Gamma_{a^{-1}b, a^{-1}\nu}$.

REMARK 4.9. Schied [11] identified the pointwise intrinsic metric $d_a$ which appears on the right-hand side of (4.6) in the particular case $a \equiv \frac{1}{2}$. To this end, define the weighted Kakutani–Hellinger distance

$$d_{KH,a}(\mu, \overline{\mu}) = \left( \int \left( \sqrt{\frac{d\mu}{d\eta}} - \sqrt{\frac{d\overline{\mu}}{d\eta}} \right)^2 a^{-1} \, d\eta \right)^{1/2}$$

for any $\mu, \overline{\mu} \in E$. Here $\eta \in E$ is any measure such that both $\mu$ and $\overline{\mu}$ are absolutely continuous w.r.t. $\eta$. Stated as his Theorem 1.2, Schied showed that

$$\sup_{F \in C_{\mathrm{b}}^1(E), \Gamma_{1/2}(F,F) \le 1} F(\mu) - F(\overline{\mu}) = d_{KH,1/2}(\mu, \overline{\mu}).$$

Here $C_{\mathrm{b}}^1(E)$ is defined as in Definition 2.2 if we replace the weak topology by the strong topology. Note that in contrast to [11] we assume in addition that $F$ is bounded. This is possible since one can approximate any $F$ by bounded $G$ with $\Gamma_a(G, G) \le \Gamma_a(F, F)$. It follows, for general $a \in C(S)_+$,

$$\sup_{F \in C_{\mathrm{b}}^1(E), \Gamma_a(F,F) \le 1} F(\mu) - F(\overline{\mu}) = d_{KH,a}(\mu, \overline{\mu}).$$

Indeed, since $F \in C_{\mathrm{b}}^1(E)$ if and only if $F_a(\mu) := F(a\mu) \in C_{\mathrm{b}}^1(E)$ and $(\partial F_a / \partial \delta_x)(\mu) = a(x)(\partial F / \partial \delta_x)(a\mu)$, it follows that $2\Gamma_{1/2}(F_a, F_a)(\mu) = \Gamma_a(F, F)(a\mu)$ and, consequently,

$$\begin{aligned} \sup_{F \in C_{\mathrm{b}}^1(E), \Gamma_a(F,F) \le 1} F(\mu) - F(\overline{\mu}) &= \sup_{F \in C_{\mathrm{b}}^1(E), \Gamma_{1/2}(F,F) \le 1/2} F(a^{-1}\mu) - F(a^{-1}\overline{\mu}) \\ &= d_{KH,a}(\mu, \overline{\mu}). \end{aligned}$$

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