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THE LOWEST CROSSING IN TWO-DIMENSIONAL CRITICAL PERCOLATION

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We study the following problem for critical site percolation on the triangular lattice. Let A and B be sites on a horizontal line e separated by distance n. Consider, in the half-plane above e, the lowest occupied crossing R_n from the half-line left of A to the half-line right of B. We show that the probability that R_n has a site at distance smaller than m from AB is of order $(\log(n/m))^{-1}$, uniformly in $1 \le m \le n/2$. Much of our analysis can be carried out for other two-dimensional lattices as well.

1. Introduction.

The idea of the "lowest" crossing between two boundary pieces of a domain is a well-known and useful tool in the study of two-dimensional percolation. Here we are interested in the question of how close the lowest crossing comes to the intermediate boundary piece it has to cross. To be specific, we fix the domain to be a half-plane and the two boundary pieces to be two disjoint half-lines.

1.1. *Statement of the main result.* Let $\mathbb{T}$ denote the triangular lattice. We note that much of our discussion applies to other lattices as well. We consider $\mathbb{T}$ as a subset of the Euclidean plane in such a way that the distance between two neighbor vertices of $\mathbb{T}$ is 1 and the integer points on the X-axis e are vertices of $\mathbb{T}$. For notational convenience, we denote these vertices on e by $\dots, -2, -1, 0, 1, 2, \dots$. Denote the site 0 by A and the site n by B. Let $l = (-\infty, A) \cap \mathbb{T}$, $r = (B, \infty) \cap \mathbb{T}$, and let $\mathbb{H}$ be the half-plane above (and including) e. Each site $v \in \mathbb{T}$ is *occupied* with probability p and *vacant* with probability $1 - p$, independently. The corresponding probability measure is denoted by Prob_p, and expectation by E_p. If S_1, S_2 are sets of sites, we say that S_1 is connected to S_2, or $S_1 \leftrightarrow S_2$, if there is a path of occupied sites that starts in S_1 and ends in S_2. We say that $S_1 \leftrightarrow S_2$ inside S_3 if all sites of the path are in S_3.

All constants below are strictly positive and finite. We write $a_n \asymp b_n$ to denote that there are constants C_1 and C_2 such that $C_1 a_n \le b_n \le C_2 a_n$. The exact values of constants denoted by C_i are not important to us, and C_i may have a different value from place to place.

REMARK. In the remainder of this paper, "path" will always mean "self-avoiding path" (i.e., a path that does not visit the same site more than once).

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THE LOWEST CROSSING. Consider all occupied paths between l and r that stay inside $\mathbb{H}$. If there is such a path, then there is a unique one closest to AB, call it R (we suppress the dependence of R on n). [See page 317 of Grimmett (1999) and Kesten (1982) for a discussion of the lowest crossing.] If R contains a site on AB, we call it a *contact point*.

We are only interested in contact points at criticality. This is because for $p < p_c$ the probability of an occupied crossing from l to r decays exponentially as $n \to \infty$. Also, it is not hard to see that for $p > p_c$ the fraction of those points on AB that are contact points is typically bounded away from 0. From now on, we set $p = 1/2$, the critical probability for site percolation on $\mathbb{T}$. We write $\mathrm{Prob}_{\mathrm{cr}}$ for $\mathrm{Prob}_{1/2}$. We note that by a Russo–Seymour–Welsh (RSW) argument [see Section 11.7 of Grimmett (1999), Theorem 6.1 of Kesten (1982), Russo (1978, 1981) and Seymour and Welsh (1978)], we have $\mathrm{Prob}_{\mathrm{cr}}(R \text{ exists}) = 1$.

Our main result is the following theorem.

THEOREM 1. *We have, uniformly in* $1 \le m \le n/2$,

$$\mathrm{Prob}_{\mathrm{cr}}(R \textit{ has distance } < m \textit{ from } AB) \asymp \big(\log(n/m)\big)^{-1}.$$

This theorem immediately implies (take $m = 1$) the following corollary.

COROLLARY 2.

$$\mathrm{Prob}_{\mathrm{cr}}(R \textit{ has a contact point}) \asymp (\log n)^{-1}, \qquad n \ge 1.$$

REMARKS. (i) Note that it is not even a priori obvious (and a new result in itself) that this probability goes to 0 as n goes to ∞ [see also (iv) below].

(ii) The statement of Theorem 1 is interesting only when m is small compared to n; when m is of the same order as n, the result follows easily from an RSW argument.

(iii) As to the case where $m \gg n$, a simple RSW argument shows that there exists an $\varepsilon > 0$ such that the probability that R has distance *larger* than λn from AB is smaller than $\lambda^{-\varepsilon}$, uniformly in n and $\lambda > 2$.

(iv) The only prerequisites needed in the proof are classical percolation results and techniques, in particular, the RSW techniques. We do not use SLE processes, which were introduced by Schramm and which have, by the work of him and other mathematicians, recently led to enormous progress [see Smirnov and Werner (2001) and the references given there]. In fact, we hope that Theorem 1 will be useful in the study of SLE_6. To illustrate this, note that Theorem 1 indicates that in the scaling limit when the lattice spacing goes to 0 and the length of AB is kept fixed (say 1), the distribution of the distance of R from AB satisfies

$$\mathrm{Prob}_{\mathrm{cr}}(R \text{ has distance } < a \text{ from } AB) \asymp \big(\log(1/a)\big)^{-1}, \qquad a < 1/2.$$

In the scaling limit, R corresponds to the boundary of the hull of the chordal SLE_6 process in the half-plane started from 0 and stopped at the first time it hits $(1, \infty)$ [see Corollary 5 of Smirnov (2001)]. In this way, one should obtain an analog of Theorem 1 in terms of SLE_6. The existence of a direct proof for SLE_6 of such a result is not known to us. Werner (private communication) has informed us that a (quite convoluted) "direct" proof of a weaker form of such a result for SLE_6 [namely, that the distance between the boundary of the hull and the interval $(0, 1)$ is a.s. strictly positive] will be included (among other results) in a joint paper by him, Lawler and Schramm.

(v) Schramm (2000) has proved that, for uniform spanning trees, the analog of the left-hand side of our Theorem 1 goes to 0 as m/n goes to 0, uniformly in n. Schramm (private communication) informed us recently that for that model the more precise behavior we obtained for percolation [i.e., the $(\log(n/m))^{-1}$ order] also seems to hold.

Apart from the above considerations, we think that Theorem 1 is interesting in itself.

1.2. *Notation, definitions and key ingredients.* The theorem follows from the proposition below. This proposition uses the knowledge of the critical exponent describing the scaling of the probability that there are two disjoint occupied paths in $\mathbb{H}$ that start at 0 and end at distance n. First, we give some additional definitions and notation.

For $n \geq 1$ and $v \in AB$, define the set

$$H_n(v) = \{u \in \mathbb{H} : |u - v| < n\},$$

where $|\cdot|$ is the graph distance from the origin. We are also going to need the half-annulus

$$H_{n,m}(v) \stackrel{\text{def}}{=} H_n(v) \setminus H_m(v) = \{u \in \mathbb{H} : m \leq |u - v| < n\}.$$

If S is a set of sites, we set

$$\partial S = \text{the set of sites in } S \text{ that have a neighbor in } S^c \cap \mathbb{H}$$

and

$$\bar{\partial} S = \text{the set of sites in } S^c \cap \mathbb{H} \text{ that have a neighbor in } S.$$

We define the event

$$D_n(v) = \bigl\{\exists \text{ two disjoint occupied paths from } \bar{\partial}\{v\} \text{ to } \partial H_n(v)\bigr\}.$$

Here, and later, "disjoint" means "vertex disjoint." We set

$$\rho(n) = \mathrm{Prob}_{\mathrm{cr}}\bigl(D_n(0)\bigr).$$

It is clear that this quantity will be important in our analysis: for a site $v \in AB$ to be a contact point, there must be two disjoint occupied paths from $\bar{\partial}\{v\}$ to the sets l and r, respectively; when v is in the bulk of AB, both sets have distance of order n from v.

We also need a version of D_n for $H_{n,m}(v)$. For $1 \le m < n$, let

$$D_{n,m}(v) = \{\exists \text{ two disjoint occupied paths from } \bar{\partial} H_m(v) \text{ to } \partial H_n(v)\},$$
$$\rho(n,m) = \mathrm{Prob}_{\mathrm{cr}}\big(D_{n,m}(0)\big).$$

We are going to need the following lemma about ρ. This lemma concerns the so-called "two-arm half-plane exponent." This exponent is exceptional in the sense that it can be derived in a quite elementary way, only using RSW, FKG and symmetry (the self-matching property of site percolation on the triangular lattice). It seems that this has been "known" for a while [see, e.g., the remark in Aizenman, Duplantier and Aharony (1999) that this exponent is "directly derivable"], but until recently there was (as far as we know) no explicit proof in the literature, although quite similar observations were made by Kesten, Sidoravicius and Zhang (1998) and Zhang (1999). Lawler, Schramm and Werner (2002), who needed such a lemma to bridge a step in the much more involved computation of other critical exponents, have included a proof in Appendix A of their paper.

LEMMA 3. (i) $\rho(n) \asymp n^{-1}, n > 1$;
(ii) $\rho(n,m) \asymp (n/m)^{-1}$, *uniformly in* $1 \le m < n$.

Finally, we state the following proposition. First, let

$$X_{n,m} = |\{0 \le k \le n/m : H_m(km) \text{ is visited by } R\}|, \qquad 1 \le m \le n/2.$$

PROPOSITION 4. *Uniformly in* $1 \le m \le n/2$, *with n a multiple of m, we have*:

(i) $E_{\mathrm{cr}} X_{n,m} \asymp 1$;
(ii) $E_{\mathrm{cr}}(X_{n,m} | X_{n,m} \ge 1) \asymp \log(n/m)$;
(iii) $E_{\mathrm{cr}} X_{n,m}^2 \asymp \log(n/m)$;
(iv) $\mathrm{Prob}_{\mathrm{cr}}(X_{n,m} \ge 1) \asymp (\log(n/m))^{-1}$.

1.3. *Outline.* The rest of the paper is organized as follows. In Section 2.1, we prove Proposition 4 from which, as we will see in Section 2.2, Theorem 1 follows immediately. The only part that uses the lattice structure in an essential way is the proof of the lemma. The rest can easily be modified to suit other two-dimensional lattices.

2. Proofs. We will make frequent use of the event defined below. We call a path π in the half-annulus $H_{n,m}(v)$ a *half-circuit* if it connects the two boundary pieces of $H_{n,m}(v)$ lying on the boundary of $\mathbb{H}$. Let

$$F_{n,m}(v) = \{\exists \text{ occupied half-circuit in } H_{n,m}(v)\}.$$

2.1. *Proof of Proposition* 4. Let R, A and B be as in Section 1 and let $1 \le m \le n/2$ with n a multiple of m. Observe that for $km \in AB$ we have

$$\begin{aligned} &\text{R visits } H_m(km) \qquad \text{if and only if} \\ &\qquad \exists \text{ occupied path from } l \text{ to } r \text{ that visits } H_m(km), \end{aligned} \tag{1}$$

and define the events

$$\begin{aligned} A_{k,n,m} &= \{\exists \text{ occupied path from } l \text{ to } r \text{ that visits } H_m(km)\} \\ &= \{R \text{ visits } H_m(km)\}, \qquad 0 \le k \le n/m. \end{aligned}$$

Since in what follows n and m are fixed, we simply write A_k for $A_{k,n,m}$. We can write

$$X_{n,m} = \sum_{0 \le k \le n/m} I[A_k],$$

where $I[\cdot]$ denotes the indicator of an event.

Throughout the proof, we will assume that $m \ge 2$. The proof for $m = 1$ is similar and, in part (ii), simpler.

PROOF OF (i). We start with a lower bound for $E_{\text{cr}} X_{n,m}$. By inclusion of events (see Figure 1) and the FKG inequality, we have

$$\begin{aligned} \text{Prob}_{\text{cr}}(A_k) &\ge \text{Prob}_{\text{cr}}\big(F_{2n,n}(km) \cap D_{2n,m/2}(km) \cap F_{m,m/2}(km)\big) \\ &\ge \text{Prob}_{\text{cr}}\big(F_{2n,n}(km)\big)\rho(2n, m/2)\text{Prob}_{\text{cr}}\big(F_{m,m/2}(km)\big) \end{aligned} \tag{2}$$

for any integer $k \in [0, n/m]$. Here and later, fractions are meant to be replaced by their integer parts whenever necessary. By an RSW argument, the first and third factors are bounded below by some constant C_1. Therefore, by Lemma 3, we have

$$E_{\text{cr}} X_{n,m} = \sum_{0 \le k \le n/m} \text{Prob}_{\text{cr}}(A_k) \ge C_1^2 C_2 (n/m)(n/m)^{-1} = C_1^2 C_2.$$

For the upper bound, we introduce the event

$$G_{n,m}(v) = \big\{\exists \text{ occupied path from } \bar{\partial} H_m(v) \text{ to } \partial H_n(v)\big\}, \qquad 1 \le m < n.$$

The scaling of $\text{Prob}_{\text{cr}}(G_{n,m})$ is known for the triangular lattice [see Theorem 3 of Smirnov and Werner (2001)]. However, for an argument valid on general lattices, we only use a power law upper bound. An RSW argument [in fact, a simple modification of Theorem 11.89 of Grimmett (1999)] shows that

$$\text{Prob}_{\text{cr}}(G_{n,m}) \le C_3 (n/m)^{-\mu} \tag{3}$$

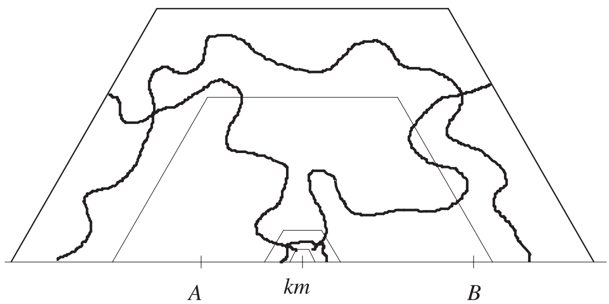

FIG. 1. *The events that force the occurrence of A_k.*

for some positive constants μ and C_3.

Let $1 \le k \le \frac{1}{2}(n/m)$ and assume that the event A_k occurs. Then it is easy to see that the events $D_{km,m}(km)$ and $G_{n/2,km}(km)$ both occur. Since these latter events are independent, we have, by Lemma 3 and (3),

$$\text{Prob}_{\text{cr}}(A_k) \le \text{Prob}_{\text{cr}}\big(D_{km,m}(km)\big)\text{Prob}_{\text{cr}}\big(G_{n/2,km}(km)\big) \le C_4 \frac{1}{k}\left(\frac{km}{n}\right)^{\mu}.$$

The sum of the right-hand side over these k's is bounded by some constant C_5 independent of n and m. A similar argument applies when $\frac{1}{2}(n/m) < k \le (n/m) - 1$. Finally, in the case $k = 0$ or $k = n/m$, we have $\text{Prob}_{\text{cr}}(A_k) \le 1$. This proves that $E_{\text{cr}} X_{n,m} \le C_6$. $\square$

PROOF OF THE LOWER BOUND IN PART (ii). The idea in this proof is, roughly speaking, as follows: if A_k occurs, there are from $H_m(km)$ disjoint occupied paths to l and r, respectively. Hence, to "let also A_j occur," it (almost) suffices to have two disjoint occupied paths from $H_m(jm)$ to the latter path, and this should, by RSW arguments, "cost" a probability of order $\text{Prob}_{\text{cr}}(D_{(j-k)m,m}(jm))$, which by the lemma is of order $1/(j-k)$. However, if one does the conditioning in a naive way, technical difficulties arise because "negative information can seep through." Therefore, the argument has to be done very carefully and an auxiliary event (which we will call F_k^* below) has to be introduced to "neutralize" this negative information. We now give the precise arguments.

Let V denote the first intersection of R with the set

$$U = \bigcup_{0 \le k \le n/m} H_m(km),$$

if such an intersection exists when R is traversed from left to right. For $v \in \partial U$, let $B_v = \{V = v\}$ and define k to be the index for which $v \in H_m(km)$, choosing the smaller if there are two of them. We prove the lower bound

$$\text{Prob}_{\text{cr}}(A_j | B_v) \ge \frac{C_1}{j-k} \qquad \text{for } k+4 \le j \le n/m - 1,\ 1 \le k \le n/(2m). \tag{4}$$

Let

$$R_1 = \text{the piece of } R \text{ to the left of } V\text{, including the site } V.$$

Also, define

$$S_1(v) = \begin{array}{l}\text{the lowest occupied path from } l \text{ to } v \text{ that is disjoint from } U, \\ \text{apart from the site } v,\end{array} \tag{5}$$

whenever there is at least one such path. Note that even when the event B_v does not hold, such paths may exist. We claim that on the event B_v we have $R_1 = S_1(v)$. Since $V = v$, we have that R_1 is disjoint from U, apart from v. If $S_1(v)$ were lower than R_1, then we would use $S_1(v)$ and the piece of R to the right of v to construct an occupied crossing lower than R, a contradiction.

For a path π, we write $\{S_1(v) = \pi\}$ as a shorthand for the event that $S_1(v)$ exists, and $S_1(v) = \pi$. The proof of the lower bound in (ii) is based on the following observation:

$$B_v = \bigcup_{\pi_1} \{S_1(v) = \pi_1\} \cap \Theta(\pi_1, v) \cap \Delta(\pi_1, v), \tag{6}$$

where

$$\Theta(\pi_1, v) = \left\{ \begin{array}{l} \exists \text{ vacant path } \pi_2^* \text{ from } \bar{\partial}\{v\} \text{ to } AB \text{ s.t. } \pi_1 \text{ is} \\ \text{the occupied path from } l \text{ to } v \text{ closest to } \pi_2^* \end{array} \right\},$$

$$\Delta(\pi_1, v) = \{\exists \text{ occupied path } \pi_3 \text{ from } \bar{\partial}\{v\} \text{ to } r \text{ disjoint from } \pi_1\},$$

and where the union is over all paths π_1 from l to v that are disjoint from U, apart from the site v. We will, for the time being, consider v as fixed, and, to simplify notation, write S_1, $\Theta(\pi_1)$ and $\Delta(\pi_1)$ instead of $S_1(v)$, and so on.

We first show that if B_v occurs, then the right-hand side of (6) occurs. Take $\pi_1 = R_1$. Then by the discussion following (5) the event $\{S_1 = \pi_1\}$ occurs. Since R is the lowest crossing, there is a vacant path from $\bar{\partial}\{v\}$ to AB. Take π_2^* to be the one closest to π_1. We claim that then π_1 is also the occupied path closest to π_2^*. Let ρ be an occupied path from l to v that is closer to π_2^* than π_1. Since π_2^* is below R, ρ is also below R. Now ρ together with the piece of R to the right of v forms an occupied crossing lower than R, a contradiction. This shows that $\Theta(\pi_1)$ occurs. Finally, taking π_3 to be the piece of R to the right of v shows that $\Delta(\pi_1)$ occurs.

Next, assume that the right-hand side of (6) occurs and choose the paths π_1, π_2^* and π_3 that show this. The fact that π_1, π_3 are occupied and that π_2^* is vacant implies that R exists and passes through v. Thus, R_1, the piece of R to the left of v, is defined. Also, R lies below the concatenation of π_1 and π_3. Since π_2^* is vacant, R_1 lies between π_1 and π_2^*. Since $\Theta(\pi_1)$ occurs, $R_1 = \pi_1 = S_1$, and hence v is the first intersection of R with U, that is, B_v occurs.

Now we are ready to start the argument for (4). By (6), we can write

$$\text{Prob}_{\text{cr}}(A_j \cap B_v) = \sum_{\pi_1} \text{Prob}_{\text{cr}}\big(\{S_1 = \pi_1\} \cap \Theta(\pi_1) \cap \Delta(\pi_1) \cap A_j\big). \tag{7}$$

Fix π_1 and on the event $\Delta(\pi_1)$ let $S_3(\pi_1)$ denote the highest occupied path from $\bar{\partial}\{v\}$ to r disjoint from π_1. The occurrence of the event $\{S_1 = \pi_1\}$ only depends on the states of v and the sites that are on or below π_1 but outside U. Let $\Omega(\pi_1)$ denote this set. For fixed π_1, the occurrence of $\{S_3(\pi_1) = \pi_3\}$ only depends on sites above the union of π_1 and π_3 and on the sites on π_3. Let $\Omega(\pi_1, \pi_3)$ denote this set. [It may happen, but is not harmful, that $\Omega(\pi_1) \cap \Omega(\pi_1, \pi_3) \neq \varnothing$.] We have

$$\Delta(\pi_1) = \bigcup_{\pi_3} \{S_3(\pi_1) = \pi_3\}.$$

Thus, we can write

$$\text{Prob}_{\text{cr}}(A_j \cap B_v) = \sum_{\pi_1} \sum_{\pi_3} \text{Prob}_{\text{cr}}\big(\{S_1 = \pi_1, S_3(\pi_1) = \pi_3\} \cap \Theta(\pi_1) \cap A_j\big). \tag{8}$$

Now we construct events $K_{k,j}$ and F_k^* such that the events $K_{k,j}$ and $\{S_1 = \pi_1, S_3(\pi_1) = \pi_3\} \cap \Theta(\pi_1)$ are conditionally independent given F_k^*, and moreover (on the event B_v) $K_{k,j}$ forces the occurrence of A_j. Let ω denote the configuration of occupied and vacant sites in $\mathbb{H}$ and define the configuration ω' by setting it equal to a new independent configuration on $\Omega(\pi_1) \cup \Omega(\pi_1, \pi_3)$ and equal to ω on $\mathbb{H} \setminus (\Omega(\pi_1) \cup \Omega(\pi_1, \pi_3))$. We let

$$F_k^* = \big\{\text{on } \omega' \ \exists \text{ vacant half-circuit in } H_{2m,m}(km)\big\}.$$

If F_k^* occurs, then there is, in the configuration ω, a vacant path π_4^* between AB and π_3 creating a block. This means that

(9) the path π_2^* in the definition of $\Theta(\pi_1)$ can be chosen to lie on the left-hand side of π_4^*.

Next, we define $K_{k,j}$ as the event that each of the following four occurs on ω':

- $\exists$ two disjoint occupied paths from $\bar{\partial} H_{m/2}(jm)$ to $\partial H_{4(j-k+2)m}(jm)$ that avoid the set $H_{2m}(km)$;
- $F_{4(j-k+2)m,2(j-k+2)m}(jm)$;
- $F_{2(j-k+2)m,(j-k+2)m}(jm)$;
- $F_{m,m/2}(jm)$.

We note that the first event we require is "almost" $D_{4(j-k+2)m,m/2}(jm)$. The only difference between these two events is the avoidance condition, and it is easy to see that their probabilities differ at most a constant factor. Observe that if $K_{k,j}$ occurs, then there is a path π_5 that is occupied on ω', visits $H_m(jm)$ and has both endpoints to the left of $H_m(km)$ on the boundary of $\mathbb{H}$. Let u be a site on π_5 that is in $H_m(jm)$. If u is above the union of π_1 and π_3, then π_3 visits $H_m(jm)$.

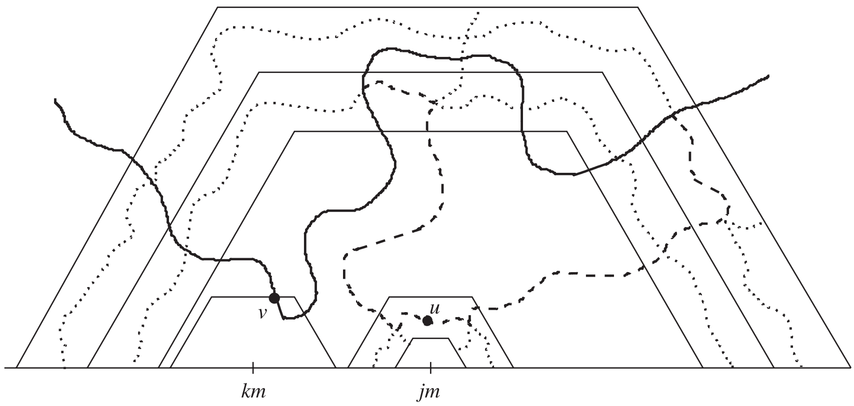

FIG. 2. *The dashed and dotted lines represent the event $K_{k,j}$ that forces the occurrence of A_j, given B_v. We used the dashed parts to construct a path that visits $H_m(jm)$.*

Otherwise, there are points $u', u'' \in \pi_5 \cap \pi_3$ separated by u, which implies that there is an occupied path (on ω) from $\bar{\partial}\{v\}$ to r that visits $H_m(jm)$ (see Figure 2). Thus, in both cases, A_j occurs.

By this observation and (8), we have

$$\begin{aligned}&\mathrm{Prob}_{\mathrm{cr}}(A_j \cap B_v)\\ &\quad \geq \sum_{\pi_1}\sum_{\pi_3} \mathrm{Prob}_{\mathrm{cr}}\big(\{S_1 = \pi_1, S_3(\pi_1) = \pi_3\} \cap \Theta(\pi_1) \cap F_k^* \cap K_{k,j}\big).\end{aligned} \tag{10}$$

By (9) and the construction of $K_{k,j}$, it follows that, given F_k^*, $K_{k,j}$ is conditionally independent of $\Theta(\pi_1) \cap \{S_1 = \pi_1,\ S_3(\pi_1) = \pi_3\}$. Moreover, $K_{k,j}$ is independent of F_k^*.

This gives that the right-hand side of (10) equals

$$\sum_{\pi_1}\sum_{\pi_3} \mathrm{Prob}_{\mathrm{cr}}\big(\{S_1 = \pi_1, S_3(\pi_1) = \pi_3\} \cap \Theta(\pi_1) \cap F_k^*\big)\mathrm{Prob}_{\mathrm{cr}}(K_{k,j}). \tag{11}$$

By the FKG inequality, Lemma 3 and RSW arguments, we have

$$\mathrm{Prob}_{\mathrm{cr}}(K_{k,j}) \geq C_2\rho\big(4(j-k+2)m, m\big) \geq \frac{C_3}{j-k}. \tag{12}$$

To deal with the rest of the expression on the right-hand side of (11), we condition on the configuration σ in $\Omega(\pi_1) \cup \Omega(\pi_1, \pi_3)$. Note that, for fixed π_1, π_3 and σ, the events $\Theta(\pi_1)$ and F_k^* are decreasing in the site variables in $\mathbb{H} \setminus (\Omega(\pi_1) \cup \Omega(\pi_1, \pi_3))$. Thus, the FKG inequality implies that

$$\begin{aligned}&\mathrm{Prob}_{\mathrm{cr}}\big(\{S_1 = \pi_1, S_3(\pi_1) = \pi_3\} \cap \Theta(\pi_1) \cap F_k^*\big)\\ &\quad \geq \mathrm{Prob}_{\mathrm{cr}}\big(\{S_1 = \pi_1, S_3(\pi_1) = \pi_3\} \cap \Theta(\pi_1)\big)\mathrm{Prob}_{\mathrm{cr}}(F_k^*)\\ &\quad \geq C_4 \mathrm{Prob}_{\mathrm{cr}}\big(\{S_1 = \pi_1, S_3(\pi_1) = \pi_3\} \cap \Theta(\pi_1)\big).\end{aligned} \tag{13}$$

The bounds (10)–(13) [and (6)] yield

$$\begin{aligned}\mathrm{Prob}_{\mathrm{cr}}(A_j \cap B_v) &\geq \frac{C_3\, C_4}{j-k} \sum_{\pi_1, \pi_3} \mathrm{Prob}_{\mathrm{cr}}\bigl(\{S_1 = \pi_1, S_3(\pi_1) = \pi_3\} \cap \Theta(\pi_1)\bigr) \\ &= \frac{C_3\, C_4}{j-k} \mathrm{Prob}_{\mathrm{cr}}(B_v).\end{aligned}$$

Summing over j gives, for v having x-coordinate at most $n/2$,

$$E_{\mathrm{cr}}(X_{n,m} | B_v) \geq C_3 \log(n/m). \tag{14}$$

Let

$$J = \{V \text{ has } x\text{-coordinate} \leq n/2\} = \bigcup_{v : v_x \leq n/2} B_v,$$

where the union is over all $v \in \partial U$ with x-coordinate at most $n/2$. By symmetry, $\mathrm{Prob}_{\mathrm{cr}}(J) \geq \frac{1}{2} \mathrm{Prob}_{\mathrm{cr}}(X_{n,m} \geq 1)$. This and (14) give

$$\begin{aligned}E_{\mathrm{cr}}(X_{n,m} | X_{n,m} \geq 1) &= \frac{E_{\mathrm{cr}}(X_{n,m}; X_{n,m} \geq 1)}{\mathrm{Prob}_{\mathrm{cr}}(X_{n,m} \geq 1)} \\ &\geq \frac{E_{\mathrm{cr}}(X_{n,m}; J)}{2\mathrm{Prob}_{\mathrm{cr}}(J)} \\ &= \frac{1}{2} E_{\mathrm{cr}}(X_{n,m} | J) \geq \frac{C_3}{2} \log\left(\frac{n}{m}\right).\end{aligned}$$

□

PROOF OF THE UPPER BOUND IN (iii). In bounding $\mathrm{Prob}_{\mathrm{cr}}(A_k \cap A_j)$, we may assume, by symmetry, that $k \leq j$ and $k \leq n/m - j$. We may further assume that $1 \leq k \leq j - 3$ by bounding $\mathrm{Prob}_{\mathrm{cr}}(A_k \cap A_j)$ by $\mathrm{Prob}_{\mathrm{cr}}(A_j)$ in the cases $k = 0, j-2, j-1, j$ and using (i). We separate three cases.

Case 1 [$j - k < 2k$]. Let $s = \lfloor (j-k-1)/2 \rfloor$ and $s' = \lfloor (j-k)/2 \rfloor$. (We have $s' = s$ if $j-k$ is odd, and $s' = s+1$ if $j-k$ is even.) It is a simple matter to check the inequalities $j - k \leq k + s' \leq n/(2m)$. It is not difficult to see that if $A_k \cap A_j$ occurs, then the following four events occur:

$$D_{sm,m}(km), \quad D_{sm,m}(jm), \quad D_{(k+s')m,(j-k)m}((k+s')m),$$

$$G_{n/2,(k+s')m}((k+s')m).$$

Also note that these events are independent. Thus, by Lemma 3 and (3),

$$\begin{aligned}\mathrm{Prob}_{\mathrm{cr}}(A_k \cap A_j) &\leq C_1 \frac{1}{s^2} \frac{j-k}{k+s'} \left(\frac{(k+s')m}{n/2}\right)^{\mu} \\ &\leq C_2 \frac{1}{(j-k)^2} \frac{j-k}{k} \left(\frac{km}{n}\right)^{\mu} \\ &= C_2 (j-k)^{-1} k^{\mu-1} \left(\frac{n}{m}\right)^{-\mu},\end{aligned} \tag{15}$$

where at the second inequality we used $k \le k + s' \le 2k$. The sum of the right-hand side of (15) over j is bounded by $C_3(\log k)k^{\mu-1}(n/m)^{-\mu}$. The sum of this quantity over k is bounded by $C_4(\log(n/m))(n/m)^{\mu}\,(n/m)^{-\mu} = C_4 \log(n/m)$.

Case 2 [$2k \le j-k \le 2(n/m-k)/3$]. Define s and s' as in Case 1. It is simple to check that $k \le s'$ and $k + s' + (j-k) \le n/m$. In this case, $A_k \cap A_j$ implies that the following independent events occur:

$$D_{km,m}(km), \quad G_{s'm,km}(km), \quad D_{sm,m}(jm), \quad G_{n-(k+s')m,(j-k)m}((k+s')m).$$

Thus, we have

$$\begin{aligned}\operatorname{Prob}_{\mathrm{cr}}(A_k \cap A_j) &\le C_5 \frac{1}{k}\left(\frac{k}{s'}\right)^{\mu} \frac{1}{s}\left(\frac{j-k}{n/m-k-s'}\right)^{\mu} \\ &\le C_6 \frac{1}{k}\left(\frac{k}{j-k}\right)^{\mu} \frac{1}{j-k}\left(\frac{j-k}{n/m}\right)^{\mu} \\ &\le C_6\, k^{\mu-1}(j-k)^{-1}\left(\frac{n}{m}\right)^{-\mu},\end{aligned} \tag{16}$$

where in the second step we used that $n/m - k - s' \ge n/(2m)$. The sum of the right-hand side over j is bounded by $C_7(\log(n/m))k^{\mu-1}(n/m)^{-\mu}$. The sum of this expression over k is bounded by $C_8(\log(n/m))(n/m)^{\mu}(n/m)^{-\mu} = C_8 \log(n/m)$.

Case 3 [$j - k > 2(n/m - k)/3$]. Our condition implies that (with s and s' as before) $k \le n/m - j < (j-k)/2$; hence, $k \le n/m - j \le s$. This time $A_k \cap A_j$ implies the following independent events:

$$D_{km,m}(km), \quad G_{sm,km}(km), \quad D_{(n/m-j)m,m}(jm), \quad G_{sm,(n/m-j)m}(jm).$$

This gives the bound

$$\begin{aligned}\operatorname{Prob}_{\mathrm{cr}}(A_k \cap A_j) &\le C_9 \frac{1}{k}\left(\frac{k}{s}\right)^{\mu} \frac{1}{n/m-j}\left(\frac{n/m-j}{s}\right)^{\mu} \\ &\le C_{10} \frac{1}{k}\left(\frac{k}{n/m}\right)^{\mu} \frac{1}{n/m-j}\left(\frac{n/m-j}{n/m}\right)^{\mu} \\ &\le C_{10} k^{\mu-1}(n/m-j)^{\mu-1}(n/m)^{-2\mu},\end{aligned} \tag{17}$$

where at the second inequality we used that $s \ge (j-k-2)/2 > (n/4m) - 1$. The sum of the right-hand side of (17) over j and k is bounded by some C_{11}.

The three cases and the remark about symmetry show that

$$E_{\mathrm{cr}} X_{n,m}^2 = \sum_{0 \le j,k \le n/m} \operatorname{Prob}_{\mathrm{cr}}(A_k \cap A_j) \le C_{12} \log(n/m).$$

□

PROOF OF (iv). From (i) and the lower bound in (ii), we get

$$\operatorname{Prob}_{\mathrm{cr}}(X_{n,m} \ge 1) = \frac{E_{\mathrm{cr}} X_{n,m}}{E_{\mathrm{cr}}(X_{n,m} | X_{n,m} \ge 1)} \le \frac{C_1}{C_2 \log(n/m)}. \tag{18}$$

On the other hand, by the Cauchy–Schwarz inequality,

$$E_{\mathrm{cr}}(X_{n,m}) = E_{\mathrm{cr}}(X_{n,m} I[X_{n,m} \geq 1]) \leq (E_{\mathrm{cr}} X_{n,m}^2)^{1/2} \big(\mathrm{Prob}_{\mathrm{cr}}(X_{n,m} \geq 1)\big)^{1/2}. \tag{19}$$

The upper bounds in (iii) and (i) imply $\mathrm{Prob}_{\mathrm{cr}}(X_{n,m} \geq 1) \geq C_3(\log(n/m))^{-1}$. □

PROOF OF THE UPPER BOUND IN (ii). The equalities in (18) and (i) and (iv) now give the upper bound in (ii). □

PROOF OF THE LOWER BOUND IN (iii). Similarly, (19) and (i) and (iv) give the lower bound in (iii). □

2.2. *Proof of Theorem* 1. The case where n is a multiple of m is (by the definition of $X_{n,m}$) clearly equivalent to part (iv) of Proposition 4. As to the general case, denote the probability in the statement of the theorem by $f(n, m)$. It is easy to see, using a simple RSW argument, that if $n' < n < n' + m$, then $f(n', m)$ and $f(n, m)$ differ at most a factor $C > 0$ which does not depend on n, n' and m. This observation, together with the special case, gives the general case.

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