The Annals of Probability
2003, Vol. 31, No. 1, 115–147

CONVERGENCE TO EQUILIBRIUM OF CONSERVATIVE PARTICLE SYSTEMS ON $\mathbb{Z}^d$

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We consider the Ginzburg–Landau process on the lattice $\mathbb{Z}^d$ whose potential is a bounded perturbation of the Gaussian potential. We prove that the decay rate to equilibrium in the variance sense is $t^{-d/2}$ up to a logarithmic correction, for any function u with finite triple norm; that is, $|||u||| = \sum_{x\in\mathbb{Z}^d} \|\partial_{\eta_x} u\|_\infty < \infty$.

1. Introduction. The rate of convergence to equilibrium is one of the main problems in the theory of Markov processes. It has recently attracted the attention of many authors in the context of symmetric conservative particle systems in finite and infinite volume [1, 2, 6, 7, 5]. In finite volume, the techniques used to obtain the rate of convergence to equilibrium rely mostly on the estimation of the spectral gap of the generator. In general, one shows that the generator of the particle system restricted to a cube of length N has a gap of order N^{-2} in any dimension. This estimate, together with standard spectral arguments, permits us to prove that the particle system restricted to a cube of size N decays to equilibrium in the variance sense at the exponential rate $\exp\{-ct/N^2\}$, that is, for any function f in L^2,

$$\|P_t f - E_\pi[f]\|_2^2 \le \exp\{-ct/N^2\}\|f - E_\pi[f]\|_2^2,$$

where $\{P_t,\ t \ge 0\}$ stands for the semigroup of the process, π for the invariant measure, $E_\pi[f]$ for the expectation of f with respect to π and $\|\cdot\|_2$ for the L^2-norm with respect to π.

In infinite volume, since the spectrum of the generator of a conservative system has no gap at the origin, instead of exponential convergence to equilibrium, one expects a polynomial convergence. By analogy with the noninteracting case and the heat equation in $\mathbb{R}^d$, which appears in the diffusive scaling limit of these models, one expects an algebraic decay to equilibrium for such models on $\mathbb{Z}^d$ at rate $O(t^{-d/2})$.

The traditional approach for algebraic decay to equilibrium is via Nash estimates [4]. The Nash inequality states that there exists a finite constant C_0 such that

$$\|f\|_2^2 \le CD(f)^{1/p}|||f|||^{2/q} \tag{1.1}$$

Received September 2000; revised January 2002.

[1]Supported in part by CNPq Grant 300358/93-8 and PRONEX 41.96.0923.00.

[2]Supported in part by NSF Grant DMS-97-03752.

AMS 2000 subject classifications. 60K35, 82A05.

Key words and phrases. Interacting particle systems, polynomial convergence to equilibrium, Nash inequality.

for all mean-zero functions f in L^2. Here, p, q are conjugate real numbers $[(1/p)+(1/q)=1]$, $D(f)$ is the Dirichlet form given by $D(f)=\langle(-L)f,f\rangle$, where L is the generator of the process, and $|||\cdot|||$ is an ad hoc norm.

The Nash inequality holds if the generator L has a positive spectral gap. In this case the L^2-norm may be taken as the triple norm $|||\cdot|||$ because $\|f\|_2^2 \le CD(f)$ for all mean-zero functions f.

A simple differential argument (cf. [10, 6]) permits deducing from the Nash inequality an algebraic decay to equilibrium in the variance sense at rate $t^{-1/(p-1)}$ provided the semigroup is a bounded operator with respect to the norm $|||\cdot|||$,

$$|||P_t f||| \le C_1 |||f||| \tag{1.2}$$

for some finite constant C_1 and all $t \ge 0$. More precisely, under assumptions (1.1) and (1.2), it is straightforward to show that there exists a finite constant C such that

$$\|P_t f\|_2^2 \le C \frac{|||f|||^2}{(1+t)^{1/(p-1)}}$$

for all mean-zero functions f with finite triple norm and all $t \ge 0$.

In the context of conservative interacting particle systems, Bertini and Zegarlinski [2] proved a Nash estimate for the speed change exclusion process with triple norm given by

$$|||f||| = \sum_{x\in\mathbb{Z}^d} \|\partial_{\eta_x} f\|_\infty,$$

where $\partial_{\eta_x} f$ measures the dependence of the function f on the site x,

$$(\partial_{\eta_x} f)(\eta) = f(\sigma^x \eta) - f(\eta).$$

Here σ^x is the operator that flips the spin at x. This model, however, is very special and the invariant measures are of product form (in other words, the infinite temperature case of lattice gases). Janvresse, Landim, Quastel and Yau [6] presented a very simple proof of the Nash inequality for a wide range of models including the lattice gases at high temperature. Furthermore, the triple norm used in [6] is weaker than the one used by [1] (and hence the result is stronger even for the simple exclusion process). Explicitly, the triple norm used in [6] for the case of the simple exclusion process is

$$|||f||| = \sum_{x\in\mathbb{Z}^d} \|\partial_{\eta_x} f\|_2.$$

Unfortunately, it seems to be a very difficult problem to show that the semigroup is a bounded operator for these triple norms. This has only been proved for the symmetric simple exclusion process. The classical approach is thus not very useful for conservative reversible dynamics and one needs to develop new ideas. Because this method is not appropriate, in [6], through ideas closely related to equilibrium fluctuations in the hydrodynamic limit theory, the authors prove that for every local function f in the symmetric zero-range model,

$$\|P_t f - \langle f\rangle\|_2^2 = \frac{C(f)}{t^{d/2}} + o(1/t^{d/2}),$$

where $C(f)$ is a constant explicitly computable.

More recently, under some strong mixing assumptions, [3] proved the existence, for any $\varepsilon > 0$ and local function f, of a finite constant $C_{f,\varepsilon}$ such that

$$\|P_t^{\tau,\Lambda,N} f - \langle f\rangle_{\tau,\Lambda,N}\|_2^2 \le C_{f,\varepsilon}\frac{1}{t^{\alpha-\varepsilon}},$$

where $P_t^{\tau,\Lambda,N}$ is the semigroup of the Kawasaki dynamics on a cube Λ with boundary condition τ and $0 \le N \le |\Lambda|$ particles, while $\alpha = 1/2$ in $d = 1$ and $\alpha = 1$ in $d \ge 2$.

2. Notation and results. Denote by $\mathbb{Z}^d$ the d-dimensional lattice. For each x in $\mathbb{Z}^d$, let η_x be the charge at site x. The charges $\eta = \{\eta_x,\ x \in \mathbb{Z}^d\}$ undergo a diffusion whose infinitesimal generator $\mathcal{L}$ is given by

$$\mathcal{L} = \frac{1}{2}\sum_{\substack{x,y\in\mathbb{Z}^d\\|x-y|=1}} (\partial_{\eta_x} - \partial_{\eta_y})^2 - \frac{1}{2}\sum_{\substack{x,y\in\mathbb{Z}^d\\|x-y|=1}} \big(V'(\eta_y) - V'(\eta_x)\big)(\partial_{\eta_y} - \partial_{\eta_x}),$$

where $V:\mathbb{R}\to\mathbb{R}$ is some potential. We shall assume that:

(H1) $\|V''\|_\infty < \infty,\ [V'(a)]^2 \le C_0(1+a^2)$ for some finite constant C_0.

(H2) $V(a) = (1/2)a^2 + F(a)$ where $F:\mathbb{R}\to\mathbb{R}$ is a smooth function such that $\|F\|_\infty < \infty,\ \|F'\|_\infty < \infty$.

We make some comments on these assumptions at the end of this section. Denote by $Z:\mathbb{R}\to\mathbb{R}$ the partition function

$$Z(\lambda) = \int_{-\infty}^{\infty} e^{\lambda a - V(a)}\,da, \tag{2.1}$$

by $R:\mathbb{R}\to\mathbb{R}$ the density function $\partial_\lambda \log Z(\lambda)$, which is smooth and strictly increasing, and by Φ the inverse of R so that

$$\alpha = \frac{1}{Z(\Phi(\alpha))}\int_{-\infty}^{\infty} a\, e^{\Phi(\alpha)a - V(a)}\,da$$

for each α in $\mathbb{R}$.

For λ in $\mathbb{R}$, denote by $\bar\nu_\lambda$ the product measure on $\mathbb{R}^{\mathbb{Z}^d}$ defined by

$$\bar\nu_\lambda(d\eta) = \prod_{x\in\mathbb{Z}^d}\frac{1}{Z(\lambda)}e^{\lambda\eta_x - V(\eta_x)}\,d\eta_x,$$

let $\nu_\alpha = \bar\nu_{\Phi(\alpha)}$ and notice that $E_{\nu_\alpha}[\eta_0] = \alpha$ for all α in $\mathbb{R}$. An elementary computation shows that the product measures $\{\nu_\alpha,\ \alpha\in\mathbb{R}\}$ are reversible for the Markov process with generator $\mathcal{L}$.

Define the triple norm $|||u|||$ of a local smooth function u by

$$|||u||| = \sum_{x\in\mathbb{Z}^d} \left\| \frac{\partial u}{\partial \eta_x} \right\|_\infty .$$

Denote by $\langle u; u\rangle_\alpha$ the variance of a local function u with respect to the measure ν_α. We claim that

$$\langle u; u\rangle_\alpha \le C_0 \sum_{j\ge 1} E_{\nu_\alpha}\left[\left(\frac{\partial u}{\partial \eta_{x_j}}\right)^2\right] \tag{2.2}$$

for some finite constant C_0. Indeed, let $\{x_j,\ j\ge 1\}$ be an enumeration of $\mathbb{Z}^d$. For $j\ge 1$, denote by $\mathcal{F}_j$ the σ-algebra generated by $\{\eta_{x_i},\ 1\le i\le j\}$ and by u_j the martingale $u_j = E_{\nu_\alpha}[u|\mathcal{F}_j]$. By convention, $u_0 = E_{\nu_\alpha}[u]$. With this notation we have that

$$\langle u; u\rangle_\alpha = \sum_{j\ge 0} E_{\nu_\alpha}\big[(u_j - u_{j+1})^2\big]. \tag{2.3}$$

Fix $j\ge 0$. Since $E_{\nu_\alpha}[(u_j - u_{j+1})^2] = E_{\nu_\alpha}[E_{\nu_\alpha}[(u_j - u_{j+1})^2|\mathcal{F}_j]]$, we will examine the latter conditional expectation, which is such that

$$E_{\nu_\alpha}\big[(u_j - u_{j+1})^2 \mid \mathcal{F}_j\big] = E_{\nu_\alpha}^{\eta_{x_{j+1}}}\big[\{u_{j+1} - u_j\}^2\big].$$

The notation $E_{\nu_\alpha}^{\eta_{x_{j+1}}}$ means that the variables $\{\eta_{x_1},\dots,\eta_{x_j}\}$ are considered as constants and are not integrated. The previous expectation concerns therefore the variable $\eta_{x_{j+1}}$ only. We have thus the variance of the function u_{j+1} of one variable. By the spectral gap for the Glauber dynamics, there exists a finite constant C_0 such that the previous expression is bounded by

$$C_0 E_{\nu_\alpha}^{\eta_{x_{j+1}}}\left[\left(\frac{\partial u_{j+1}}{\partial \eta_{x_{j+1}}}\right)^2\right] = C_0 E_{\nu_\alpha}^{\eta_{x_{j+1}}}\left[\left(E_{\nu_\alpha}\left[\frac{\partial u}{\partial \eta_{x_{j+1}}} \,\Big|\, \mathcal{F}_{j+1}\right]\right)^2\right]$$

because $\partial_{\eta_{x_{j+1}}} u_{j+1} = E_{\nu_\alpha}[\partial_{\eta_{x_{j+1}}} u|\mathcal{F}_{j+1}]$. In particular, by the Schwarz inequality,

$$E_{\nu_\alpha}\big[(u_j - u_{j+1})^2 \mid \mathcal{F}_j\big] \le C_0 E_{\nu_\alpha}\left[\left(\frac{\partial u}{\partial \eta_{x_{j+1}}}\right)^2 \,\Big|\, \mathcal{F}_j\right].$$

In view of this estimate and of (2.3),

$$\langle u; u\rangle_\alpha \ \le\ C_0 \sum_{j\ge 1} E_{\nu_\alpha}\left[\left(\frac{\partial u}{\partial \eta_{x_j}}\right)^2\right],$$

which proves the claim. In particular, by definition of the triple norm,

$$\langle u; u\rangle_\alpha \le C_0 |||u|||^2. \tag{2.4}$$

Denote by P_t the semigroup associated to the generator $\mathcal{L}$ in $L^2(\nu_\alpha)$ and by $\mathrm{Var}(\nu, u)$ the variance, with respect to a probability measure ν, of a function u in $L^2(\nu)$. The main theorem of this article states that the process relaxes to equilibrium in $L^2(\nu_\alpha)$ at rate $t^{-d/2}$ with some logarithmic corrections.

THEOREM 2.1. *There exists a constant C depending only on d, α and the potential V such that for every $L^2(\nu_\alpha)$ function u with finite triple norm,*

$$\mathrm{Var}(\nu_\alpha, [P_t u]) \le \frac{C|||u|||^2(\log\{2+t\})^5}{(1+t)^{d/2}}$$

for all $t \ge 0$.

We conclude this section with some observations concerning the assumptions, the result and the approach.

REMARK 2.2. The logarithmic correction which appears in the statement of the theorem is spurious and comes from the method. It is probably possible to improve the exponent 5.

The proof of this theorem relies on a sharp estimate for the spectral gap of the generator restricted to finite cubes. In the context of the present article this result has been proved by [8] under assumptions (H2) using the martingale method introduced by [12]. It is only for this reason that we impose (H2). On the other hand, for technical reasons, we are led to show in the proof that the Ginzburg–Landau process starting from a Dirac measure in finite time reaches a finite entropy state (relative to some invariant measure). To prove such a result, which has been considered before in [11], we need assumptions (H1). Both assumptions can certainly be weakened.

In this article we have made some arbitrary choices to present the simplest proof of Theorem 2.1. We present here alternative approaches and indicate the modifications needed.

We first reduce the proof to a finite volume situation (i.e., to a case where the diffusion occurs in a cube with periodic boundary conditions). We could have kept to the lattice $\mathbb{Z}^d$. In this case we would have to estimate the global entropy with respect to ν_α at time t of the a process that starts from a Dirac measure. In order to avoid this estimate, we preferred to use the known fact that the lattice dynamics can be approximated by the finite volume one to reduce the original decay problem to the finite volume setting. The problem is to show that all estimates depend on the functions only through their triple norm.

The second idea is to introduce a small nonconservative Glauber dynamics. The reason for this is because our proof requires some sharp estimates on covariance terms. These estimates are obtained through the multiscale analysis and the large deviation techniques developed in [12, 13] to prove a logarithmic Sobolev inequality for conservative dynamics. In Sections 4 and 5 we prove such estimates with respect to the grand canonical measure ν_α. It is the presence of the Glauber dynamics, which drives a process starting from a Dirac measure to a grand canonical measure in finite time, that makes these estimates with respect to ν_α enough for our needs. Without the Glauber dynamics, the measure $\nu_{\Lambda_L,\mathbf{M}_q} P_t^{\Lambda_L}$ of the next section would be singular with respect to ν_α, so that we would have to compare $\nu_{\Lambda_L,\mathbf{M}_q} P_t^{\Lambda_L}$ with a canonical measure. This forces us to prove the inequalities presented in Sections 4 and 5 for canonical instead of grand canonical measures.

3. Decay to equilibrium. Fix an integer $L \geq 1$ and denote by Λ_L the cube $\{-L, \dots, L\}^d$. We shall consider the process $\eta(t)$ as evolving on Λ_L with periodic boundary conditions. We will prove Theorem 2.1 in this context with all estimates being uniform over L. This uniformity will permit extending the result to the lattice $\mathbb{Z}^d$.

For $L \geq 1$ fixed, consider the diffusion process $\{\eta_x(t),\ x \in \Lambda_L\}$ with generator $\mathcal{L}_{\Lambda_L}$ given by

$$\mathcal{L}_{\Lambda_L} = \frac{1}{2} \sum_{\substack{x,y\in\Lambda_L\\|x-y|=1}} (\partial_{\eta_x} - \partial_{\eta_y})^2 - \frac{1}{2} \sum_{\substack{x,y\in\Lambda_L\\|x-y|=1}} \big(V'(\eta_y) - V'(\eta_x)\big)(\partial_{\eta_y} - \partial_{\eta_x}).$$

In this formula, summation is performed modulo Λ_L. Denote by $P_t^{\Lambda_L}$ the semigroup associated to this generator and by $\nu_\alpha^{\Lambda_L}$ the product measure on $\mathbb{R}^{\Lambda_L}$ with marginals equal to the marginals of ν_α. Most of the time we will omit the superscript Λ_L of $P_t^{\Lambda_L}$, $\nu_\alpha^{\Lambda_L}$.

Fix α in $\mathbb{R}$ and a local function u in $L^2(\nu_\alpha^{\Lambda_L})$ with finite triple norm. Assume without loss of generality that u has mean zero with respect to ν_α: $E_{\nu_\alpha}[u] = 0$. We claim that there exists a constant C, independent of L, such that

$$\operatorname{Var}\big(\nu_\alpha, [P_t^{\Lambda_L} u]\big) \leq \frac{C |||u|||^2 (\log\{2+t\})^5}{(1+t)^{d/2}} \tag{3.1}$$

for all $t \geq 0$. Theorem 2.1 is a simple consequence of this result.

PROOF OF THEOREM 2.1. Fix a mean-zero function u in $L^2(\nu_\alpha)$ with finite triple norm. For $k \geq 1$, denote by $\mathcal{G}_k$ the σ-algebra generated by $\{\eta_x,\ x \in \Lambda_k\}$. Let $u_k = E[u|\mathcal{G}_k]$. By the Schwarz inequality,

$$E_{\nu_\alpha}\big[(P_t u)^2\big] \leq 2E_{\nu_\alpha}\big[(P_t u_k)^2\big] + 2E_{\nu_\alpha}\big[(P_t u - P_t u_k)^2\big].$$

Recall the definition of $P_t^{\Lambda_L}$. Since the infinite volume dynamics is approximated by the finite volume dynamics and since $|||u_k||| \le |||u|||$, it follows from (3.1) that

$$E_{\nu_\alpha}\big[(P_t u_k)^2\big] = \lim_{L\to\infty} E_{\nu_\alpha^{\Lambda_L}}\big[(P_t^{\Lambda_L} u_k)^2\big]$$
$$\le C_0 \frac{(\log\{2+t\})^5}{(1+t)^{d/2}} |||u_k|||^2 \le C_0 \frac{(\log\{2+t\})^5}{(1+t)^{d/2}} |||u|||^2.$$

Since this inequality holds for all k and since $\lim_{k\to\infty}\langle (P_t u - P_t u_k)^2\rangle \le \lim_{k\to\infty}\langle (u-u_k)^2\rangle = 0$ because u belongs to $L^2(\nu_\alpha)$,

$$E_{\nu_\alpha}\big[(P_t u)^2\big] \le C_0 \frac{(\log\{2+t\})^5}{(1+t)^{d/2}} |||u|||^2 + \lim_{k\to\infty} 2E_{\nu_\alpha}\big[(P_t u - P_t u_k)^2\big]$$
$$= C_0 \frac{(\log\{2+t\})^5}{(1+t)^{d/2}} |||u|||^2,$$

which proves Theorem 2.1. □

We now turn to the proof of (3.1). To detach the main ideas, the argument is divided in several steps. We specify in each step the assumptions needed on the potential V. We have two possible choices right at the beginning, either to prove the multiscale estimates presented in Sections 4 and 5 for canonical measures or to introduce a small nonconservative Glauber dynamics in the model and prove the multiscale estimates for grand canonical measures. While the first approach can be considered more natural, the second one is certainly simpler because the multiscale estimates, as we will see, are the main technical difficulty of the article and are already quite subtle for grand canonical measures. We therefore adopted the second approach.

For $\varepsilon > 0$, denote by $\mathcal{L}_{\Lambda_L}^{G,\varepsilon}$ the Glauber generator defined by

$$\mathcal{L}_{\Lambda_L}^{G,\varepsilon} = \frac{\varepsilon}{2} \sum_{x\in\Lambda_L} \partial_{\eta_x}^2 - \frac{\varepsilon}{2} \sum_{x\in\Lambda_L} [\Phi(\alpha) - V'(\eta_x)]\partial_{\eta_x}.$$

Notice that from our choice of the drift term, $\nu_\alpha^{\Lambda_L}$ is the unique invariant measure for the diffusion process with generator $\mathcal{L}_{\Lambda_L} + \mathcal{L}_{\Lambda_L}^{G,\varepsilon}$. Denote by $\{P_t^{\varepsilon,\Lambda_L},\ t\ge 0\}$ the semigroup associated to this latter generator.

We claim that (3.1) follows from the same estimate with $P_t^{\varepsilon,\Lambda_L}$ in place of $P_t^{\Lambda_L}$ if ε is appropriately chosen. This statement, which will be made precise below, is a consequence of the following result which states that $P_t^{\varepsilon,\Lambda_L} u$ and $P_t^{\Lambda_L} u$ are not too far in the L^2 distance.

LEMMA 3.1. *Assume that the potential V has a bounded second derivative $\|V''\|_\infty < \infty$. Fix $\varepsilon > 0$ and a function u in $L^2(\nu_\alpha^{\Lambda_L})$ with finite triple norm. There* *exists a constant C_0, depending only on the potential V and the dimension d, such that*

$$\langle (P_t^{\varepsilon,\Lambda_L} u - P_t^{\Lambda_L} u)^2 \rangle \le C_0 \, |||u|||^2 \varepsilon t(1+t)$$

for all $t \ge 0$. Here $\langle \cdot \rangle$ represents expectation with respect to ν_α.

PROOF. Let $S(t) = S^\varepsilon(t) = \langle (P_t^{\varepsilon,\Lambda_L} u - P_t^{\Lambda_L} u)^2 \rangle$. S is a positive function that vanishes at time 0. An elementary computation shows that S' is bounded above by

$$2\langle P_t^{\varepsilon,\Lambda_L} u, \mathcal{L}_{\Lambda_L}^{G,\varepsilon} P_t^{\varepsilon,\Lambda_L} u \rangle - 2\langle P_t^{\Lambda_L} u, \mathcal{L}_{\Lambda_L}^{G,\varepsilon} P_t^{\varepsilon,\Lambda_L} u \rangle,$$

where $\langle \cdot, \cdot \rangle$ stands for the inner product in $L^2(\nu_\alpha)$. Notice that the first term of this expression is negative. Schwarz inequality gives that the previous difference is bounded above by

$$-\langle P_t^{\Lambda_L} u, \mathcal{L}_{\Lambda_L}^{G,\varepsilon} P_t^{\Lambda_L} u \rangle = \frac{\varepsilon}{2} \sum_{x \in \Lambda_L} E_{\nu_\alpha}\left[\left(\frac{\partial P_t^{\Lambda_L} u}{\partial \eta_x} \right)^2 \right].$$

Denote by $R(t)$ the sum on the right-hand side. By Lemma 3.7 below, $R(t) \le C_0(1+t)|||u|||^2$ for some finite constant C_0. Therefore,

$$S'(t) \le C_0 \varepsilon (1+t) |||u|||^2.$$

This concludes the proof of the lemma because $S(0) = 0$. □

Fix a time $T > 0$, a mean-zero function u in $L^2(\nu_\alpha)$ with finite triple norm and set $\varepsilon = (1+T)^{-2-(d/2)}$. Assume that

$$\operatorname{Var}(\nu_\alpha, [P_T^{\varepsilon,\Lambda_L} u]) \le \frac{C |||u|||^2 (\log\{2+T\})^5}{(1+T)^{d/2}} \tag{3.2}$$

for some constant C depending only on the dimension d, on the potential V and on the density α. We claim that in this case (3.1) holds. Indeed, since $\langle [P_T^{\Lambda_L} u]^2 \rangle$ is less than or equal to $2\langle [P_T^{\varepsilon,\Lambda_L} u]^2 \rangle + 2\langle [P_T^{\varepsilon,\Lambda_L} u - P_T^{\Lambda_L} u]^2 \rangle$, by (3.2) and Lemma 3.1,

$$\langle [P_T^{\Lambda_L} u]^2 \rangle \le \frac{C |||u|||^2 (\log\{2+T\})^5}{(1+T)^{d/2}} + C\varepsilon (1+T)^2 |||u|||^2.$$

By definition of ε, the right-hand side is bounded by $C|||u|||^2(\log\{2+T\})^5 \times (1+T)^{-d/2}$, proving (3.1). Therefore, to prove (3.1) and, in consequence, Theorem 2.1, we only have to check that (3.2) holds.

We now turn to the proof of (3.2). Fix a time $T \ge 1$, a mean-zero function u in $L^2(\nu_\alpha)$ with finite triple norm and set $\varepsilon = (1+T)^{-2-(d/2)}$. Denote by $\mathcal{L}_{\Lambda_L}^\varepsilon$ the generator $\mathcal{L}_{\Lambda_L} + \mathcal{L}_{\Lambda_L}^{G,\varepsilon}$,

$$\mathcal{L}_{\Lambda_L}^\varepsilon = \mathcal{L}_{\Lambda_L} + \mathcal{L}_{\Lambda_L}^{G,\varepsilon}$$

and recall that we denote by $P_t^{\varepsilon,\Lambda_L}$ the associated semigroup.

Set $u_t = P_t^{\varepsilon,\Lambda_L} u$ and notice that u_t is the solution of the backward equation

$$\partial_t u_t = \mathcal{L}^{\varepsilon}_{\Lambda_L} u_t, \qquad u_0 = u. \tag{3.3}$$

Fix two constants $t_0 > 1$ and $b_0 > 2$. For $n \geq 1$, let $t_n = b_0^n t_0$. For $t_0 < t \leq T$, denote by $n(t)$ the largest integer n such that $t_n \leq t$. To keep notation simple we shall convey that $t_{n(t)+1} = t$. With this notation, we may write

$$\begin{aligned}(1+t)^{(d+2)/2}\langle u_t^2\rangle &- (1+t_0)^{(d+2)/2}\langle u_{t_0}^2\rangle \\ &= \sum_{j=0}^{n(t)} (1+t_{j+1})^{(d+2)/2}\langle u_{t_{j+1}}^2\rangle - (1+t_j)^{(d+2)/2}\langle u_{t_j}^2\rangle.\end{aligned}$$

Denote by $\{\tau_x,\ x \in \mathbb{Z}^d\}$ the group of translations on the configuration space: $\tau_x\eta$ is the configuration whose spin at y is given by $(\tau_x\eta)_y = \eta_{x+y}$ for all y in $\mathbb{Z}^d$. The translations extend naturally to functions and measures. Since the dynamics is translation invariant, τ_x and $P_t^{\varepsilon,\Lambda_L}$, $\mathcal{L}^{\varepsilon}_{\Lambda_L}$ commute, so that $\tau_x u_s = \tau_x P_s^{\varepsilon,\Lambda_L} u = P_s^{\varepsilon,\Lambda_L}\tau_x u = (\tau_x u)_s$. In particular, since ν_α is also translation invariant, the previous expression is equal to

$$\begin{aligned}\sum_{j=0}^{n(t)} \frac{1}{|\Lambda_L|} \sum_{x\in\Lambda_L} &\big\{(1+t_{j+1})^{(d+2)/2}\langle[(\tau_x u)_{t_{j+1}}]^2\rangle \\ &\qquad - (1+t_j)^{(d+2)/2}\langle[(\tau_x u)_{t_j}]^2\rangle\big\}.\end{aligned}$$

It follows from (3.3) and the previous formula that $(1+t)^{(d+2)/2}\langle u_t^2\rangle - (1+t_0)^{(d+2)/2}\langle u_{t_0}^2\rangle$ is bounded above by

$$\begin{aligned}\sum_{j=0}^{n(t)} \frac{1}{|\Lambda_L|} \sum_{x\in\Lambda_L} &\left\{\frac{(d+2)}{2}\int_{t_j}^{t_{j+1}} ds\,(1+s)^{d/2}\langle(\tau_x u_s)^2\rangle \right.\\ &\left.\qquad - 2\int_{t_j}^{t_{j+1}} ds\,(1+s)^{(d+2)/2} D_{\Lambda_L}(\nu_\alpha, \tau_x u_s)\right\}.\end{aligned}$$

In this formula, $D_{\Lambda_L}(\nu_\alpha,\cdot)$ stands for Dirichlet form associated to $\mathcal{L}_{\Lambda_L}$, ν_α,

$$D_{\Lambda_L}(\nu_\alpha, f) = -\int f\,\mathcal{L}_{\Lambda_L} f\, d\nu_\alpha.$$

In the case where Λ_L is replaced by $\mathbb{Z}^d$, we denote $D_{\Lambda_L}(\nu_\alpha, f)$ simply by $D(\nu_\alpha, f)$.

Notice that we neglected in the previous estimate a negative term, which corresponds to $\langle\tau_x u_s, \mathcal{L}^{G,\varepsilon}_{\Lambda_L}\tau_x u_s\rangle$. Indeed, until the end of the proof, we will never use the piece of the generator corresponding to the Glauber dynamics in our estimates. This nonconservative dynamics will only be used to turn a singular measure (with respect to the grand canonical measure ν_α) into an absolutely continuous measure after any positive time. Without this Glauber part, what we would have instead is that the singular measure would become instantaneously absolutely continuous with respect to an appropriate canonical measure. However, as we said earlier, we would like to avoid working with canonical measures.

Since the dynamics is translation invariant, $D_{\Lambda_L}(\nu_\alpha, \tau_x u_s) = D_{\Lambda_L}(\nu_\alpha, u_s)$ for all x. We may thus rewrite the previous expression as

$$\sum_{j=0}^{n(t)} \frac{1}{|\Lambda_L|} \sum_{x\in\Lambda_L} \frac{(d+2)}{2} \int_{t_j}^{t_{j+1}} ds\,(1+s)^{d/2}\langle(\tau_x u_s)^2\rangle \tag{3.4}$$
$$- 2\int_{t_0}^{t} ds\,(1+s)^{(d+2)/2} D_{\Lambda_L}(\nu_\alpha, u_s).$$

For a finite subset Λ of $\mathbb{Z}^d$, denote by ν_α^Λ the product measure on $\mathbb{R}^\Lambda$ with marginals equal to the marginals of ν_α. For each M in $\mathbb{R}$, let $\nu_{\Lambda,M}$ be the canonical measure on $\mathbb{R}^\Lambda$ with total spin equal to M. This is the product measure ν_α^Λ conditioned that the total spin on Λ is M,

$$\nu_{\Lambda,M}(\cdot) = \nu_\alpha^\Lambda\biggl(\cdot \biggm| \sum_{x\in\Lambda}\eta_x = M\biggr).$$

Note that the right-hand side does not depend on the particular choice of the parameter α. Expectation with respect to $\nu_{\Lambda,M}$ is denoted by $E_{\Lambda,M}$. From [8] we have the following spectral gap estimate.

THEOREM 3.2. *Under the assumptions* (H2) *stated in Section* 2 *on the potential V, there exists an universal constant $R_0 > 1$ such that for all $\ell \geq 2$ and all M in $\mathbb{R}$,*

$$E_{\nu_{\Lambda_\ell,M}}\bigl[(v - E_{\nu_{\Lambda_\ell,M}}[v])^2\bigr] \leq R_0 \ell^2 D_{\Lambda_\ell}(\nu_{\Lambda_\ell,M}, v)$$

for all v in $L^2(\nu_{\Lambda_\ell,M})$.

The second step in the proof of Theorem 2.1 consists in applying the spectral gap for the dynamics restricted to finite boxes in order to replace u_s by a function which depends only on the density of particles on boxes of length $O(\sqrt{s})$.

Let $\ell = \ell(s) = \sqrt{(2/[d+2]R_0)t_j}$ if s belongs to the interval $[t_j, t_{j+1})$. Let $\mathcal{R} = \{(2\ell+1)x,\ x \in \mathbb{Z}^d\}$ and consider an enumeration of this set: $\mathcal{R} = \{x_1, x_2, \dots\}$ such that $|x_j| \leq |x_k|$ for $j \leq k$. Let $\Omega_j = x_j + \Lambda_\ell$ and let $M_j = M_j(\eta)$ be the sum of η on Ω_j:

$$M_j = \sum_{x\in\Omega_j} \eta_x.$$

Let q denote the total number of cubes in Λ_L. Note that $q = O((L/\ell)^d)$. For each $j \geq 1$, denote by $\mathbf{M}_j$ the vector $(M_1, \dots, M_j)$.

For a function v in $L^2(\nu_\alpha^{\Lambda_L})$, denote by $B_{\ell,L} v$ the conditional expectation of v given $\mathbf{M}_q$,

$$B_{\ell,L} v = E_{\nu_\alpha}[v | M_1, \dots, M_q]. \tag{3.5}$$

LEMMA 3.3. *Under the assumptions* (H2) *stated in Section* 2 *on the potential* V, *for any* $v \in L^2(\nu_\alpha^{\Lambda_L})$,

$$E_{\nu_\alpha^{\Lambda_L}}\big[(v - B_{\ell,L} v)^2\big] \leq R_0 \ell^2 D_{\Lambda_L}(\nu_\alpha, v).$$

This lemma follows from the spectral gap stated in Theorem 3.2 and its proof can be found in [6]. From this lemma, the translation invariance of the Dirichlet form and the choice of ℓ we have that

$$\begin{aligned} &\frac{d+2}{2} \int_{t_j}^{t_{j+1}} ds\, (1+s)^{d/2} \big\langle (\tau_x u_s - B_{\ell,L} \tau_x u_s)^2 \big\rangle \\ &\qquad \leq \int_{t_j}^{t_{j+1}} (1+s)^{(d+2)/2} D(\nu_\alpha, u_s)\, ds \end{aligned}$$

for all x in Λ_L and all $0 \leq j \leq n(t)$. Since $\langle (\tau_x u_s)^2 \rangle = \langle (\tau_x u_s - B_{\ell,L} \tau_x u_s)^2 \rangle + \langle (B_{\ell,L} \tau_x u_s)^2 \rangle$, (3.4) is now bounded above by

$$\frac{d+2}{2} \sum_{j=0}^{n(t)} \frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} \int_{t_j}^{t_{j+1}} ds\, (1+s)^{d/2} \big\langle (B_{\ell,L} \tau_x u_s)^2 \big\rangle.$$

Up to this point, we have proved that

$$\begin{aligned} &(1+t)^{(d+2)/2} \langle u_t^2 \rangle - (1+t_0)^{(d+2)/2} \langle u_{t_0}^2 \rangle \\ &\qquad \leq \frac{d+2}{2} \int_{t_0}^{t} ds\, (1+s)^{d/2} \frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} \big\langle (B_{\ell,L} \tau_x u_s)^2 \big\rangle. \end{aligned} \tag{3.6}$$

To conclude the proof of Theorem 2.1, it remains to estimate the right-hand side of this formula. Fix an interval $[t_j, t_{j+1})$ in which $\ell(s)$ is constant. We shall estimate

$$\frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} \int_{t_j}^{t_{j+1}} ds\, (1+s)^{d/2} \big\langle (B_{\ell,L} \tau_x u_s)^2 \big\rangle. \tag{3.7}$$

Recall the definition of the canonical measures $\nu_{\Lambda,M}$, the one of the product measure ν_α^Λ and the decomposition of Λ_L into subcubes $\Omega_1, \dots, \Omega_q$ of side length $2\ell + 1$ with $\mathbf{M}_q = (M_1, \dots, M_q)$ the number of particles in each. Fix a vector $\mathbf{M}_q$ and denote by $\nu_{\Lambda_L, \mathbf{M}_q}$ the measure $d\nu_{\Omega_1, M_1} \otimes \cdots \otimes d\nu_{\Omega_q, M_q}$. Notice that $\nu_{\Lambda_L, \mathbf{M}_q}$ is not absolutely continuous with respect to ν_α, but that $\nu_{\Lambda_L, \mathbf{M}_q} P_t^{\varepsilon, \Lambda_L}$ has this property for any $t > 0$, due to the presence of the Glauber dynamics. For $t > 0$, let $f_t = f_t^{\varepsilon,\ell,L,\mathbf{M}_q}$ be the Radon–Nikodym derivative of $\nu_{\Lambda_L,\mathbf{M}_q} P_t^{\varepsilon,\Lambda_L}$ with respect to ν_α,

$$f_t(\eta) = f_t^{\varepsilon,\ell,L,\mathbf{M}_q}(\eta) = \frac{d\nu_{\Lambda_L,\mathbf{M}_q} P_t^{\varepsilon,\Lambda_L}}{d\nu_\alpha^{\Lambda_L}}. \tag{3.8}$$

Since ν_α is translation invariant and reversible, and since the dynamics is translation invariant, we have that

$$B_{\ell,L}\tau_x u_s = E_{\nu_\alpha}[\tau_x u_s \mid \mathbf{M}_q] = E_{\nu_\alpha}[u\,\tau_{-x} f_s]. \tag{3.9}$$

With this notation, we have that (3.7) is equal to

$$\frac{1}{|\Lambda_L|} \sum_{x\in\Lambda_L} \int_{t_j}^{t_{j+1}} ds\,(1+s)^{d/2} E_{\nu_\alpha}\Big[\big(E_{\nu_\alpha}\big[u\tau_{-x} f_s^{\varepsilon,\ell,L,\mathbf{M}_q}\big]\big)^2\Big],$$

where the first expectation is the expectation of the variable $\mathbf{M}_q$ with respect to ν_α.

Fix $\mathbf{M}_q$. By the Schwarz inequality, $\{E_{\nu_\alpha}[u\tau_{-x} f_s^{\varepsilon,\ell,L,\mathbf{M}_q}]\}^2$ is bounded above by

$$2\Big(E_{\nu_\alpha}\big[(u - B_{\ell,L}u)\big(\tau_{-x} f_s^{\varepsilon,\ell,L,\mathbf{M}_q}\big)\big]\Big)^2 + 2\Big(E_{\nu_\alpha}\big[(B_{\ell,L}u)\big(\tau_{-x} f_s^{\varepsilon,\ell,L,\mathbf{M}_q}\big)\big]\Big)^2. \tag{3.10}$$

We first estimate the second term of this expression. By definition of $f_s^{\varepsilon,\ell,L,\mathbf{M}_q}$,

$$E_{\nu_\alpha}\big[B_{\ell,L}u\tau_{-x} f_s^{\varepsilon,\ell,L,\mathbf{M}_q}\big] = B_{\ell,L}(P_s^{\varepsilon,\Lambda_L}\tau_x B_{\ell,L}u).$$

Therefore, since $B_{\ell,L}$ is a projection, since the semigroup $\{P_s^{\varepsilon,\Lambda_L},\ s \ge 0\}$ is a contraction in $L^2(\nu_\alpha)$ and since the dynamics is translation invariant,

$$\begin{aligned} E_{\nu_\alpha}\Big[\big(E_{\nu_\alpha}\big[B_{\ell,L}u\tau_{-x} f_s^{\varepsilon,\ell,L,\mathbf{M}_q}\big]\big)^2\Big] &= E_{\nu_\alpha}\big[\big(B_{\ell,L}(P_s^{\varepsilon,\Lambda_L}\tau_x B_{\ell,L}u)\big)^2\big] \\ &\le E_{\nu_\alpha}\big[(P_s^{\varepsilon,\Lambda_L}\tau_x B_{\ell,L}u)^2\big] \le E_{\nu_\alpha}\big[(B_{\ell,L}u)^2\big]. \end{aligned}$$

Recall the definition of the triple norm $|||\cdot|||$. We claim the following.

PROPOSITION 3.4. *Under the assumptions* (H2) *stated in Section* 2 *on the potential* V, *there exists a finite constant* C *depending only on* α, d *and the potential* V *such that*

$$E_{\nu_\alpha}\big[(B_{\ell,L}u)^2\big] \le C\ell^{-d}(\log\ell)^2\,|||u|||^2$$

for all mean-zero functions u *with finite triple norm.*

The factor $(\log\ell)^2$ in this formula is spurious and comes from the method. We postpone the proof of this proposition to Section 4. It follows from this result and the definition of $\ell(s)$ that the contribution to (3.7) of the second term of (3.10) is bounded by

$$C_0(\log t_j)^2\,|||u|||^2(t_{j+1} - t_j) \tag{3.11}$$

for some finite constant C_0 because $(1+t_{j+1}) \le 2t_{j+1} \le 2b_0 t_j$.

We now estimate the first term of (3.10). It relies on the next proposition, whose statement requires some notation. An unoriented bond b is a pair $b = \{b_1, b_2\}$ of nearest neighbor sites, so that b_1, b_2 are sites of $\mathbb{Z}^d$ and $\|b_2 - b_1\| = 1$ for the Euclidean norm. For an unoriented bond b, a measure μ and a density f with respect to μ, let $\mathfrak{D}_b(\mu, f)$ be the functional defined by

$$\mathfrak{D}_b(\mu, f) = 2\int \sqrt{f}\mathcal{L}_b\sqrt{f}\, d\nu_\alpha = E_\mu\big[\{\partial_{\eta_{b_1}}\sqrt{f} - \partial_{\eta_{b_2}}\sqrt{f}\}^2\big],$$

where $\mathcal{L}_b$ stands for the piece of the generator $\mathcal{L}_{\Lambda_L}$ (hence, without the nonconservative Glauber part) corresponding to the diffusion over the bond b. For $K \ge 1$, let

$$\mathfrak{D}_{\Lambda_K}(\mu, f) = \sum_{b\in\Lambda_K} \mathfrak{D}_b(\mu, f),$$

where the summation is carried over all unoriented bonds b in Λ_K.

PROPOSITION 3.5. *Under the assumptions* (H2) *stated in Section* 2 *on the potential V, there exists a finite constant C_0 depending only on α, d and the potential V such that for every density f and every function u with finite triple norm*,

$$\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L} \big(E_{\nu_\alpha}[(u - B_{\ell,L}u)\tau_x f]\big)^2 \le C_0 \frac{\ell^2(\log \ell)^4}{|\Lambda_L|} |||u|||^2 \mathfrak{D}_{\Lambda_L}(\nu_\alpha, f). \tag{3.12}$$

Here again, the logarithmic correction which appears in the statement is spurious and comes from the method. The proof of this proposition is postponed to Section 5.

It follows from this result that the contribution to (3.7) of the first term of (3.10) is bounded above by

$$\begin{aligned} C_0 |||u|||^2\, t_j(\log t_j)^4 \frac{|\Lambda_\ell|}{|\Lambda_L|}\int_{t_j}^{t_{j+1}} ds\, E_{\nu_\alpha}\big[\mathfrak{D}_{\Lambda_L}\big(\nu_\alpha, f_s^{\varepsilon,\ell,L,\mathbf{M}_q}\big)\big] \\ \le C_0 |||u|||^2 t_j(\log t_j)^4 \frac{|\Lambda_\ell|}{|\Lambda_L|}\int_{t_j}^{t_{j+1}} ds\, E_{\nu_\alpha}\big[\mathfrak{D}_{\varepsilon,\Lambda_L}\big(\nu_\alpha, f_s^{\varepsilon,\ell,L,\mathbf{M}_q}\big)\big], \end{aligned}$$

where

$$\mathfrak{D}_{\varepsilon,\Lambda_L}(\nu_\alpha, f) = \langle\sqrt{f}, (-\mathcal{L}^\varepsilon_{\Lambda_L})\sqrt{f}\rangle.$$

Notice that in the previous formula we replaced the functional $\mathfrak{D}$ restricted to $\Lambda_{2\ell}$ by the global functional.

For a density f with respect to ν_α, denote by $H(f)$ the entropy of $f\, d\nu_\alpha$ with respect to ν_α. It is well known that $\int_s^t dr\, \mathfrak{D}_{\varepsilon,\Lambda_L}(\nu_\alpha, f_r)$ is less than or equal to $H(f_s)$. The previous expression is therefore bounded above by

$$C_0 |||u|||^2 t_j(\log t_j)^4 \frac{|\Lambda_\ell|}{|\Lambda_L|} E_{\nu_\alpha}\big[H\big({f_{t_j}}^{\varepsilon,\ell,L,\mathbf{M}_q}\big)\big].$$

PROPOSITION 3.6. *Assume assumptions* (H1) *on the potential* V. *Let* $f_t^{\varepsilon,\ell,L,\mathbf{M}_q}$ *be defined as in* (3.8) *for* $t>0$. *There exists a finite constant* $C_0 = C_0(\alpha, d, V)$ *such that*

$$E_{\nu_\alpha}\big[H\big(f_t^{\varepsilon,\ell,L,\mathbf{M}_q}\big)\big] \le C_0\bigg(\frac{L}{\ell}\bigg)^d \{\log \ell + \log \varepsilon^{-1}\}$$

for all $t \ge 1$. *Here the expectation concerns the random vector* $\mathbf{M}_q$.

It is here and only here that we will use the Glauber part of the dynamics. The proof of this proposition is postponed to Section 6. It follows from this result and the previous estimate that the contribution to (3.7) of the first term of (3.10) is bounded above by

$$C_0 |||u|||^2 t_j (\log\{2+T\})^5$$

because $\varepsilon^{-1} = T^{2+(d/2)}$, $t_j \le t \le T$. In view of this bound and (3.11), we get that (3.7) is bounded above by

$$\begin{aligned} C_0 |||u|||^2 \{(\log t_j)^2 (t_{j+1} - t_j) + t_j (\log\{2+T\})^5\} \\ \le C_0\, |||u|||^2 (\log\{2+T\})^5 (t_{j+1} - t_j) \end{aligned}$$

because we assumed that $b_0 > 2$ so that $t_j \le t_{j+1} - t_j$ for $j < n(t)$. Summing over j, we obtain from (3.6) that

$$(1+t)^{(d+2)/2}\langle u_t^2\rangle - (1+t_0)^{(d+2)/2}\langle u_{t_0}^2\rangle \le C_0 |||u|||^2 t (\log\{2+T\})^5.$$

Since $P_t^{\varepsilon,\Lambda_L}$ is a contraction and u has mean zero, $\langle u_{t_0}^2\rangle \le \langle u; u\rangle$. Replace t by T in this formula. Equation (3.1) follows now from the previous estimate and (2.4).

We conclude this section with an estimate used above. For a function u in $L^2(\nu_\alpha)$, denote by $|||u|||_0$ the weak triple norm defined by

$$|||u|||_0^2 = \sum_{x\in\Lambda_L} E_{\nu_\alpha}\bigg[\bigg\{\frac{\partial u}{\partial \eta_x}\bigg\}^2\bigg].$$

LEMMA 3.7. *Let* u *be a mean-zero function in* $L^2(\nu_\alpha)$ *such that* $|||u|||_0^2 < \infty$. *Assume that the potential* V *has a bounded second derivative*: $\|V''\|_\infty < \infty$. *Then there exists a finite constant* C_0 *depending only on* $\|V''\|_\infty$ *and the dimension* d *such that*

$$R(t) = \sum_{x\in\Lambda_L} E_{\nu_\alpha}\bigg[\bigg\{\frac{\partial P_t^{\Lambda_L} u}{\partial \eta_x}\bigg\}^2\bigg] \le C_0(1+t)|||u|||_0^2.$$

This result has, of course, a similar version for the infinite volume dynamics. It may have an interest on its own. Since $\|\!|u|\!\|_0 \le \|\!|u|\!\|$, the result remains in force if we replace the weak triple norm defined above by the strong one.

PROOF OF LEMMA 3.7. To keep notation simple, let $u_t = P_t^{\Lambda_L} u$ and observe that $\partial_t u_t = \mathcal{L}_{\Lambda_L} u_t$. In particular,

$$R'(t) = 2 \sum_{x \in \Lambda_L} \left\langle \frac{\partial u_t}{\partial \eta_x}, \frac{\partial \mathcal{L}_{\Lambda_L} u_t}{\partial \eta_x} \right\rangle.$$

An elementary computation shows that this expression is bounded above by

$$2 \sum_{|x-y|=1} E_{\nu_\alpha}\left[\frac{\partial u_t}{\partial \eta_x} V''(\eta_x) \left\{ \frac{\partial u_t}{\partial \eta_y} - \frac{\partial u_t}{\partial \eta_x} \right\} \right].$$

Since we assumed V'' to be a bounded function, by the elementary inequality $2ab \le Aa^2 + A^{-1}b^2$, the previous expression is bounded above by

$$\frac{1}{A} R(t) + C_0 A \sum_{|x-y|=1} E_{\nu_\alpha}\left[\left\{ \frac{\partial u_t}{\partial \eta_y} - \frac{\partial u_t}{\partial \eta_x} \right\}^2 \right] \tag{3.13}$$

for all $A > 0$ and some finite constant depending only on the dimension d and $\|V''\|_\infty$. Notice that the second sum is $\langle u_t, (-\mathcal{L}_{\Lambda_L}) u_t \rangle$. Since the time derivative of $\langle u_t, u_t \rangle$ is $2\langle u_t, \mathcal{L}_{\Lambda_L} u_t \rangle$, we have that

$$\int_0^t ds \langle u_s, (-\mathcal{L}_{\Lambda_L}) u_s \rangle \le \langle u, u \rangle.$$

Therefore, recalling that $R'(t)$ is bounded by (3.13), taking time integrals in that formula, we obtain from the previous estimate that

$$R(t) \le R(0) + \frac{1}{A} \int_0^t ds\, R(s) + C_0 A \langle u, u \rangle,$$

or minimizing over A, that

$$R(t) \le R(0) + C_0 \|u\|_2 \left\{ \int_0^t ds\, R(s) \right\}^{1/2}$$

for all $t \ge 0$. It is not difficult to deduce from this differential inequality that

$$R(t) \le C_0 \{ R(0) + \langle u, u \rangle t \}.$$

The lemma follows from this inequality and estimate (2.2), which obviously also holds in the finite volume case. □

4. Proof of Proposition 3.4. We assume throughout this section hypothesis (H2) and only this one on the potential V.

For $0 \le j \le q$, denote by $u_j(M_1, \dots, M_j)$ the conditional expectation $E_{\nu_\alpha}[u| M_1, \dots, M_j]$. By convention $u_0 = E_{\nu_\alpha}[u]$, which vanishes because u has mean zero with respect to ν_α. Since $B_{\ell,L} u = u_q$, we may write

$$(4.1)\quad E_{\nu_\alpha}\big[(B_{\ell,L}u)^2\big] = E_{\nu_\alpha}\left[\left(\sum_{j=0}^{q-1} u_{j+1} - u_j\right)^2\right] = \sum_{j=0}^{q-1} E_{\nu_\alpha}\big[(u_{j+1} - u_j)^2\big].$$

Fix $0 \le j < q$ and notice that

$$E_{\nu_\alpha}\big[(u_{j+1} - u_j)^2\big] = E_{\nu_\alpha}\big[E_{\nu_\alpha}\big[(u_{j+1} - u_j)^2 \big| M_1, \dots, M_j\big]\big].$$

We will estimate the conditional expectation $E_{\nu_\alpha}[(u_{j+1} - u_j)^2 | M_1, \dots, M_j]$ in which the variables $M_1, \dots, M_j$ are fixed. Therefore, in the lines below the dependence of the functions on $M_1, \dots, M_j$ will be most of the time omitted. Let

$$R = E_{\nu_\alpha}\big[u | M_1, \dots, M_j, \{\eta_x,\ x \in \Omega_{j+1}\}\big].$$

Then $u_{j+1}(M_1, \dots, M_{j+1}) = E_{\nu_\alpha}[R | M_1, \dots, M_j, M_{j+1}] = E_{\Omega_{j+1}, M_{j+1}}[R]$. In the last expectation, as we announced before, the variables $M_1, \dots, M_j$ are treated as fixed constants and do not appear therefore in the expectation. From now on, R is considered as a function of $\{\eta_x,\ x \in \Omega_{j+1}\}$ only. With this convention, we have that

$$\begin{aligned} &E_{\nu_\alpha}\big[(u_{j+1} - u_j)^2 \big| M_1, \dots, M_j\big] \\ &\quad = E_{\Omega_{j+1},\alpha}\big[(E_{\Omega_{j+1}, M_{j+1}}[R] - E_{\Omega_{j+1},\alpha}[R])^2\big]. \end{aligned}$$

In this formula, $E_{\Omega_{j+1},\alpha}$ represents the expectation with respect to $\nu_\alpha^{\Omega_{j+1}}$.

By the spectral gap for the Glauber dynamics, there exists a finite constant C_0 such that

$$\begin{aligned} &E_{\Omega_{j+1},\alpha}\big[(E_{\Omega_{j+1}, M_{j+1}}[R] - E_{\Omega_{j+1},\alpha}[R])^2\big] \\ &\quad \le C_0 \sum_{x \in \Omega_{j+1}} E_{\Omega_{j+1},\alpha}\left[\left(\frac{\partial}{\partial \eta_x} E_{\Omega_{j+1}, M_{j+1}}[R]\right)^2\right]. \end{aligned}$$

Of course, $\partial_{\eta_x} E_{\Omega_{j+1}, M_{j+1}}[R]$ does not depend on x and a simple computation shows that

$$\begin{aligned} &\frac{\partial}{\partial \eta_x} E_{\Omega_{j+1}, M_{j+1}}[R] \\ &\quad = \frac{1}{|\Lambda_\ell|} \sum_{x \in \Omega_{j+1}} E_{\Omega_{j+1}, M_{j+1}}\left[\frac{\partial R}{\partial \eta_x}\right] - \frac{1}{|\Lambda_\ell|} E_{\Omega_{j+1}, M_{j+1}}\left[R; \sum_{x \in \Omega_{j+1}} V'(\eta_x)\right]. \end{aligned}$$

In this formula, $E[f;g]$ stands for the covariance of f and g: $E[f;g]=E[fg]-E[f]E[g]$. In particular, the variance of $E_{\Omega_{j+1},M_{j+1}}[R]$ is bounded above by

$$
\begin{aligned}
&2C_0|\Lambda_\ell|^{-1}E_{\Omega_{j+1},\alpha}\left[\left(\sum_{x\in\Omega_{j+1}}E_{\Omega_{j+1},M_{j+1}}\left[\frac{\partial R}{\partial\eta_x}\right]\right)^2\right]\\
&\qquad+2C_0|\Lambda_\ell|^{-1}E_{\Omega_{j+1},\alpha}\left[\left(E_{\Omega_{j+1},M_{j+1}}\left[R;\sum_{x\in\Omega_{j+1}}V'(\eta_x)\right]\right)^2\right].
\end{aligned}
\tag{4.2}
$$

The first term in (4.2) is easy to estimate. It is bounded by

$$2C_0|\Lambda_\ell|^{-1}\left(\sum_{x\in\Omega_{j+1}}\left\|\frac{\partial R}{\partial\eta_x}\right\|_\infty\right)^2.$$

By Lemma 4.1 below with $v=R$, $\Lambda_\ell=\Omega_{j+1}$ and $M=M_{j+1}$, the second term is bounded above by

$$C_1[\log\ell]^2|\Lambda_\ell|^{-1}\left(\sum_{x\in\Omega_{j+1}}\left\|\frac{\partial R}{\partial\eta_x}\right\|_\infty\right)^2$$

for some constant C_1 depending only on the potential V.

For x in Ω_{j+1}, by definition of R, $\|\partial_{\eta_x}R\|_\infty\le\|\partial_{\eta_x}u\|_\infty$. It follows from this observation and from the two previous estimates that the variance of the expectation $E_{\Omega_{j+1},M_{j+1}}[R]$, which is equal to $E_{\nu_\alpha}[(u_{j+1}-u_j)^2|M_1,\dots,M_j]$, is bounded above by

$$C_0[\log\ell]^2\ell^{-d}\left(\sum_{x\in\Omega_{j+1}}\left\|\frac{\partial u}{\partial\eta_x}\right\|_\infty\right)^2.$$

In view of (4.1), to conclude the proof of the lemma it remains to sum over j and to notice that

$$\sum_{j=0}^{q-1}\left(\sum_{x\in\Omega_{j+1}}\left\|\frac{\partial u}{\partial\eta_x}\right\|_\infty\right)^2\le|||u|||^2.$$

This concludes the proof of Proposition 3.4. □

The remainder of this section is devoted to an estimate that was needed above and that will be used again in the proof of Proposition 3.5.

LEMMA 4.1. *Consider a function v defined on a cube Λ_ℓ with finite triple norm. There exists a finite constant C_0 depending only on the potential V such that*

$$E_{\Lambda_\ell,M}\left[v;\sum_{x\in\Lambda_\ell}V'(\eta_x)\right]\le C_0\log\ell\sum_{x\in\Lambda_\ell}\left\|\frac{\partial v}{\partial\eta_x}\right\|_\infty\tag{4.3}$$

for all M in $\mathbb{R}$.

PROOF. This proof requires some notation. Fix an integer $m > 1$. Assume without loss of generality that $2\ell + 1 = (2m+1)^\gamma$ for some $\gamma \ge 1$. For $0 \le k \le \gamma$, let $p_k = [(2m+1)^k - 1]/2$, so that the cube Λ_{p_k} has length $(2m+1)^k$. For each $k \ge 0$, we decompose $\mathbb{Z}^d$ in a disjoint union of cubes of length $(2m+1)^k$. This is done in the following way. Let $A^k = \{x(2m+1)^k,\ x \in \mathbb{Z}^d\}$, let $\{x_j^k,\ j \ge 1\}$ be an enumeration of A^k such that $|x_j^k| \le |x_i^k|$ if $j \le i$ and let $\Omega_j^k = x_j^k + \Lambda_{p_k}$. In particular, for each $0 \le k \le \gamma$,

$$x_1^k = 0, \qquad \bigcup_{j \ge 1} \Omega_j^k = \mathbb{Z}^d, \qquad \Omega_j^k \cap \Omega_i^k = \phi \qquad \text{if } i \ne j.$$

The previous decomposition of $\mathbb{Z}^d$ in disjoint cubes can be transformed in a decomposition of Λ_ℓ because we assumed that $2\ell + 1 = (2m+1)^\gamma$. Let $q_k = [(2\ell+1)/(2m+1)^k]^d = (2m+1)^{d(\gamma-k)}$. Then, by construction, for each $0 \le k \le \gamma$,

$$\bigcup_{j=1}^{q_k} \Omega_j^k = \Lambda_\ell.$$

Moreover, the decomposition of Λ_ℓ at level $k < \gamma$ can be considered as a subdecomposition of the one of level $k+1$. More precisely, for $1 \le k \le \gamma$ and $1 \le j \le q_k$, let $A_j^k = A^{k-1} \cap \Omega_j^k$. Thus A_j^k is the set of points in A^{k-1} that belongs to the cube Ω_j^k. By construction,

$$\Omega_j^k = \bigcup_{i;x_i \in A_j^k} \Omega_i^{k-1}.$$

We will below index the sets $\{A_j^k,\ 1 \le j \le q_k\}$, $\{\Omega_j^k,\ 1 \le j \le q_k\}$ by sites, that is, for x in A^k, Ω_x^k will represent the set Ω_j^k if $x = x_j^k$ and A_x^k will represent the set A_j^k.

We are now in a position to prove the lemma. For $1 \le k \le \gamma$ and M in $\mathbb{R}$, let $\tilde{V}_k(M)$ be the expectation of V' with respect to the canonical measure $\nu_{\Lambda_{p_k},M}$:

$$\tilde{V}_k(M) = E_{\Lambda_{p_k},M}[V'(\eta_x)].$$

In particular, $\tilde{V}_\gamma(M) = E_{\Lambda_\ell,M}[V'(\eta_x)]$. By convention, $\tilde{V}_0 = V'$. Since we can add constants in a covariance,

$$E_{\Lambda_\ell,M}\left[v;\ \sum_{x \in \Lambda_\ell} V'(\eta_x)\right] = E_{\Lambda_\ell,M}\left[v;\ \sum_{x \in \Lambda_\ell} V'(\eta_x) - \tilde{V}_\gamma(M)\right]. \tag{4.4}$$

With the notation introduced previously,

$$\begin{aligned}\sum_{x\in\Lambda_\ell}&\{V'(\eta_x)-\tilde V_\gamma(M)\}\\&=\sum_{x\in\Lambda_\ell}V'(\eta_x)-(2m+1)^{\gamma d}\tilde V_\gamma(M)\\&=\sum_{k=0}^{\gamma-1}\left\{\sum_{x\in A^k}(2m+1)^{kd}\tilde V_k(M_x^k)-\sum_{x\in A^{k+1}}(2m+1)^{(k+1)d}\tilde V_{k+1}(M_x^{k+1})\right\}.\end{aligned}$$

Here, for x in A^k, say $x=x_j^k$, M_x^k represents $\sum_{y\in\Omega_j^k}\eta_y$ and we adopted the convention that $M_x^0=\eta_x$. The previous sum can still be written as

$$\sum_{k=0}^{\gamma-1}\sum_{x\in A^{k+1}}(2m+1)^{kd}\left\{\sum_{y\in A_x^{k+1}}\tilde V_k(M_y^k)-(2m+1)^d\tilde V_{k+1}(M_x^{k+1})\right\}.$$

In conclusion,

$$\begin{aligned}\sum_{x\in\Lambda_\ell}&\{V'(\eta_x)-\tilde V_\gamma(M)\}\\&=\sum_{k=0}^{\gamma-1}\sum_{x\in A^{k+1}}(2m+1)^{kd}\left\{\sum_{y\in A_x^{k+1}}\tilde V_k(M_y^k)-(2m+1)^d\tilde V_{k+1}(M_x^{k+1})\right\}.\end{aligned}$$

Equation (4.4) is thus equal to

$$\sum_{k=0}^{\gamma-1}\sum_{x\in A^{k+1}}(2m+1)^{kd}D_{k,x},\tag{4.5}$$

where

$$D_{k,x}=E_{\Lambda_\ell,M}\left[v;\sum_{y\in A_x^{k+1}}\tilde V_k(M_y^k)-(2m+1)^d\tilde V_{k+1}(M_x^{k+1})\right].$$

We now estimate $D_{k,x}$ for fixed k, x. The second member of the covariance is measurable with respect to $\{\eta_z,\ z\in\Omega_x^{k+1}\}$ and has mean zero with respect to all canonical measures on this set. In particular, $D_{k,x}$ is equal to

$$E_{\Lambda_\ell,M}\left[v\left\{\sum_{y\in A_x^{k+1}}\tilde V_k(M_y^k)-(2m+1)^d\tilde V_{k+1}(M_x^{k+1})\right\}\right].$$

Moreover, taking conditional expectation with respect to the σ-algebra generated by M_x^{k+1}, $\{\eta_z,\ z\in\Lambda_\ell-\Omega_x^{k+1}\}$, we obtain that the previous expression is equal to

$$E_{\Lambda_\ell,M}\left[E_{\Omega_x^{k+1},M_x^{k+1}}\left[v\left\{\sum_{y\in A_x^{k+1}}\tilde V_k(M_y^k)-(2m+1)^d\tilde V_{k+1}(M_x^{k+1})\right\}\right]\right].$$

In this formula, we first integrate the variables $\{\eta_z,\ z \in \Omega_x^{k+1}\}$ with respect to the canonical measure $\nu_{\Omega_x^{k+1}, M_x^{k+1}}$ and then we integrate M_x^{k+1} and the remaining variables $\{\eta_z,\ z \in \Lambda_\ell - \Omega_x^{k+1}\}$ with respect to $\nu_{\Lambda_\ell, M}$.

With respect to $\nu_{\Omega_x^{k+1}, M_x^{k+1}}$, $\tilde{V}_{k+1}(M_x^{k+1})$ is a constant. Therefore,

$$E_{\Omega_x^{k+1}, M_x^{k+1}}\left[v \left\{ \sum_{y \in A_x^{k+1}} \tilde{V}_k(M_y^k) - (2m+1)^d \tilde{V}_{k+1}(M_x^{k+1}) \right\} \right]$$
$$= E_{\Omega_x^{k+1}, M_x^{k+1}}\left[v; \sum_{y \in A_x^{k+1}} \tilde{V}_k(M_y^k) \right].$$

Let $G_k(M_y^k) = \tilde{V}_k(M_y^k) - \tilde{V}(m_x^{k+1}) - \tilde{V}'(m_x^{k+1})[m_y^k - m_x^{k+1}]$, where $\tilde{V}(\alpha) = E_{\nu_\alpha}[V(\eta_0)]$, $\tilde{V}'(\alpha)$ is the derivative of $\tilde{V}$ with respect to α and m_z^j is the empirical density given by $m_z^j = M_z^j / |\Omega_1^j|$. Since we may add constants in a covariance, the previous expectation is equal to

$$E_{\Omega_x^{k+1}, M_x^{k+1}}\left[v; \sum_{y \in A_x^{k+1}} G_k(M_y^k) \right]. \tag{4.6}$$

By the Schwarz inequality, the square of this expression is bounded above by

$$E_{\Omega_x^{k+1}, M_x^{k+1}}[v; v] E_{\Omega_x^{k+1}, M_x^{k+1}}\left[\sum_{y \in A_x^{k+1}} G_k(M_y^k); \sum_{y \in A_x^{k+1}} G_k(M_y^k) \right]. \tag{4.7}$$

On the one hand, the first variance is bounded above by

$$E_{\Omega_x^{k+1}, M_x^{k+1}}[\{v - v(m_x^{k+1})\}^2].$$

In this formula, in the second term v, we replaced all variables $\{\eta_z,\ z \in \Omega_x^{k+1}\}$, which are being integrated, by the value m_x^{k+1}. Of course, the difference $v - v(m_x^{k+1})$ is absolutely bounded by

$$\sum_{z \in \Omega_x^{k+1}} \left\| \frac{\partial v}{\partial \eta_z} \right\|_\infty |\eta_z - m_x^{k+1}|.$$

In particular, the first variance of (4.7) is bounded by

$$\left(\sum_{z \in \Omega_x^{k+1}} \left\| \frac{\partial v}{\partial \eta_z} \right\|_\infty \{E_{\Omega_x^{k+1}, M_x^{k+1}}[\{\eta_z - m_x^{k+1}\}^2]\}^{1/2} \right)^2.$$

By Corollary 5.5 and Lemma 5.1 in [8], $E_{\Omega_x^{k+1}, M_x^{k+1}}[\{\eta_z - m_x^{k+1}\}^2]$ is uniformly bounded because $m > 1$ (and therefore $|\Omega_1^{k+1}| \geq 3^d$ for all $k \geq 0$). The first variance in (4.7) is thus bounded above by

$$C_1\Big(\sum_{z\in\Omega_x^{k+1}}\Big\|\frac{\partial v}{\partial\eta_z}\Big\|_\infty\Big)^2$$

for some finite constant C_1 depending only on the potential V.

By formula (3.10) in [8], the second variance in (4.7) is bounded above by $C_2|\Omega_1^k|^{-2}=C_2(2m+1)^{-2kd}$ for some finite constant C_2, which depends only on the potential V and on m. This proves that the covariance (4.6), and therefore $D_{k,x}$, is bounded above by

$$C_3(2m+1)^{-kd}\sum_{z\in\Omega_x^{k+1}}\Big\|\frac{\partial v}{\partial\eta_z}\Big\|_\infty$$

for some finite constant C_3 depending only on the potential V. In view of (4.5), this shows that (4.4) is bounded by

$$C_4\sum_{k=0}^{\gamma-1}\sum_{x\in A^{k+1}}\sum_{z\in\Omega_x^{k+1}}\Big\|\frac{\partial v}{\partial\eta_z}\Big\|_\infty\le C_5\log\ell\sum_{x\in\Lambda_\ell}\Big\|\frac{\partial v}{\partial\eta_z}\Big\|_\infty$$

because $\gamma=\log(2\ell+1)/\log(2m+1)$. This proves the lemma. □

5. Proof of Proposition 3.5. We assume throughout this section hypothesis (H2) and only this one on the potential V.

To avoid heavy notation and to detach the main ideas, we present the proof in dimension 2. We first introduce some notation.

The following enumeration $\{x_j,\ 1\le j\le|\Lambda_\ell|\}$ of the cube Λ_ℓ will be used repeatedly. Let

$$x_j=\begin{cases}(-\ell+j-1,-\ell), & \text{for } 1\le j\le 2\ell+1,\\ (\ell,-3\ell+j-1), & \text{for } 2\ell+1\le j\le 4\ell+1.\end{cases}$$

If we remove the sites $\{x_j,\ 1\le j\le 4\ell+1\}$ from Λ_ℓ, we obtain a cube of side length 2ℓ. We may now set $x_{4\ell+2}=(-\ell+1,-\ell+1)$ and repeat the same procedure. This defines an enumeration $\{x_1,\dots,x_{|\Lambda_\ell|}\}$ of Λ_ℓ.

For $1\le j\le|\Lambda_\ell|$, denote by Γ_j the set $\{x_j,\dots,x_{|\Lambda_\ell|}\}$. We define a path from x_j to a site y in Γ_{j+1}. Denote by $\{e_1,e_2\}$ the canonical basis of $\mathbb{R}^2$. Fix x_j in Λ_ℓ, y in Γ_{j+1}. By construction, the set Γ_{j+1} is a square minus a few points a the bottom or at the right side. If x_j belongs to the basis of the set Γ_{j+1}, in the sense that x_j-e_1 does not belong to Γ_{j+1}, the path from x_j to y moves first along the y-axis and then along the x-axis. More precisely, the path, denoted by $\gamma(x_j,y)$, is a sequence $z_0=x_j,\dots,z_n=y$ of distinct points for which there exists $-1\le n_0<n$ such that $z_{j+1}-z_j=e_2$ for $0\le j\le n_0$, $z_{j+1}-z_j=\pm e_1$ for $n_0<j<n$. Notice that the path is uniquely determined by the previous properties. In the other case,

that is, if $x_j - e_1$ belongs to Γ_{j+1}, the path moves first along the x-axis and then along the y-axis. Thus, the path $\gamma(x_j, y)$ is now a sequence $z_0 = x_j, \dots, z_n = y$ of distinct points for which there exists $-1 \le n_0 < n$ such that $z_{j+1} - z_j = -e_1$ for $0 \le j \le n_0$, $z_{j+1} - z_j = \pm e_2$ for $n_0 < j < n$. In all cases the length of the path is not longer than $d\ell$, where d is the dimension.

We now turn to the proof of Proposition 3.5. We first estimate

$$E_{\nu_\alpha}[(u - B_{\ell,L}u)f] \tag{5.1}$$

for some density f. Recall that $B_{\ell,L}$ stands for the conditional expectation $E[u|M_1, \dots, M_q]$. For $0 \le k \le q$, denote by $\mathcal{G}_k$ the σ-algebra generated by $M_1, \dots, M_k, \{\eta_z,\ z \in \bigcup_{k+1 \le i \le q} \Omega_i\}$. With this notation,

$$u - B_{\ell,L}u = \sum_{0 \le k \le q-1} \{E[u|\mathcal{G}_k] - E[u|\mathcal{G}_{k+1}]\}. \tag{5.2}$$

We shall estimate

$$E_{\nu_\alpha}[\{E[u|\mathcal{G}_k] - E[u|\mathcal{G}_{k+1}]\}f]$$

for $0 \le k \le q - 1$.

The difference between $E[u|\mathcal{G}_k]$ and $E[u|\mathcal{G}_{k+1}]$ is that the first conditional expectation depends on $\{\eta_z,\ z \in \Omega_{k+1}\}$ while in the second one these variables have been integrated on each hyperplane with a fixed total number of particles M_{k+1}.

By Lemma 5.1 below, there exists a finite constant C_0 such that for each k, $E_{\nu_\alpha}[\{E[u|\mathcal{G}_k] - E[u|\mathcal{G}_{k+1}]\}f]$ is bounded above by

$$C_0 \sum_{j=1}^{|\Lambda_\ell|} \left\{ \left\| \frac{\partial u}{\partial \eta_{x_j^k}} \right\|_\infty + \frac{\log \ell}{|\Gamma_{j+1}^k|} \sum_{y \in \Gamma_{j+1}^k} \left\| \frac{\partial u}{\partial \eta_y} \right\|_\infty \right\} \sqrt{W_j^k(f)},$$

where $\{x_j^k,\ 1 \le j \le |\Lambda_\ell|\}$ is the set $\{x_j,\ 1 \le j \le |\Lambda_\ell|\}$ suitably translated to be an enumeration of Ω_k and where Γ_j^k is the set Γ_j translated in the same way. Moreover,

$$W_j^k(f) = \frac{\ell}{|\Gamma_{j+1}^k|} \sum_{y \in \Gamma_{j+1}^k} \sum_{b \in \gamma(x_j^k, y)} \mathfrak{D}_b(\nu_\alpha, f) + \frac{\ell^2}{|\Gamma_{j+1}^k|} \sum_{b \in \Gamma_{j+1}^k} \mathfrak{D}_b(\nu_\alpha, f).$$

Let

$$R_{k,j}(u) = \left\| \frac{\partial u}{\partial \eta_{x_j^k}} \right\|_\infty + \frac{\log \ell}{|\Gamma_{j+1}^k|} \sum_{y \in \Gamma_{j+1}^k} \left\| \frac{\partial u}{\partial \eta_y} \right\|_\infty.$$

In view of the previous estimate and of decomposition (5.2), by the Schwarz inequality,

$$
\begin{aligned}
&\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L}\big(E_{\nu_\alpha}[(u-B_{\ell,L}u)\tau_x f]\big)^2\\
&\quad\le C_0\Bigg\{\sum_{k=0}^{q-1}\sum_{j=1}^{|\Lambda_\ell|}R_{k,j}(u)\Bigg\}\Bigg\{\sum_{k=0}^{q-1}\sum_{j=1}^{|\Lambda_\ell|}R_{k,j}(u)\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L}W_j^k(\tau_x f)\Bigg\}\qquad(5.3)\\
&\quad\le C_0\Bigg\{\sum_{k=0}^{q-1}\sum_{j=1}^{|\Lambda_\ell|}R_{k,j}(u)\Bigg\}^2\max_{k,j}\Bigg\{\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L}W_j^k(\tau_x f)\Bigg\}.
\end{aligned}
$$

We estimate separately the sum and the maximum. Clearly, the sum is bounded above by

$$
2|||u|||^2+2(\log\ell)^2\Bigg(\sum_{k=1}^{q}\sum_{j=1}^{|\Lambda_\ell|}\frac{1}{|\Gamma_{j+1}^k|}\sum_{y\in\Gamma_{j+1}^k}\Big\|\frac{\partial u}{\partial\eta_y}\Big\|_\infty\Bigg)^2.
$$

In the second term, fix k and change the order of the two remaining sums to obtain that

$$
\sum_{j=1}^{|\Lambda_\ell|}\frac{1}{|\Gamma_{j+1}^k|}\sum_{y\in\Gamma_{j+1}^k}\Big\|\frac{\partial u}{\partial\eta_y}\Big\|_\infty=\sum_{y\in\Omega_k}\Big\|\frac{\partial u}{\partial\eta_y}\Big\|_\infty\sum_{j;\,y\in\Gamma_{j+1}^k}\frac{1}{|\Gamma_{j+1}^k|}.
$$

The last sum is less than or equal to $\sum_{1\le i\le|\Lambda_\ell|}i^{-1}$. The previous expression is thus bounded above by $C_0(\log\ell)\sum_{y\in\Omega_k}\|\partial_{\eta_y}u\|_\infty$. This shows that the sum part in (5.3) is bounded above by $C_0(\log\ell)^4|||u|||^2$.

We turn now to the maximum part of (5.3). The maximum can be divided in two terms, the first one being

$$
\begin{aligned}
&\max_{j,k}\Bigg\{\frac{\ell^2}{|\Gamma_{j+1}^k|}\sum_{b\in\Gamma_{j+1}^k}\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L}\mathfrak{D}_b(\nu_\alpha,\tau_x f)\Bigg\}\\
&\qquad\le\frac{\ell^2}{|\Lambda_L|}\mathfrak{D}_{\Lambda_L}(\nu_\alpha,f)\max_{j,k}\Bigg\{\frac{1}{|\Gamma_{j+1}^k|}\sum_{b\in\Gamma_{j+1}^k}\Bigg\}
\end{aligned}
$$

because $\mathfrak{D}_b(\nu_\alpha,\tau_x f)=\mathfrak{D}_{b-x}(\nu_\alpha,f)$. This expression is obviously bounded by $C_1\ell^2|\Lambda_L|^{-1}\mathfrak{D}_{\Lambda_L}(\nu_\alpha,f)$ for some finite constant C_1 depending only on the dimension.

The second term in the maximum of formula (5.3) is

$$
\begin{aligned}
&\max_{j,k}\Bigg\{\frac{\ell}{|\Gamma_{j+1}^k|}\sum_{y\in\Gamma_{j+1}^k}\sum_{b\in\gamma(x_j^k,y)}\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L}\mathfrak{D}_b(\nu_\alpha,\tau_x f)\Bigg\}\\
&\qquad\le\frac{\ell}{|\Lambda_L|}\mathfrak{D}_{\Lambda_L}(\nu_\alpha,f)\max_{j,k}\Bigg\{\frac{1}{|\Gamma_{j+1}^k|}\sum_{y\in\Gamma_{j+1}^k}\sum_{b\in\gamma(x_j^k,y)}\Bigg\}
\end{aligned}
$$

by definition of the canonical paths, for each k, j, the expression inside braces in the previous formula is bounded above by

$$\frac{1}{|\Gamma_{j+1}^k|}\sum_{b\in\Omega_k}\sum_{\substack{y\in\Gamma_{j+1}^k\\ y;b\in\gamma(x_j^k,y)}} \le C_1\ell$$

for some finite constant C_1. The second term in the maximum is thus bounded above by $C_1\ell^2|\Lambda_L|^{-1}\mathfrak{D}_{\Lambda_L}(\nu_\alpha, f)$.

Putting together the four previous estimates, we obtain that (5.3) is bounded above by $C_0(\log\ell)^4\ell^2|\Lambda_L|^{-1}|||u|||^2\mathfrak{D}_{\Lambda_{\Lambda_L}}(\nu_\alpha, f)$, which proves the proposition. $\square$

LEMMA 5.1. *Fix $\ell \ge 1$ and recall the enumeration of Λ_ℓ. Let u be a local function, let $M = \sum_{x\in\Lambda_\ell}\eta_x$, let f be a density with respect to ν_α and let $\mathcal{G}$ be the σ-algebra generated by M, $\{\eta_z,\ z\notin\Lambda_\ell\}$. Then,*

$$\begin{aligned}&E_{\nu_\alpha}\big[\{u - E[u|\mathcal{G}]\}f\big]\\ &\qquad \le C_0\sum_{j=1}^{|\Lambda_\ell|}\left\{\left\|\frac{\partial u}{\partial\eta_{x_j}}\right\|_\infty + \frac{\log\ell}{|\Gamma_{j+1}|}\sum_{y\in\Gamma_{j+1}}\left\|\frac{\partial u}{\partial\eta_y}\right\|_\infty\right\}\sqrt{W_j(f)},\end{aligned}$$

where

$$W_j(f) = \frac{\ell}{|\Gamma_{j+1}|}\sum_{y\in\Gamma_{j+1}}\sum_{b\in\gamma(x_j,y)}\mathfrak{D}_b(\nu_\alpha, f) + \frac{\ell^2}{|\Gamma_{j+1}|}\sum_{b\in\Gamma_{j+1}}\mathfrak{D}_b(\nu_\alpha, f).$$

PROOF. For $0\le j\le|\Lambda_\ell|$, let $M_j = \sum_{j\le k\le|\Lambda_\ell|}\eta_{x_k}$ and denote by $\mathcal{F}_j$ the σ-algebra generated by $\{\eta_{x_k},\ k\le j\}$, M_{j+1} and $\{\eta_x,\ x\notin\Lambda_\ell\}$. Denote by $\{u_j,\ 0\le j\le|\Lambda_\ell|\}$ the martingale

$$u_j = E[u|\mathcal{F}_j].$$

With this notation $u_0 = E[u|\mathcal{G}]$, $u_{|\Lambda_\ell|} = u$ and we may write

$$\begin{aligned}E_{\nu_\alpha}\big[\{u - E[u|\mathcal{G}]\}f\big] &= \sum_{j=0}^{|\Lambda_\ell|-1} E_{\nu_\alpha}\big[\{u_{j+1} - u_j\}f\big]\\ &= \sum_{j=0}^{|\Lambda_\ell|-1} E_{\nu_\alpha}\left[f_j E_{\nu_\alpha}\left[\{u_{j+1}-u_j\}\frac{f_{j+1}}{f_j}\,\Big|\,\mathcal{F}_j\right]\right],\end{aligned}$$

where $f_j = E_{\nu_\alpha}[f|\mathcal{F}_j]$.

Recall that for a finite subset Λ of $\mathbb{Z}^d$ and a real number M, $\nu_{\Lambda,M}$ stands for the canonical measure on Λ concentrated on configurations such that $\sum_{x\in\Lambda}\eta_x = M$ and that $E_{\Lambda,M}$ stands for the expectation with respect to $\nu_{\Lambda,M}$. A straightforward computation shows that $E_{\nu_\alpha}[h|\mathcal{F}_j] = E_{\Gamma_{j+1},M_{j+1}}[h]$ so that

$$E_{\nu_\alpha}\left[\{u_{j+1}-u_j\}\frac{f_{j+1}}{f_j}\,\Big|\,\mathcal{F}_j\right] = E_{\Gamma_{j+1},M_{j+1}}\left[\{u_{j+1}-u_j\}\frac{f_{j+1}}{f_j}\right].$$

Notice that in the previous expectation, only the variables $\{\eta_{x_i},\ j+1\le i\le|\Lambda_\ell|\}$ are integrated, all the others remaining fixed. We may think therefore that the function u_{j+1} is a function of $\eta_{x_{j+1}}$ only

$$u_{j+1} = E_{\Gamma_{j+2},M_{j+1}-\eta_{x_{j+1}}}[u] \tag{5.4}$$

because $M_{j+2} = M_{j+1} - \eta_{x_{j+1}}$. Moreover, its expectation with respect to $\nu_{\Gamma_{j+1},M_{j+1}}$ is equal to u_j. In particular, by Corollary 6.4 in [8] and the logarithmic Sobolev inequality for Glauber dynamics (Lemma 4.1 in [8]), there exists a finite constant C_0 such that

$$\begin{aligned} &E_{\Gamma_{j+1},M_{j+1}}\left[(u_{j+1}-u_j)\frac{f_{j+1}}{f_j}\right] \\ &\qquad\le C_0\|\partial_{\eta_{x_{j+1}}}u_{j+1}\|_\infty\sqrt{\mathfrak{D}^G_{x_{j+1}}(\nu_{\Gamma_{j+1},M_{j+1}}, f_{j+1}/f_j)}. \end{aligned} \tag{5.5}$$

In this formula, $\mathfrak{D}^G_{x_{j+1}}$ stands for the Glauber–Dirichlet form defined by

$$\mathfrak{D}^G_{x_{j+1}}(\nu_{\Gamma_{j+1},M_{j+1}}, h) = E_{\Gamma_{j+1},M_{j+1}}\left[\left(\partial_{\eta_{x_{j+1}}}\sqrt{h}\right)^2\right].$$

We shall estimate separately each term appearing on the right-hand side of (5.5).

By Lemma 5.2 below,

$$\left\|\frac{\partial u_{j+1}}{\partial\eta_{x_{j+1}}}\right\|_\infty \le \left\|\frac{\partial u}{\partial\eta_{x_{j+1}}}\right\|_\infty + C_0(\log\ell)\frac{1}{|\Gamma_{j+2}|}\sum_{y\in\Gamma_{j+2}}\left\|\frac{\partial u}{\partial\eta_y}\right\|_\infty \tag{5.6}$$

for some finite constant C_0.

We now estimate $\mathfrak{D}^G_{x_{j+1}}(\nu_{\Gamma_{j+1},M_{j+1}}, f_{j+1}/f_j)$. By (5.8) below, $\partial_{\eta_{x_{j+1}}}f_{j+1}$ is equal to

$$\begin{aligned} &E_{\Gamma_{j+2},M_{j+2}}\left[\frac{1}{|\Gamma_{j+2}|}\sum_{y\in\Gamma_{j+2}}\left\{\partial_{\eta_{x_{j+1}}}f - \partial_{\eta_y}f\right\}\right] \\ &\qquad + E_{\Gamma_{j+2},M_{j+2}}\left[f;\ \frac{1}{|\Gamma_{j+2}|}\sum_{y\in\Gamma_{j+2}}V'(\eta_y)\right]. \end{aligned}$$

In particular, $4 f_j \mathfrak{D}^G_{x_{j+1}}(\nu_{\Gamma_{j+1},M_{j+1}}, f_{j+1}/f_j)$, which is equal to $E_{\Gamma_{j+1},M_{j+1}}[f_{j+1}^{-1} \times \{\partial_{\eta_{x_{j+1}}} f_{j+1}\}^2]$, is bounded above by

$$
\begin{aligned}
&2E_{\Gamma_{j+1},M_{j+1}}\left[\frac{1}{f_{j+1}}\left(E_{\Gamma_{j+2},M_{j+2}}\left[\frac{1}{|\Gamma_{j+2}|}\sum_{y\in\Gamma_{j+2}}\{\partial_{\eta_{x_{j+1}}} f - \partial_{\eta_y} f\}\right]\right)^2\right] \\
&\quad + 2E_{\Gamma_{j+1},M_{j+1}}\left[\frac{1}{f_{j+1}}\left(E_{\Gamma_{j+2},M_{j+2}}\left[f; \frac{1}{|\Gamma_{j+2}|}\sum_{y\in\Gamma_{j+2}} V'(\eta_y)\right]\right)^2\right].
\end{aligned}
\tag{5.7}
$$

We use the paths $\gamma(x_j, y)$ introduced in the beginning of the section in order to estimate the first term in (5.7). Notice first that $\partial_{\eta_x} f - \partial_{\eta_y} f$ can be rewritten as $2\sqrt{f}[\partial_{\eta_x}\sqrt{f} - \partial_{\eta_y}\sqrt{f}]$. Doing this, rewriting the difference $\partial_{\eta_{x_{j+1}}} - \partial_{\eta_y}$ as $\sum_{z_j, z_{j+1}\in\gamma(x_{j+1},y)}\{\partial_{\eta_{z_j}} - \partial_{\eta_{z_{j+1}}}\}$ and then applying the Schwarz inequality, we obtain that the first term in (5.7) is bounded above by

$$
\frac{8d\ell}{|\Gamma_{j+2}|}\sum_{y\in\Gamma_{j+2}}\sum_{b\in\gamma(x_{j+1},y)} \mathfrak{D}_b(\nu_{\Gamma_{j+1},M_{j+1}}, f),
$$

where the second summation is performed over all unoriented bonds belonging to the path $\gamma(x_{j+1}, y)$.

By Corollary 6.3 in [8], by the logarithmic Sobolev inequality for the Ginzburg–Landau process (Theorem 2.2 in [8]) and by the Schwarz inequality, the second term in (5.7) is bounded above by

$$
\frac{C_0\ell^2}{|\Gamma_{j+2}|}\sum_{b\in\Gamma_{j+2}} \mathfrak{D}_b(\nu_{\Gamma_{j+1},M_{j+1}}, f)
$$

for some finite constant C_0. Here the sum is carried over all bonds b in Γ_{j+2}. The two previous estimates together with (5.7) show that the square of the second member of (5.5) is bounded above by

$$
\begin{aligned}
&\frac{1}{f_j}\frac{C_0\ell}{|\Gamma_{j+2}|}\sum_{y\in\Gamma_{j+2}}\sum_{b\in\gamma(x_{j+1},y)} \mathfrak{D}_b(\nu_{\Gamma_{j+1},M_{j+1}}, f) \\
&\quad + \frac{1}{f_j}\frac{C_0\ell^2}{|\Gamma_{j+2}|}\sum_{b\in\Gamma_{j+2}} \mathfrak{D}_b(\nu_{\Gamma_{j+1},M_{j+1}}, f)
\end{aligned}
$$

for some finite constant C_0.

Putting together this estimate with (5.6) and recalling (5.4) and the definition of $W_{j+1}(f)$ given in the statement of the lemma, we get, by the Schwarz inequality (to move the square root outside the expectation with respect to ν_α) that

$E_{\nu_\alpha}[\{u_{j+1}-u_j\}f]$ is bounded above by

$$C_0\left\{\left\|\frac{\partial u}{\partial\eta_{x_{j+1}}}\right\|_\infty+\frac{\log\ell}{|\Gamma_{j+2}|}\sum_{y\in\Gamma_{j+2}}\left\|\frac{\partial u}{\partial\eta_y}\right\|_\infty\right\}\sqrt{W_{j+1}(f)},$$

which proves the lemma. □

LEMMA 5.2. *Recall the definition of the martingale* $\{u_j,\ 0\le j\le|\Lambda_\ell|\}$ *introduced in the beginning of this section. Then,*

$$\left\|\frac{\partial u_j}{\partial\eta_{x_j}}\right\|_\infty\le\left\|\frac{\partial u}{\partial\eta_{x_j}}\right\|_\infty+C_0(\log\ell)\frac{1}{|\Gamma_{j+1}|}\sum_{y\in\Gamma_{j+1}}\left\|\frac{\partial u}{\partial\eta_{x_j}}\right\|_\infty$$

for some finite constant C_0.

PROOF. Recall from (5.4) that $u_j=E_{\Gamma_{j+1},M_j-\eta_{x_j}}[u]$. In particular,

$$\begin{aligned}\frac{\partial u_j}{\partial\eta_{x_j}}&=E_{\Gamma_{j+1},M_j-\eta_{x_j}}[\partial_{\eta_{x_j}}u]-E_{\Gamma_{j+1},M_j-\eta_{x_j}}\left[\frac{1}{|\Gamma_{j+1}|}\sum_{y\in\Gamma_{j+1}}\partial_{\eta_y}u\right]\\&\quad+E_{\Gamma_{j+1},M_j-\eta_{x_j}}\left[u;\frac{1}{|\Gamma_{j+1}|}\sum_{y\in\Gamma_{j+1}}V'(\eta_y)\right].\end{aligned}\tag{5.8}$$

Here again, $E[f;g]$ stands for the covariance of f, g. The first term on the right-hand side of this formula is clearly bounded by $\|\partial_{\eta_{x_j}}u\|_\infty$, while the second one is bounded by $|\Gamma_{j+1}|^{-1}\sum_{y\in\Gamma_{j+1}}\|\partial_{\eta_y}u\|_\infty$. Finally, by Lemma 4.1, the third one is bounded by $C_0(\log\ell)|\Gamma_{j+1}|^{-1}\sum_{y\in\Gamma_{j+1}}\|\partial_{\eta_y}u\|_\infty$ for some finite constant C_0. This concludes the proof of the lemma. □

6. Proof of Proposition 3.6. We assume throughout this section hypothesis (H1) and only this one on the potential V.

Fix a vector $\mathbf{M}_q=(M_1,\dots,M_q)$ and recall the definition of the density $f_t=f_t^{\varepsilon,\ell,L,\mathbf{M}_q}$. To avoid confusion between the vector $\mathbf{M}_q$, which is fixed and which characterizes the initial state, and the variables $\sum_{x\in\Omega_k}\eta_x$, we denote the latter ones by $\mathbf{N}_q=(N_1,\dots,N_q)$.

The main problem in the proof of Proposition 3.6 is that $\nu_{\Lambda_L,\mathbf{M}_q}$ is not absolutely continuous with respect to ν_α. In particular, the entropy at time 0 is infinite. The idea of the proof is the following. We shall introduce a function g_t of the total spins in each cube $\{\Omega_j,\ 1\le j\le q\}$: $g_t=g_t(\mathbf{N}_q)$. With this function, we decompose the entropy $H(f_t)$ as

$$\int f_t\log\frac{f_t}{g_t}\,d\nu_\alpha+\int f_t\log g_t\,d\nu_\alpha.$$

We shall choose g_t so that at time 0, g_t is a Dirac measure on the density $\mathbf{M}_q$. In particular, the first term above vanishes at $t = 0$ because f_0 and g_0 coincide. This observation permits estimating this term by taking a time derivative. A simple computation, related to the relative entropy method, shows that the time derivative of this expression is bounded by

$$\int f_t \, \frac{\mathcal{L}^{\varepsilon}_{\Lambda_L} g_t - \partial_t g_t}{g_t} \, d\nu_\alpha. \tag{6.1}$$

We want now to choose a function $g_t = g_t(\mathbf{N}_q)$ for which most of the terms in the difference $\mathcal{L}^{\varepsilon}_{\Lambda_L} g_t - \partial_t g_t$ cancel. Once this choice is made, we must check that there are enough cancellations for the previous integral to be small for t small. We need also to check that this choice of g_t is so that the second piece in the decomposition of the entropy is not too large for t small (notice that it must diverges as $t \downarrow 0$ because at time 0 the integral is infinite, but we expect a logarithmic divergence in time).

We need some notation in order to define the function $g_t(\mathbf{N}_q)$. Let $2K + 1 = (2L + 1)/(2\ell + 1)$. Denote by $\mathbb{T}^d_K$ the set Λ_K viewed as a d-dimensional torus with $(2K + 1)^d$ points. Points of $\mathbb{T}^d_K$ are denote by the letters $x = (x_1, \dots, x_d)$. Denote by Δ the Laplacian operator on $\mathbb{T}^d_K$, so that for a function $f : \mathbb{T}^d_K \to \mathbb{R}$, $(\Delta f)(x) = \sum_{1 \le j \le d} \{f(x + e_j) + f(x - e_j) - 2f(x)\}$, where $\{e_j,\ 1 \le j \le d\}$ represents the canonical basis of $\mathbb{R}^d$. Δ can be viewed as a matrix with entries $\{\Delta_{x,y},\ x, y \in \mathbb{T}^d_K\}$ given by $\Delta_{x,y} = -2d\mathbf{1}\{x = y\} + \mathbf{1}\{|x - y| = 1\}$. Denote by $X = X_{\varepsilon,\ell}$ the strictly positive operator $X = (2\ell + 1)^{d-1}\{(2\ell + 1)\varepsilon I - \Delta\}$, where I stands for the identity.

For $t \ge 0$, denote by $g_t(\mathbf{N}_q)$ the Gaussian kernel defined by

$$g_t(\mathbf{N}_q) = \frac{1}{(|2\pi t \Sigma^{-1}|)^{1/2}} \exp\left\{-\frac{1}{2t}[\mathbf{N}_q - \mathbf{M}_q]\, \Sigma\, [\mathbf{N}_q - \mathbf{M}_q]\right\},$$

where $\Sigma = X^{-1}$ and $|\Sigma|$ stands for the determinant of Σ.

We can write the entropy $H(f_t)$ as

$$\int f_t \log \frac{f_t}{g_t} \, d\nu_\alpha + \int f_t \log g_t \, d\nu_\alpha. \tag{6.2}$$

We shall estimate these two terms separately. By definition of g_t, the second one is bounded above by

$$\begin{aligned} -\tfrac{1}{2} \log\{|2\pi t \Sigma^{-1}|\} &- (1/2t) E_{\Lambda_L, \mathbf{M}_q}\big[\{\mathbf{M}_q(t) - \mathbf{M}_q\}\Sigma\{\mathbf{M}_q(t) - \mathbf{M}_q\}\big] \\ &\le -\tfrac{1}{2} \log\{|2\pi t\, \Sigma^{-1}|\} \end{aligned}$$

because Σ is a positive operator. Since $\Sigma = X^{-1}$, by definition of X, $|2\pi t \Sigma^{-1}| = [2\pi t(2\ell + 1)^{d-1}]^q |\delta I - \Delta|$, where $\delta = \varepsilon(2\ell + 1)$. Since $-\Delta$ is a positive operator, all eigenvalues of $\delta I - \Delta$ are bounded below by δ. In particular, $|\delta I - \Delta| \ge \delta^q$.

Hence, $|2\pi t\Sigma^{-1}| \ge [2\pi t|\Lambda_\ell|\varepsilon]^q$ and

$$\int f_t \log g_t \, d\nu_\alpha \le -\frac{|\Lambda_L|}{2|\Lambda_\ell|} \log\{2\pi t|\Lambda_\ell|\varepsilon\}. \tag{6.3}$$

We now turn to the first term in (6.2). In the derivation of this estimate, two quantities will appear naturally. For $t \ge 0$, let

$$W(t) = E_{\Lambda_L,\mathbf{M}_q}\bigg[\sum_{x\in\Lambda_L} \eta_x(t)^2\bigg], \qquad U(t) = \sum_{k=1}^{q} E_{\Lambda_L,\mathbf{M}_q}\big[\{M_k(t) - M_k\}^2\big]. \tag{6.4}$$

We prove below in Lemmas 6.1 and 6.2 two estimates on $W(t)$, $U(t)$.

A straightforward calculation shows that the time derivative of the first term in (6.2) is bounded above by (6.1). To compute $\mathscr{L}^\varepsilon_{\Lambda_L} g_t$, notice that $\partial_{\eta_x} g_t = \partial_{M_j} g_t$ if x belongs to Ω_j. It is now easy to see that g_t is so defined that all the second order operators in $\mathscr{L}^\varepsilon_{\Lambda_L}$ cancel with the time derivative of g_t and only the drift terms remain:

$$\begin{aligned} \frac{(\mathscr{L}^\varepsilon_{\Lambda_L} - \partial_t) g_t(\mathbf{N}_q)}{g_t(\mathbf{N}_q)} &= \frac{1}{2t} \sum_{k\sim j} \big\{(\Sigma[\mathbf{N}-\mathbf{M}])_k - (\Sigma[\mathbf{N}-\mathbf{M}])_j\big\}\{V'_{k,j} - V'_{j,k}\} \\ &\quad + \frac{\varepsilon}{2t} \sum_k (\Sigma[\mathbf{N}-\mathbf{M}])_k \sum_{x\in\Omega_k} V'(\eta_x). \end{aligned} \tag{6.5}$$

In this formula, the summation $k \sim j$ is performed over all indices k, j such that the cubes Ω_k, Ω_j are at distance 1. In this case, $V'_{k,j}$ stands for $\sum_x V'(\eta_x)$, where the summation is carried over all sites x in Ω_k that are at distance 1 from Ω_j. Finally, to keep notation simple, we assume without loss of generality the chemical potential λ to be 0. We can do it because the only assumptions we will use are (H1) that are fulfilled for $-\lambda a + V(a)$ as soon as they are fulfilled by $V(a)$.

We claim that

$$E_{\Lambda_L,\mathbf{M}_q}\bigg[\frac{(\mathscr{L}^\varepsilon_{\Lambda_L} - \partial_t) g_t(\mathbf{M}_q(t))}{g_t(\mathbf{M}_q(t))}\bigg] \le \frac{C_0}{(\varepsilon\ell t)^{1/2}}\{W(0) + L^d\} \tag{6.6}$$

for $0 \le t \le 1$ and some finite constant C_0.

In view of (6.5), there are two terms we need to estimate. The first one, by the elementary inequality $2ab \le Aa^2 + A^{-1}b^2$ and by the Schwarz inequality, is bounded above by

$$\begin{aligned} &\frac{1}{4At} E_{\Lambda_L,\mathbf{M}_q}\bigg[\sum_{k\sim j} \big\{\big(\Sigma[\mathbf{M}(t) - \mathbf{M}]\big)_k - \big(\Sigma[\mathbf{M}(t) - \mathbf{M}]\big)_j\big\}^2\bigg] \\ &\quad + \frac{2dA(2\ell+1)^{d-1}}{t} \sum_{x\in\Lambda_L} E_{\Lambda_L,\mathbf{M}_q}\big[\{V'(\eta_x(t))\}^2\big] \end{aligned} \tag{6.7}$$

for every $A > 0$. A Fourier computation shows that the first term in this sum is less than or equal to

$$\frac{C_0 \ell^{1-2d}}{A\varepsilon t} \sum_{k=1}^{q} E_{\Lambda_L, \mathbf{M}_q}\big[[M_k(t) - M_k]^2 \big]$$

for some finite constant C_0 that depends only on d. By (6.10), this sum is bounded above by $C_0\{W(0) + L^d\}/A\ell^d\varepsilon$ for $0 \le t \le 1$. On the other hand, by hypothesis (H1) and by (6.9), the second term in (6.7) is bounded above by $C_0 \ell^{d-1} t^{-1}\{W(0) + L^d\}$. Minimizing over A, we show the (6.7) is less than or equal to $C_0(\varepsilon\ell t)^{-1}[W(0) + L^d]$.

We now examine the expectation of the second term in the decomposition (6.5). By the very same arguments, the expectation of this expression is bounded above by

$$\frac{\varepsilon A}{4t} \sum_k E_{\Lambda_L, \mathbf{M}_q}\big[\{(\Sigma[\mathbf{M}(t) - \mathbf{M}])_k\}^2\big] + \frac{C_0 \varepsilon \ell^d}{4At} \sum_{x \in \Lambda_L} E_{\Lambda_L, \mathbf{M}_q}\big[[V'(\eta_x(t))]^2\big] \tag{6.8}$$

for every $A > 0$. A Fourier computation and assumption (H1) permit bounding this expression by

$$\frac{C_0 A}{\varepsilon \ell^{2d} t} \sum_k E_{\Lambda_L, \mathbf{M}_q}\big[\{M_k(t) - M_k\}^2\big] + \frac{C_0 \varepsilon \ell^d}{At} \left\{ L^d + \sum_{x \in \Lambda_L} E_{\Lambda_L, \mathbf{M}_q}\big[\eta_x(t)^2\big] \right\}.$$

By (6.9) and (6.10) below, this expression is bounded above by

$$\frac{C_0 A}{\varepsilon \ell^{d+1}} \{L^d + W(0)\} + \frac{C_0 \varepsilon \ell^d}{At} \{L^d + W(0)\}.$$

It remains to minimize over A to show that (6.8) is less than or equal to $C_0(\ell t)^{-1}[W(0) + L^d]$. This estimate together with the one of (6.7) proves the claim (6.6).

Since the left-hand side of (6.6) is an upper bound for the time derivative of the first term in (6.2), integrating in time we get that

$$\int f_t \log \frac{f_t}{g_t}\, d\nu_\alpha \le \frac{C_0 \sqrt{t}}{\sqrt{\varepsilon \ell}} \{W(0) + L^d\}.$$

In view of this estimate and of (6.2), (6.3),

$$E_{\nu_\alpha}[H(f_t)] \le \frac{C_0 \sqrt{t}}{\sqrt{\varepsilon \ell}} \{E_{\nu_\alpha}[W(0)] + L^d\} - \frac{|\Lambda_L|}{2|\Lambda_\ell|} \log\{2\pi t |\Lambda_\ell| \varepsilon\}.$$

By definition,

$$E_{\nu_\alpha}[W(0)] = |\Lambda_\ell| \sum_{k=1}^{q} E_{\nu_\alpha}\big[E_{\Omega_k, M_k}[\eta_0^2]\big] = |\Lambda_L| E_{\nu_\alpha}[\eta_0^2].$$

The previous expression is thus bounded by

$$\frac{C_0\sqrt{t}L^d}{\sqrt{\varepsilon\ell}} - \frac{|\Lambda_L|}{2|\Lambda_\ell|}\log\{2\pi t|\Lambda_\ell|\varepsilon\}.$$

For $t^* = \varepsilon\ell^{1-2d}$, this expression is less than or equal to $C_0(L/\ell)^d \log\{\varepsilon^{-2}\ell^{d-1}\}$ so that

$$E_{\nu_\alpha}[H(f_{t^*})] \le C_0\left(\frac{L}{\ell}\right)^d \{\log\ell + \log\varepsilon^{-1}\}.$$

By definition of ε and since $\ell \ge 1$, $t^* \le 1$. Since $H(f_t)$ decreases in time, for every $t \ge 1$,

$$E_{\nu_\alpha}[H(f_t)] \le C_0\left(\frac{L}{\ell}\right)^d \{\log\ell + \log\varepsilon^{-1}\},$$

which proves the proposition. □

We conclude this section with the derivation of two estimates needed above. We first claim the following.

LEMMA 6.1. *There exists a finite constants C_0 depending only on B_1, d such that*

$$W(t) \le W(0)e^{C_0 t} + L^d(e^{C_0 t} - 1)$$

for all $t \ge 0$.

PROOF. The result follows from Gronwall inequality and an elementary computation. First of all, it is easy to show that $\mathcal{L}^\varepsilon_{\Lambda_L}\sum_{x\in\Lambda_L}\eta_x^2$ is equal to

$$(4d+\varepsilon)|\Lambda_L| - \varepsilon\sum_x \eta_x V'(\eta_x) - \sum_{|x-y|=1}[\eta_y - \eta_x][V'(\eta_y) - V'(\eta_x)].$$

We estimate the second and third terms of this expression by applying the elementary inequality $2ab \le a^2 + b^2$. Now, recalling that $\varepsilon \le 1$ and from assumption (H1) that $V'(a)^2 \le B_1[1 + a^2]$, we get that the previous sum is less than or equal to

$$C(d, B_1)\left\{L^d + \sum_x \eta_x^2\right\}.$$

It follows from this estimate that $W'(t) \le C\{L^d + W(t)\}$. To conclude the proof of the lemma, one just needs to apply the Gronwall inequality. □

Notice that the right-hand side of the inequality presented in the statement of the previous lemma is bounded by $C_1\{W(0) + tL^d\}$ for $t \le 1$, so that

$$W(t) \le C_1\{W(0) + tL^d\} \tag{6.9}$$

for all $0 \le t \le 1$.

Recall the definition of $U(t)$ given in (6.4). We claim the following.

LEMMA 6.2. *There exists a finite constant C_0 depending only on B_1, d such that*

$$U(t) \le \ell^{d-1}\{W(0) + L^d\}(e^{C_0 t} - 1)$$

for all $0 \le t \le 1$.

PROOF. The strategy here is similar to that of the previous lemma and relies on the Gronwall lemma and on explicit computations. Notice that for any smooth function H, $\partial_{\eta_x} H(M_k) = \mathbf{1}\{x \in \Omega_k\}\partial_{M_k} H(M_k)$. In particular, it is not difficult to show that $\mathcal{L}^{\varepsilon}_{\Lambda_L} \sum_k M_k^2$ is equal to

$$\frac{|\Lambda_L|}{|\Lambda_\ell|}\big(\varepsilon + 4d|\Lambda_\ell|^{(d-1)/d}\big) - \sum_{k\sim j}[M_k - M_j][V'_{k,j} - V'_{j,k}] \\ - \varepsilon \sum_k M_k \sum_{x\in\Omega_k} V'(\eta_x).$$

Here, the factor ℓ^{d-1} in the first term comes from the fact that $\sum_k M_k^2$ changes only due to the diffusion at the boundary of the squares $\{\Omega_i,\ 1 \le i \le q\}$. The fact that $\varepsilon \le \ell^{-1}$, the elementary inequality $2ab \le a^2 + b^2$, the Schwarz inequality and assumption (H1) permits bounding the previous expression by

$$C_0(d, B_1)\left\{L^d \ell^{d-1} + \sum_k M_k^2 + \ell^{d-1} \sum_x \eta_x^2\right\}.$$

This estimate shows that $U'(t)$ is bounded by $C_0\{L^d\ell^{d-1} + \ell^{d-1}W(t) + U(t)\}$. From (6.9), $W(t) \le C_1\{W(0) + L^d\}$ for $t \le 1$. Therefore, $U'(t) \le C_0\{\ell^{d-1}W(0) + L^d\ell^{d-1} + U(t)\}$. Since $U(0) = 0$, to conclude the proof of the lemma we just need to apply the Gronwall inequality. □

It follows from the lemma that there exists a finite constant C_0 depending only on d, B_1 such that

$$U(t) \le C_0 \ell^{d-1}\{W(0) + L^d\}t \tag{6.10}$$

for all $0 \le t \le 1$.

Acknowledgment. We thank the referee for many suggestions which helped to improve the clarity of this manuscript.

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