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ASYMPTOTIC PROPERTIES OF ADDITIVE FUNCTIONALS OF BROWNIAN MOTION

BY MASAYOSHI TAKEDA AND TUSHENG ZHANG[1]

Osaka University and Universität Bielefeld

In this paper we study the asymptotic behavior of additive functionals of Brownian motion which are not necessarily of bounded variation. The result is then applied to the Hilbert transform of the Brownian local time.

1. Introduction and framework. A Borel measure μ on R^d is said to be in the Kato class K_d if:

(i) $$\lim_{\alpha\to 0}\left[\sup_x \int_{|y-x|\le\alpha} \frac{|\mu|(dy)}{|x-y|^{d-2}}\right] = 0 \quad \text{when } d \ge 3;$$

(ii) $$\lim_{\alpha\to 0}\left[\sup_x \int_{|y-x|\le\alpha} |\mu|(dy)|\ln(|y-x|^{-1})|\right] = 0 \quad \text{when } d = 2;$$

(iii) $$\sup_x \int_{|y-x|\le 1} |\mu|(dy) < +\infty \quad \text{when } d = 1.$$

Let $\rho \in H_2^1(R^d)$ with $|\nabla\rho|^2 \in K_d$, and let μ be a measure in the Kato class K_d. We introduce the following quadratic form on $L^2(R^d)$:

$$\begin{aligned} Q(u,v) &= \tfrac{1}{2}\int_{R^d} \nabla u(x)\cdot\nabla v(x)\,dx + \tfrac{1}{2}\int_{R^d}\nabla(uv)\,\nabla\rho(x)\,dx \\ &\quad + \int_{R^d} u(x)v(x)\mu(dx), \\ D(Q) &= H_2^1(R^d) \quad \text{(the Sobolev space of order 1)}. \end{aligned} \tag{1.1}$$

This quadratic form is studied in [9] when $d \ge 3$. It is shown that $(Q, D(Q))$ is a closed semibounded form on $L^2(R^d)$. Although the results in [9] are stated for $d \ge 3$, they still hold for $d \le 2$. For our applications we would like to mention several facts in the case $d = 1$. It is well known that for any $\varepsilon > 0$ there exists $c(\varepsilon)$ such that

$$|u(x)|^2 \le \varepsilon\int_x^{x+1} u'(y)^2\,dy + c(\varepsilon)\int_x^{x+1}|u(y)|^2\,dy, \qquad u \in H_2^1(R). \tag{1.2}$$

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Given any $\mu \in K_d$, integrating both sides of (1.2) with respect to μ, we get

$$\begin{aligned}
&\int_R |u(x)|^2 |\mu|(dx) \\
&\quad \le \varepsilon \int_R u'(y)^2\, dy \int_{y-1}^{y} |\mu|(dx) + c(\varepsilon) \int_R u^2(y)\, dy \int_{y-1}^{y} |\mu|(dx) \\
&\quad \le \varepsilon \sup_x \int_{|x-y|\le 1} |\mu|(dy) \int_R u'(y)^2\, dy \\
&\qquad + c(\varepsilon) \sup_x \int_{|x-y|\le 1} \mu(dy) \int_R u^2(y)\, dy.
\end{aligned} \tag{1.3}$$

Applying (1.3) to $\rho'(x)^2$, we obtain, for any $\varepsilon > 0$,

$$\begin{aligned}
\left| \int u'(x) v(x) \rho'(x)\, dx \right| &\le \left(\int u'(x)^2\, dx \right)^{1/2} \left(\int v(x)^2 \rho'(x)^2\, dx \right)^{1/2} \\
&\le \frac{1}{2} \varepsilon \int u'(x)^2\, dx + \frac{1}{2} \frac{1}{\varepsilon} \int v(x)^2 \rho'(x)^2\, dx \\
&\le \frac{1}{2} \varepsilon \int u'(x)^2\, dx + \frac{1}{2} \frac{1}{\varepsilon} \sup_x \int_{|x-y|\le 1} \rho'(y)^2\, dy \\
&\qquad \times \left[\varepsilon^2 \int v'(x)^2\, dx + c(\varepsilon^2) \int v(x)^2\, dx \right] \\
&\le \frac{1}{2} \varepsilon \int u'(x)^2\, dx + \frac{1}{2} \left(\sup_x \int_{|x-y|\le 1} \rho'(y)^2\, dy \right) \\
&\qquad \times \left[\varepsilon \int v'(x)^2\, dx + \frac{c(\varepsilon^2)}{\varepsilon} \int v(x)^2\, dx \right].
\end{aligned} \tag{1.4}$$

From this it is easy to deduce that the quadratic form $(Q, D(Q))$ is well defined, closed and lower semibounded.

Let (Ω, F, F_t, P) be a probability space with filtration F_t satisfying the usual conditions.

DEFINITION 1. A real-valued stochastic process A_t on (Ω, P) is said to be of zero quadratic variation if

$$\lim_{n\to\infty} \sum_{t_i \in \tau^n} \left(A_{t_{i+1}} - A_{t_i} \right)^2 = 0 \tag{1.5}$$

in measure with respect to P for any sequence τ^n of partitions on $[0, T]$ with $\delta(\tau^n) \to 0$, where $T > 0$ is any fixed constant; $\delta(\tau^n)$ denotes the maximum length of the partition τ^n.

DEFINITION 2. We say that a continuous stochastic process Y_t is a Dirichlet process if the following decomposition holds:

$$Y_t = M_t + A_t, \tag{1.6}$$

where M_t is a continuous local martingale and A_t is of zero quadratic variation.

It is clear from the definition that the decomposition (1.6) is unique for a given Dirichlet process. Let $M = (\Omega, X_t, P_x)$ denote the Brownian motion on R^d. It is well known that the Dirichlet form associated with M_t is given by

$$\varepsilon^0(u, v) = \int_{R^d} \nabla u \cdot \nabla v \, dx, \qquad D(\varepsilon^0) = H_2^1(R^d). \tag{1.7}$$

For $u \in H_2^1(R^d)$, let $\tilde{u}$ denote a quasi-continuous version of u. Then Fukushima [8] showed that the additive functional $A_t^{[u]} = \tilde{u}(X_t) - \tilde{u}(X_0)$ admits the following decomposition:

$$A_t^{[u]} = M_t^u + N_t^u \quad P_x\text{-a.s., for q.e. } x \in R^d, \tag{1.8}$$

where M^u is a martingale additive functional of finite energy and the process N^u is a continuous additive functional of zero energy. In particular, A_t^u is a Dirichlet process under $P_m(\cdot) = \int_{R^d} P_x(\cdot)\, dx$. Let $\rho \in H_2^1(R^d)$. We assume that ρ is bounded and continuous. Let $\mu \in K_d$. The additive functional with Revuz measure μ is denoted by $A_t^{[\mu]}$. Let N_t^ρ be the continuous additive functional of zero energy defined in the Fukushima decomposition (1.8) for $A_t^{[\rho]}$. Introduce the following generalized Schrödinger semigroup:

$$T_t f(x) = E_x\left[\exp(N_t^\rho - A_t^\mu) f(X_t)\right], \qquad f \in B_b(R^d). \tag{1.9}$$

The following theorem is the main result proved in [9].

THEOREM 1.1. (i) *T_t extends to a strongly continuous semigroup on $L^2(R^d)$.*

(ii) *T_t has a symmetric continuous integral kernel $q(t, x, y)$ such that*

$$q(t, x, y) \le c e^{\beta t} t^{-d/2}. \tag{1.10}$$

(iii) *The quadratic form associated with T_t is given by $(Q, D(Q))$ in (1.1).*

In this paper we are concerned with the asymptotic properties of additive functionals of zero energy. More precisely, we are going to study the limits of the following type:

$$\lim_{t\to\infty} \frac{1}{t} \log E_x\left[\exp(N_t^\rho - A_t^\mu)\right]. \tag{1.11}$$

If $\rho = 0$, the previous type of asymptotic behavior has been studied in [12]. We note that N_t^ρ in general is not of bounded variation any more. Our strategy is the same as that in [12]. We first prove a modified Donsker–Varadhan large deviation principle. It is then applied to the study of the limit of type (1.11). This will be carried out in Section 2. We apply our results in Section 3 to the Hilbert transform of Brownian local time, which has been extensively studied by both mathematicians and physicists (see [2], [6], [7] and [11]).

2. A large deviation principle. Let $(L, D(L))$ be the generator of the semigroup T_t in $L^2(R^d)$. Then $D(L) \subset D(Q) = H_2^1(R^d)$, and $Q(u, v) = (-Lu, v)$, $u \in D(L)$, $v \in D(Q)$. Denote by $\{R_\alpha, \alpha > \beta\}$ the resolvent operators associated with $(Q, D(Q))$. For $f \in B_b(R^d)$, it holds that

$$R_\alpha f(x) = E_x\left[\int_0^\infty \exp(-\alpha t + N_t^\rho - A_t^\mu) f(X_t)\, dt\right] \tag{2.1}$$

and

$$L(R_\alpha f) = \alpha R_\alpha f - f.$$

Define

$$D_+^2(L) = \{R_\alpha f,\ \alpha > \beta,\ f \in C_0(R^d),\ f \geq 0,\ f \not\equiv 0\}. \tag{2.2}$$

We note that by the property of Brownian motion, any function ϕ in $D_+^2(L)$ is strictly positive. Now, we can state the following crucial result.

PROPOSITION 2.1. *For $\phi = R_\alpha f \in D_+^2(L)$, let M^ϕ denote the martingale part in Fukushima's decomposition* (1.8). *Then*

$$C_t^\phi = \exp(N_t^\rho - A_t^\mu)\frac{\phi(X_t)}{\phi(X_0)}\exp\left(-\int_0^t \frac{L\phi}{\phi}(X_s)\, ds\right)$$

is a supermartingale multiplicative functional and

$$C_t^\phi = \exp\left(M_t - \tfrac{1}{2}\langle M\rangle_t\right), \quad P_x\text{-}a.s.,\ q.e.\ x \in R^d, \tag{2.3}$$

where

$$M_t = \int_0^t \frac{1}{\phi(X_s)}\, dM_s^\phi.$$

PROOF. We first prove that (2.3) holds under P_m. Define

$$\begin{aligned} M_t^{\rho,\phi} = {}& \exp(N_t^\rho - A_t^\mu)\phi(X_t) - \phi(X_0) \\ & - \int_0^t \exp(N_s^\rho - A_s^\mu) L\phi(X_s)\, ds. \end{aligned} \tag{2.4}$$

Then it follows from the semigroup property that

$$E_x[M_t^{\rho,\phi}] = 0 \quad \text{and} \quad M_{t+s}^{\rho,\phi} = M_s^{\rho,\phi} + \exp(N_s^\rho - A_s^\mu) M_t^{\rho,\phi}(\theta_s).$$

This particularly implies that $M_t^{\rho,\phi}$ is a martingale. Applying Itô's formula to the semimartingale $\exp(N_t^\rho - A_t^\mu)\phi(X_t)$, we get

$$\begin{aligned} &\log(\exp(N_t^\rho - A_t^\mu)\phi(X_t)) - \log(\phi(X_0)) \\ &\quad = \int_0^t \exp(-N_s^\rho + A_s^\mu)\frac{1}{\phi(X_s)}\, dM_s^{\rho,\phi} + \int_0^t \frac{1}{\phi(X_s)} L\phi(X_s)\, ds \\ &\qquad - \frac{1}{2}\int_0^t \exp(-2N_s^\rho + 2A_s^\mu)\frac{1}{\phi^2(X_s)}\, d\langle M^{\rho,\phi}\rangle_s. \end{aligned} \tag{2.5}$$

It is easy to see from (2.5) that (2.3) will follow if we can prove that

$$M_t^{\rho,\phi} = \int_0^t \exp(N_s^\rho - A_s^\mu)\, dM_s^\phi. \tag{2.6}$$

We note that N_s^ρ is not a bounded variation process in general; (2.6) cannot be directly derived from Itô's formula. By the uniqueness of the decomposition of the Dirichlet process, in order to prove (2.6) it suffices to show that $Y_t := \exp(N_t^\rho - A_t^\mu)\phi(X_t) - \phi(X_0)$ is a Dirichlet process and the martingale part is given by $\int_0^t \exp(N_s^\rho - A_s^\mu)\, dM_s^\phi$. This follows if we can show that

$$B_t := Y_t - \int_0^t \exp(N_s^\rho - A_s^\mu)\, dM_s^\phi$$

is a process of zero quadratic variation. Now let $\tau^n = \{0 = t_0^n < t_1^n < t_2^n < \cdots < t_{K_n}^n = T\}$ be any sequence of partitions on $[0, T]$ (T is a fixed constant) with $\delta(\tau^n) \to 0$ as $n \to +\infty$. We have

$$\begin{aligned}
&\sum_{t_i \in \tau^n} \left(B_{t_{i+1}} - B_{t_i}\right)^2 \\
&\quad= \sum_{t_i \in \tau^n} \Bigg[\exp\left(N_{t_{i+1}}^\rho - A_{t_{i+1}}^\mu\right)\phi\left(X_{t_{i+1}}\right) - \exp\left(N_{t_i}^\rho - A_{t_i}^\mu\right)\phi\left(X_{t_i}\right) \\
&\qquad\qquad - \int_{t_i}^{t_{i+1}} \exp(N_s^\rho - A_s^\mu)\, dM_s^\phi\Bigg]^2 \\
&\quad= \sum_{t_i \in \tau^n} \Bigg[\left(\exp\left(N_{t_{i+1}}^\rho - A_{t_{i+1}}^\mu\right) - \exp\left(N_{t_i}^\rho - A_{t_i}^\mu\right)\right)\phi\left(X_{t_{i+1}}\right) \\
&\qquad\qquad + \exp\left(N_{t_i}^\rho - A_{t_i}^\mu\right)\left(\phi\left(X_{t_{i+1}}\right) - \phi\left(X_{t_i}\right)\right) \\
&\qquad\qquad - \exp\left(N_{t_i}^\rho - A_{t_i}^\mu\right)\left(M_{t_{i+1}}^\phi - M_{t_i}^\phi\right) \\
&\qquad\qquad + \int_{t_i}^{t_{i+1}} \left[\exp\left(N_{t_i}^\rho - A_{t_i}^\mu\right) - \exp(N_s^\rho - A_s^\mu)\right] dM_s^\phi\Bigg]^2 \\
&\quad\le c\Bigg[\sum_{t_i \in \tau^n} \left(\exp\left(N_{t_{i+1}}^\rho - A_{t_{i+1}}^\mu\right) - \exp\left(N_{t_i}^\rho - A_{t_i}^\mu\right)\right)^2 \phi^2\left(X_{t_{i+1}}\right) \\
&\qquad\qquad + \sum_{t_i \in \tau^n} \exp\left(2N_{t_i}^\rho - 2A_{t_i}^\mu\right)\left(N_{t_{i+1}}^\phi - N_{t_i}^\phi\right)^2 \\
&\qquad\qquad + \sum_{t_i \in \tau^n} \left(\int_{t_i}^{t_{i+1}} \left(\exp\left(N_{t_i}^\rho - A_{t_i}^\mu\right) - \exp(N_s^\rho - A_s^\mu)\right) dM_s^\phi\right)^2\Bigg],
\end{aligned} \tag{2.7}$$

where $N_t^\phi = \phi(X_t) - \phi(X_t) - M_t^\phi$ is defined in the Fukushima decomposition (1.8).

We use I_n, II_n and III_n to denote the terms on the right-hand side of the preceding inequality. We are going to prove that each term goes to 0 in P_m.

First

$$
\begin{aligned}
I_n &\le c \sum_{t_i \in \tau^n} \sup_{s \le T} \phi^2(X_s) \exp\Big(2 \sup_{s \le T} |N_s^\rho - A_s^\mu|\Big) \\
&\qquad \times \Big[N_{t_{i+1}}^\rho - A_{t_{i+1}}^\mu - \big(N_{t_i}^\rho - A_{t_i}^\mu\big)\Big]^2 \\
&\le c \sup_{s \le T} \phi^2(X_s) \exp\Big(2 \sup_{s \le T} |N_s^\rho - A_s^\mu|\Big) \\
&\qquad \times \sum_{t_i \in \tau^n} \Big[\big(N_{t_{i+1}}^\rho - A_{t_{i+1}}^\mu\big) - \big(N_{t_i}^\rho - A_{t_i}^\mu\big)\Big]^2.
\end{aligned} \tag{2.8}
$$

Since $N_t^\rho - A_t^\mu$ is of zero quadratic variation,

$$I_n \to 0 \quad \text{in } P_m.$$

By the same argument,

$$II_n \to 0 \quad \text{in } P_m.$$

To deal with III_n, we introduce some notation. Define

$$Z_t^n = \sum_{t_i \in \tau^n} 1_{[t_i, t_{i+1})}(t) \exp\big(N_{t_i}^\rho - A_{t_i}^\mu\big). \tag{2.9}$$

It is clear that $Z_t^n \to \exp(N_t^\rho - A_t^\mu)$. Put

$$T^m = \inf\Big\{t \ge 0, \sup_{s \le t} \exp(N_s^\rho - A_s^\mu) > m\Big\} \quad \text{for } m \in N. \tag{2.10}$$

Then T^m is a stopping time, $T^m \to +\infty$ as $m \to +\infty$ and

$$|Z_{t \wedge T^m}^n| \le m. \tag{2.11}$$

Thus, for any $\varepsilon > 0$,

$$
\begin{aligned}
P_m(III_n > \varepsilon) &\le P_m\big(III_m > \varepsilon, T < T^k\big) + P_m(T^k \le T) \\
&\le P_m\Bigg(\sum_{t_i \in \tau^n} \Bigg(\int_{t_i}^{t_{i+1}} 1_{[s < T^k]} \Big(\exp\big(N_{t_i}^\rho - A_{t_i}^\mu\big) \\
&\qquad\qquad - \exp\big(N_s^\rho - A_s^\mu\big)\Big)\, dM_s^\phi \Bigg)^2 > c\varepsilon \Bigg) \\
&\qquad + P_m(T^k \le T) \\
&\le \frac{1}{c\varepsilon} \sum_{t_i \in \tau^n} P_m \Bigg[\int_{t_i}^{t_{i+1}} 1_{[s < T^k]} \Big(\exp\big(N_{t_i}^\rho - A_{t_i}^\mu\big) \\
&\qquad\qquad - \exp\big(N_s^\rho - A_s^\mu\big)\Big)^2 |\nabla \phi|^2(X_s)\, ds \Bigg] \\
&\qquad + P_m(T^k \le T) \\
&= \frac{1}{c\varepsilon} P_m \Bigg[\int_0^T 1_{[s < T^k]} \big(Z_s^n - \exp\big(N_s^\rho - A_s^\mu\big)\big)^2 |\nabla \phi|^2(X_s)\, ds \Bigg] \\
&\qquad + P_m(T^k \le T).
\end{aligned} \tag{2.12}
$$

Note that

$$1_{[s<T^k]}\big(Z_s^n - \exp(N_s^\rho - A_s^\mu)\big)^2 |\nabla\phi|^2(X_s)\,ds \le (2k)^2 |\nabla\phi|^2(X_s)\,ds$$

and

$$P_m\left[\int_0^T |\nabla\phi|^2(X_s)\,ds\right] = T\int_{R^d} |\nabla\phi|^2(x)\,dx.$$

Applying the dominated convergence theorem, letting first $n \to \infty$, then $k \to \infty$, we get from (2.12) that

$$III_n \to 0 \quad \text{in } P_m.$$

Namely, we have proven that B_t is of zero quadratic variation. Thus we obtain

$$C_t^\phi = \exp\big(M_t - \tfrac{1}{2}\langle M\rangle_t\big), \quad P_x\text{-a.s., for almost all } x \in R^d. \tag{2.13}$$

We are now going to refine the preceding equality to quasi-everywhere $x \in R^d$. Set

$$\Lambda = \big\{\omega\colon \text{there exists } t \ge 0,\, C_t^\phi \ne \exp\big(M_t - \tfrac{1}{2}\langle M\rangle\big)_t\big\}.$$

Then $\theta_t\Lambda \subset \Lambda$, $\theta_t\Lambda \uparrow \Lambda$ as $t \downarrow 0$. This implies that the function $P_x(\Lambda)$ is excessive, and hence quasi-continuous. On the other hand, $P_x(\Lambda) = 0$ almost surely by (2.13). By the properties of quasi-continuous functions, we conclude that $P_x(\Lambda) = 0$ quasi-everywhere, which completes the proof. □

PROPOSITION 2.2. (i) *For any $p > 0$, there exist constants M_p and c such that*

$$E_x\big[\exp(pN_t^\rho - pA_t^\mu)\big] \le c\exp(M_p t). \tag{2.14}$$

(ii) *The operator T_t maps $B_b(R^d)$ into $C_b(R^d)$.*

(iii) *There exists a constant $\varepsilon > 0$ such that*

$$R_\alpha 1(x) \ge \varepsilon > 0 \quad \text{for } \alpha > \beta. \tag{2.15}$$

PROOF. (i) Note that $N_t^\rho = \rho(X_t) - \rho(X_0) - \int_0^t \nabla\rho(X_s)\,dX_s$. We have

$$\begin{aligned}
&E_x\left\{\exp\left[p\rho(X_t) - p\rho(X_0) - p\int_0^t (\nabla\rho)(X_s)\,dX_s - pA_t^\mu\right]\right\}\\
&\quad\le c_1 E_x\left\{\exp\left[-p\int_0^t \nabla\rho(X_s)\,dX_s - p^2\int_0^t |\nabla\rho|^2(X_s)\,ds\right.\right.\\
&\qquad\qquad\left.\left. + p^2\int_0^t |\nabla\rho|^2(X_s)\,ds - pA_t^\mu\right]\right\}\\
&\quad\le c_1 E_x\left\{\exp\left[-2p\int_0^t \nabla\rho(X_s)\,dX_s - 2p^2\int_0^t |\nabla\rho|^2(X_s)\,ds\right]\right\}^{1/2}\\
&\qquad\times E_x\left\{\exp\left[2p^2\int_0^t |\nabla\rho|^2(X_s)\,ds - 2pA_t^\mu\right]\right\}^{1/2}\\
&\quad\le c\exp(M_p t),
\end{aligned}$$

where we have used the fact that $\exp[-2p\int_0^t \nabla\rho(X_s)\,dX_s - 2p^2\int_0^t|\nabla\rho|^2(X_s)\,ds]$ is a martingale and

$$E_x\left[\exp\left(2p^2\int_0^t |\nabla\rho|^2(X_s)\,ds - 2pA_t^\mu\right)\right] \le \exp(c_p t)$$

(which follows from a general result for additive functionals corresponding to a Kato class measure ν_ρ).

(ii) Fix any $f \in B_b(R^d)$. We see from (i) that $T_t f(x) \in B_b(R^d)$. It remains to prove that $T_t f(x)$ is continuous. To this end, we fix a point x_0 in R^d. Let $x_n \in R^d$ be any sequence such that $x_n \to x_0$. We need to show

$$T_t f(x_n) \to T_t f(x_0). \tag{2.16}$$

For any $c > 0$, we have

$$\begin{aligned} T_t f(x_n) &= \int q(t, x_n, y) f(y)\,dy \\ &= \int_{|y|>c} q(t, x_n, y) f(y)\,dy + \int_{|y|\le c} q(t, x_n, y) f(y)\,dy. \end{aligned} \tag{2.17}$$

From Theorem 1.1 and the dominated convergence theorem,

$$\lim_{x_n \to x_0} \int_{|y|\le c} q(t, x_n, y) f(y)\,dy = \int_{|y|\le c} q(t, x_0, y) f(y)\,dy \quad \text{for any } c > 0.$$

Thus, to prove (2.16), it suffices to show that for any $\varepsilon > 0$ there exists c_ε such that

$$\sup_n \left|\int_{|y|>c_\varepsilon} q(t, x_n, y) f(y)\,dy\right| \le \varepsilon. \tag{2.18}$$

In fact, we can see (2.18) by

$$\begin{aligned} &\left|\int_{|y|>c} q(t, x_n, y) f(y)\,dy\right| \\ &\quad = \left| E_{x_n}\left[\exp(N_t^\rho - A_t^\mu) f(X_t) 1_{[|X_t|>c]}\right]\right| \\ &\quad \le E_{x_n}\left[\exp(2N_t^\rho - 2A_t^\mu)\right]^{1/2} E_{x_n}\left[f^2(X_t) 1_{[|X_t|>c]}\right]^{1/2} \\ &\quad \le \exp(M_2 t)\|f\|_\infty \frac{1}{c^2} E_{x_n}\left[|X_t|^2\right] \le \exp(M_2 t)\|f\|_\infty^2 \frac{2t + x_n^2}{c^2}. \end{aligned}$$

(iii) Applying (i) to $-\rho$ and $-\mu$, we have

$$\begin{aligned} E_x\left[\exp(N_t^\rho - A_t^\mu)\right] &\ge \frac{1}{E_x\left[\exp(-N_t^\rho + A_t^\mu)\right]} \\ &\ge \frac{1}{\exp(Mt)} = \exp(-Mt). \end{aligned} \tag{2.19}$$

Thus

$$R_\alpha 1(x) = \int_0^\infty \exp(-\alpha t) E_x\left[\exp(N_t^\rho - A_t^\mu)\right] dt$$
$$\geq \int_0^\infty \exp(-\alpha t)\exp(-Mt)\, dt = \frac{1}{\alpha + M} = \varepsilon > 0.$$

Now we introduce some notation. Let $M_1(R^d)$ denote the space of all probability measures on R^d. Define a rate function $L_{\varepsilon^\mu}(\nu)$ on $M_1(R^d)$ by

$$I_{\varepsilon^\mu}(\nu) = \begin{cases} Q(\phi, \phi), & \text{if } \nu \ll dx \text{ and } \phi = \sqrt{\dfrac{d\nu}{dx}} \in D(Q), \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.20}$$

Let

$$L_t(A) = \frac{1}{t}\int_0^t \chi_A(X_s)\, ds, \qquad A \in B_b(R^d),$$

denote the occupation measure of Brownian motion.

After preparing the previous two crucial propositions, using the same arguments as in [3], [4] and [12], we have the following result.

THEOREM 2.3. (i) *For any open set* $G \in M_1(R^d)$ *and* $x \in R^d$,

$$\lim_{t \to +\infty} \frac{1}{t} \log E_x\left[\exp(N_t^\rho - A_t^\mu), L_t \in G\right] \geq -\inf_{\nu \in G} I_{\varepsilon^\mu}(\nu). \tag{2.21}$$

(ii) *For any compact set* $K \in M_1(R^d)$ *and* $x \in R^d$,

$$\limsup_{t \to \infty} \frac{1}{t} \log E_x\left[\exp(N_t^\rho - A_t^\mu); L_t \in K\right] \leq -\inf_{\nu \in K} I_{\varepsilon^\mu}(\nu).$$

PROOF. (i) Let $\phi = R_\alpha f \in D_+^2(L)$ with $\phi^2\, dx \in G$. Denote by $M^\phi = (\Omega, X_t, P_x^\phi, \xi)$ the transformed process of M by the supermartingale multiplicative functional C_t^ϕ defined in Proposition 2.1, that is, $P_x^\phi = C_t^\phi P_x$. Then it is shown in [12] that M^ϕ is ergodic with invariant measure $\phi^2 dx$. We have

$$\begin{aligned} &E_x\left[\exp(N_t^\rho - A_t^\mu), L_t \in G\right] \\ &\quad = E_x^\phi\left[\left(C_t^\phi\right)^{-1}\exp(N_t^\rho - A_t^\mu), L_t \in G\right] \\ &\quad \geq \exp\left(t\int_{R^d} \phi L\phi\, dx - \varepsilon\right)\frac{\phi(x)}{\|\phi\|_\infty}\left(1 - P_x^\phi(\Omega - S(t, \varepsilon))\right), \end{aligned} \tag{2.22}$$

where

$$S(t, \varepsilon) = \left\{\omega \in \Omega : \left|\int_{R^d} \frac{L\phi}{\phi}(X_s) L_t(\omega, dx) - \int_{R^d} \phi L\phi\, dx\right| < \varepsilon\right\}$$
$$\cap \{\omega : L_t(\omega) \in G\}.$$

By the ergodic property,

$$P_x^{\phi}(\Omega - S(t, \varepsilon)) \to 0 \quad \text{as } t \to \infty \text{ for all } x \in R^d.$$

Thus, by (2.22),

$$\liminf_{t \to +\infty} \frac{1}{t} \log E_x\big[\exp(N_t^{\rho} - A_t^{\mu}); L_t \in G\big] \geq \int_{R^d} \phi L \phi \, dx - \varepsilon.$$

Since ε is arbitrary and $D_+^2(L)$ is dense in $D(Q)$, (i) follows.

To prove (ii), we define a modified I-functional. Put $D_{++}(L) = \{\phi = R_\alpha g,$ $\alpha > \beta,\ g \in C_b(R^d),\ g \geq \varepsilon$ for some $\varepsilon > 0\}$. Introduce a functional I on $M_1(R^d)$ by

$$I(\nu) = -\inf_{\phi \in D_{++}(L)} \int_{R^d} \frac{L\phi}{\phi} \, d\nu.$$

(ii) Let $\phi \in D_{++}(L)$. Since $E_x[C_t^{\phi}] \leq 1$, we have

$$E_x\left[\exp\left(N_t^{\rho} - A_t^{\mu} - \int_0^t \frac{L\phi}{\phi}(X_s)\, ds\right)\right] \leq \frac{\phi(x)}{\inf_{x \in R^d} \phi(x)}.$$

Note that $\inf \phi(x) > 0$ because of Proposition 2.2. Hence, for any Borel set $C \in M_1(R^d)$,

$$\limsup_{t \to \infty} \frac{1}{t} \log E_x\big[\exp(N_t^{\rho} - A_t^{\mu}); L_t \in C\big] \leq \inf_{\phi \in D_{++}(L)} \sup_{\nu \in C} \int_{R^d} \frac{L\phi}{\phi} \, d\nu. \tag{2.23}$$

Now let K be a compact subset of $M_1(R^d)$. Put

$$l = \sup_{\nu \in K} \inf_{\phi \in D_{++}(L)} \int_{R^d} \frac{L\phi}{\phi} \, d\nu.$$

Then for any $\delta > 0$ and using the fact that $L\phi \in C_b(R^d)$ and K is compact, we can find a finite number of $\nu_1, \dots, \nu_k \in K$, $\phi_{\nu_i} \in D_{++}(L)$ and neighborhoods $N(\nu_i)$ of ν_i such that $K \subseteq \bigcup_{j=1}^k N(\nu_j)$ and

$$\sup_{\nu \in N(\nu_i)} \int_{R^d} \frac{L\phi_{\nu_i}}{\phi_{\nu_i}} \, d\nu \leq l + 2\delta.$$

Thus,

$$\begin{aligned} \limsup_{t \to +\infty} \frac{1}{t} \log E_x\big[\exp(N_t^{\rho} - A_t^{\mu}), L_t \in K\big] &\leq \max_{1 \leq j \leq k} \inf_{\phi \in D_{++}(L)} \sup_{\nu \in N_j} \int \frac{L\phi}{\phi} \, d\nu \\ &\leq l + 2\delta. \end{aligned}$$

Since δ is arbitrary, we have

$$\limsup_{t \to \infty} \frac{1}{t} \log E_x\big[\exp(N_t^{\rho} - A_t^{\mu}), L_t \in K\big] \leq l.$$

Therefore, (ii) follows from the fact that $I(\nu) = I_{\varepsilon^{\mu}}(\nu)$, which can be proven similarly as in [12]. $\square$

COROLLARY 2.4.

$$\lim_{t\to\infty}\frac{1}{t}\log E_x\left[\exp\left(N_t^{\rho}-A_t^{\mu}\right)\right] = -\inf_{u\in D(Q),\,\|u\|_2=1} Q(u,u). \tag{2.24}$$

By noting that $\|T_t\|_{1,\infty}<\infty$, the proof of this corollary is just a repetition of that of Theorem 6.1 in [12].

3. Applications.

3.1. *Asymptotic property of Cauchy principal values of Brownian local time.* Let $M_t=(\Omega, X_t, P_x, x\in R)$ be the one-dimensional Brownian motion. Define

$$H_t = \lim_{\varepsilon\to 0}\int_0^t X_s^{-1}1_{[|X_s|\geq\varepsilon]}\,ds. \tag{3.1}$$

Then it is well known that H_t defines a continuous additive functional, which is called the Cauchy principal value of Brownian local time. We have the following result.

THEOREM 3.1.

$$\lim_{t\to\infty}\frac{1}{t}\log E_x\left[\exp(H_t)\right] = -\inf_{u\in D(Q),\,\|u\|_2=1} Q(u,u), \tag{3.2}$$

where

$$Q(u,v) = \frac{1}{2}\int\nabla u\cdot\nabla v\,dx + \text{p.v.}\int uv\frac{1}{x}\,dx,$$
$$D(Q) = H_2^1(R).$$

PROOF. It is known (see [8]) that the additive functional H_t is the zero energy part N_t^{ϕ} of Fukushima's decomposition (1.8) for $A_t^{[\phi]}$, where $\phi(x) = x\log|x| - x$. Since $\phi'(x)^2 = (\log|x|)^2$ is not in the Kato class, we cannot directly apply Corollary 2.4. However, we can write ϕ as

$$\phi(x) = \rho_1(x) + \rho_2(x), \tag{3.3}$$

where $\rho_1(x) = \phi_R(x)\phi(x)$, $\rho_2(x) = (1-\phi_R(x))\phi(x)$, $R>0$, and $\phi_R(x)$ is any function in $C_0^\infty(R)$ such that $\phi_R(x)=1$ on $|x|\leq R$, $\phi_R(x)=0$ on $|x|\geq R+1$. Note now that $(\rho'(x))^2 = [\phi_R'(x)\phi(x)+\phi'(x)\phi_R]^2$ is in the Kato class, and $\rho_2''(x) = \phi'(x)(1-\phi_R(x)) + 2\phi_R'(x)\phi'(x) + \phi_R''(x)\phi(x)$ is bounded. Hence $\mu = -\frac{1}{2}\rho_2''(x)\,dx$ is in the Kato class. Then it follows that

$$H_t = N_t^{\phi} = N_t^{\rho_1} + N_t^{\rho_2} = N_t^{\rho_1} - A_t^{\mu}.$$

Now we can apply Corollary 2.4 to get

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\log E_x\left[\exp(H_t)\right] &= \lim_{t\to\infty}\frac{1}{t}\log E_x\left[\exp\left(N_t^{\rho_1}-A_t^{\mu}\right)\right]\\ &= -\inf_{u\in D(Q),\,\|u\|_2=1} Q(u,u),\end{aligned}$$

where

$$Q(u,u) = \frac{1}{2}\int \nabla u \cdot \nabla v\, dx + \frac{1}{2}\int \nabla(uv)\cdot\nabla\rho_1(x)\, dx - \frac{1}{2}\int \rho_2''(x)uv\, dx$$
$$= \frac{1}{2}\int \nabla u \cdot \nabla v\, dx + \text{p.v.}\int uv\frac{1}{x}\, dx.$$

This proves Theorem 3.1. □

3.2. *Cauchy principal values of local time*: *more general case.* Let $\alpha > -\frac{3}{2}$ and $\alpha \neq 1$. Define

$$H_t^\alpha = \lim_{\varepsilon\to 0}\int_0^t |X_s|^\alpha(\operatorname{sgn}(X_s))1_{[|X_s|\geq\varepsilon]}\, ds.$$

Then it is a known fact that H^α is well defined and continuous. It is the so-called Hilbert transform of the local time. We have the following result.

THEOREM 3.2.

$$\lim_{t\to\infty}\frac{1}{t}\log E_x\left[\exp(H_t^\alpha)\right] = -\inf_{u\in D(Q^\alpha),\, \|u\|_2=1} Q^\alpha(u,u),$$

where Q^α is defined as

$$Q^\alpha(u,v) = \int_R \nabla u\cdot\nabla v\, dx + \text{p.v.}\int_R (uv)|x|^\alpha \operatorname{sgn}(x)\, dx,$$
$$D(Q^\alpha) = H_2^1(R).$$

PROOF. We define

$$\phi(x) = \frac{1}{\alpha+1}\int_0^x |y|^{\alpha+1}\, dy = \begin{cases} \dfrac{x^{\alpha+2}}{(\alpha+2)(\alpha+1)}, & x\geq 0,\\[2ex] \dfrac{(-x)^{\alpha+2}}{(\alpha+2)(\alpha+1)}, & x<0.\end{cases}$$

By [13] and [14], $H_t = N_t^\phi$. [Note that $\phi\in H^1_{2,\mathrm{loc}}(R)$.] As we did in (3.3), we write $\phi = \rho_1+\rho_2$, where $\rho_1(x) = \phi(x)\phi_1(x)$, $\rho_2(x) = \phi(x)(1-\phi_1(x))$. Since $\alpha > -\frac{3}{2}$, it follows that $\rho_1'(x)^2\in L^1(R)$; hence $\rho_1'(x)^2$ is in the Kato class. Note that $\mu := -\frac{1}{2}\rho_2''(x)\, dx$ is in the Kato class [this is due to the fact that $\rho_2''(x)\in B_b(R)$]. Applying Corollary 2.4 to $N_t^\phi = N_t^{\rho_1} - A_t^\mu$, we finish the proof. □

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DEPARTMENT OF MATHEMATICAL SCIENCES
FACULTY OF ENGINEERING SCIENCE
OSAKA UNIVERSITY
OSAKA
JAPAN

FAKULTÄT FÜR MATHEMATIK
UNIVERSITÄT BIELEFELD
POSTFACH 100131
33501 BIELEFELD
GERMANY