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PERCOLATION AND CONTACT PROCESSES WITH LOW-DIMENSIONAL INHOMOGENEITY

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We consider inhomogeneous nearest neighbor Bernoulli bond percolation on $\mathbb{Z}^d$ where the bonds in a fixed s-dimensional hyperplane ($1 \le s \le d-1$) have density p_1 and all other bonds have fixed density, $p_c(\mathbb{Z}^d)$, the homogeneous percolation critical value. For $s \ge 2$, it is natural to conjecture that there is a new critical value, $p_c^s(\mathbb{Z}^d)$, for p_1, strictly between $p_c(\mathbb{Z}^d)$ and $p_c(\mathbb{Z}^s)$; we prove this for large d and $2 \le s \le d-3$. For $s = 1$, it is natural to conjecture that $p_c^1(\mathbb{Z}^d) = 1$, as shown for $d = 2$ by Zhang; we prove this for large d. Related results for the contact process are also presented.

0. Introduction. We begin with some general background and some motivation for studying the type of inhomogeneous percolation models considered in this paper. In independent nearest neighbor bond percolation on $\mathbb{Z}^d$, the bonds (i.e., the nearest neighbor edges) b of $\mathbb{Z}^d$ are independently open (respectively, closed) with probability p_b (respectively, $1 - p_b$). Clusters are maximal collections of sites connected to each other by (nearest neighbor) paths of open bonds. The existence of an infinite cluster is a tail event and hence its probability must be zero or one; in the latter case percolation is said to occur. The standard percolation model is the homogeneous one in which $p_b = p$ for every b, and the basic result of percolation theory [see, e.g., Grimmett (1989)] is that for $d \ge 2$, there is a critical value $p_c = p_c(\mathbb{Z}^d)$ *strictly* between 0 and 1 such that percolation occurs (respectively, is absent) for $p > p_c$ (respectively, for $p < p_c$). It is a major open problem (except for $d = 2$ and for large d, as discussed below) to prove absence of percolation at $p = p_c$.

One of the recurring themes in work on this and related problems has been the relation between percolation on the full space $\mathbb{Z}^d$ and percolation on the half space $\mathbb{Z}^+ \times \mathbb{Z}^{d-1}$. For example, this relation played an important role in the results of Harris (1960) which, when combined with those of Kesten (1980), imply absence of percolation at the critical point in $\mathbb{Z}^2$. For general d, absence of critical percolation was reduced by Barsky, Grimmett and

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Newman (1991) to the problem of showing that for any p, percolation in the full space implies percolation in the half space.

There is a natural way to interpolate continuously between the full space and the half space models; namely, by using the inhomogeneous model in which $p_b = p$ for all bonds b except those between a fixed pair of adjacent $(d-1)$-dimensional hyperplanes, where $p_b = p'$. When $p' = p$, this is just the homogeneous model on the full space while for $p' = 0$, this becomes a pair of independent models on half spaces. This interpolating feature suggests that the study of the percolation properties of models with similar types of inhomogeneity may be of some value, even if one's ultimate goal is to obtain results about the standard homogeneous models.

In this paper, we study d-dimensional models with lower-dimensional inhomogeneity, but in a different direction than what would be needed to prove absence of critical percolation in the standard models. Here, we will use results already known for the standard models (in high dimension) to obtain results about the inhomogeneous models. What one would like to do, to prove absence of critical percolation, is set $p = p_c$ and then show that absence of percolation when $p' = 0$ implies its absence when $p' = p_c$; what we do instead is more in the direction of showing that (something stronger than) absence of percolation when $p' = p_c$ implies its absence when $p' > p_c$.

In Section 1, we state precisely the inhomogeneous percolation models we consider and our results about those models. In Section 2 we present related results for inhomogeneous contact processes on $\mathbb{Z}^d$. We remark that it is possible to extend our results to inhomogeneous models on other lattices than $\mathbb{Z}^d$ and this will be done in a future paper. Such models include percolation on the product of $\mathbb{Z}$ with a homogeneous tree [see Grimmett and Newman (1990) and Wu (1993)] and the contact process on a homogeneous tree [see Pemantle (1992), Morrow, Schinazi and Zhang (1994), Liggett (1995) and Wu (1995)].

To avoid confusion, it should be noted that the percolation models we treat in Section 1 differ from the ones described above (but in a manner insignificant for our results) in that the inhomogeneous bonds are taken *within* a hyperplane rather than *between* two adjacent hyperplanes. A more significant feature is that our results are not applicable to hyperplanes of dimension $d-1$ but only for dimensions strictly below $d-2$.

1. Percolation. We consider the following inhomogeneous nearest neighbor independent bond percolation model on $\mathbb{Z}^d$ with $d \geq 2$: each bond in $\{0\}^{d-s} \times \mathbb{Z}^s$ (where $1 \leq s \leq d-1$) is open (respectively, closed) with probability p_1 (respectively, $1 - p_1$) and each remaining bond in $\mathbb{Z}^d$ is open (respectively closed) with probability p_2 (respectively $1 - p_2$): all bonds of $\mathbb{Z}^d$ are independent of each other. We call this a (p_1, p_2)-model and shall assume that $0 < p_1, p_2 < 1$ unless stated otherwise. The resulting probability measure will be denoted by P_{p_1, p_2} and expectation with respect to P_{p_1, p_2} denoted by E_{p_1, p_2}. Two sites, x and y, of $\mathbb{Z}^d$ are said to be connected if there is a path of open bonds in $\mathbb{Z}^d$ from x to y, and we denote this event by $x \leftrightarrow y$. The open cluster $C(x)$ of x is defined to be the random set of sites connected to x. The number of sites in $C(x)$ is denoted by $|C(x)|$. The percolation probability θ^s (or simply θ) is then defined by

$$\theta(p_1, p_2) = P_{p_1, p_2}(|C(o)| = \infty), \tag{1.1}$$

where o is the origin of $\mathbb{Z}^d$. Although the probability $P_{p_1, p_2}(|C(x)| = \infty)$ depends on the site x because of the inhomogeneity of the model, it is easy to show [by Fortuin–Kasteleyn–Ginibre (FKG) inequalities] that either $P_{p_1, p_2}(|C(x)| = \infty) > 0$ for every x in $\mathbb{Z}^d$ or it is identically 0. Percolation occurs if $\theta(p_1, p_2) > 0$. When $p_1 = p_2 = p$, the model becomes the standard homogeneous one. As mentioned above, it is a fundamental result that there exists a critical value $p_c = p_c(\mathbb{Z}^d)$ in $(0, 1)$ such that

$$\begin{aligned} \theta(p, p) &= 0 \quad \text{if } p < p_c, \\ \theta(p, p) &> 0 \quad \text{if } p > p_c. \end{aligned} \tag{1.2}$$

[See e.g., Grimmett (1989) or Kesten (1982).] It has been long conjectured that

$$\theta(p_c, p_c) = 0. \tag{1.3}$$

However, (1.3) is only proved for $d = 2$ [see Harris (1960) and Kesten (1980)] and for sufficiently large d [$d \geq 19$ is large enough; see Hara and Slade (1990)].

In this paper, we consider the model in which p_2 is fixed to be $p_c = p_c(\mathbb{Z}^d)$ while p_1 varies. For $s \geq 2$, it is natural to conjecture that there is a new critical value p_c^s (for p_1) with $p_c < p_c^s < p_c(\mathbb{Z}^s)$ such that

$$\theta(p_1, p_c) = 0 \quad \text{if } p_1 < p_c^s, \tag{1.4a}$$

$$\theta(p_1, p_c) > 0 \quad \text{if } p_1 > p_c^s. \tag{1.4b}$$

We will prove this for d sufficiently large [as in Hara and Slade (1990)] and $2 \leq s \leq d - 3$. Notice that (1.4a) (with $p_c^s > p_c$) is a stronger statement than (1.3). For $s = 1$, we will show that for d sufficiently large,

$$\theta(p_1, p_c) = 0 \quad \text{for any } p_1 \in [0, 1). \tag{1.5}$$

Madras, Schinazi and Schonmann (1994) have considered the $s = 1$ case from a different point of view. They proved, among many other things, that for any fixed p_1 in $(p_c, 1)$ the model (with p_2 as the single parameter) has the same critical value as that of the homogeneous model on $\mathbb{Z}^d$. That is, for any $p_1 \in (p_c, 1)$

$$\theta(p_1, p_2) = 0 \quad \text{if } p_2 < p_c, \tag{1.6a}$$

$$\theta(p_1, p_2) > 0 \quad \text{if } p_2 > p_c. \tag{1.6b}$$

[Of course, (1.6b) is obvious.] Equation (1.5) is one of their conjectures. In two dimensions, (1.5) has been proved by Zhang (1994) by a careful extension of the dual contour argument. We remark that (1.6) is also valid for $0 \leq p_1 \leq p_c$. The $p_1 < p_c$ case of (1.6a) is obvious while the $p_1 < p_c$ case of (1.6b) follows from the fact that the critical probability for the half space equals the critical probability for the full space [see Harris (1960) for $d = 2$ and Grimmett and Marstrand (1990) for $d > 2$]; this is so because $\theta(0, p_2)$ is positive if there is percolation in a half space at $p = p_2$. Finally, the $p_1 = p_c$ case follows by simple comparisons to the $p_1 > p_c$ and $p_1 < p_c$ cases.

We state our results in the following two theorems and their corollaries after introducing some more notation. Define

$$\tau_{p_1, p_2}(x, y) = P_{p_1, p_2}(x \leftrightarrow y)$$

to be the connectivity function between two sites x and y of $\mathbb{Z}^d$ and write $\tau_{p_c}(x, y)$ for $\tau_{p_c, p_c}(x, y)$, the critical connectivity function in the homogeneous model. Note that by uniqueness of the infinite cluster and the FKG inequalities,

$$\begin{aligned}\tau_{p_c}(x, y) &= P_{p_c, p_c}(C(x) = C(y))\\ &\geq P_{p_c, p_c}(C(x) = C(y), |C(x)| = \infty = |C(y)|)\\ &= P_{p_c, p_c}(|C(x)| = \infty, |C(y)| = \infty)\\ &\geq P_{p_c, p_c}(|C(x)| = \infty) P_{p_c, p_c}(|C(y)| = \infty) = [\theta(p_c, p_c)]^2;\end{aligned}$$

thus the hypotheses of either of our two theorems automatically imply that $\theta(p_c, p_c) = 0$.

THEOREM 1. *Let $d \geq 2$ and $s = 1$. If*

$$\sum_{l \leq 0,\, m \geq 1} \tau_{p_c}((0, \dots, 0, l), (0, \dots, 0, m)) < \infty,$$

then

$$\theta(p_1, p_c) = 0 \quad \text{for any } p_1 \in [0, 1). \tag{1.7}$$

THEOREM 2. *Let $d \geq 2$ and $(d >) s \geq 2$. If $\sum_{x \in \{0\}^{d-s} \times \mathbb{Z}^s} \tau_{p_c}(o, x) < \infty$, then there exists a new critical value p_c^s in $(p_c, p_c(\mathbb{Z}^s))$ such that*

$$\begin{aligned}\theta(p_1, p_c) &= 0 \quad \text{if } p_1 < p_c^s,\\ \theta(p_1, p_c) &> 0 \quad \text{if } p_1 > p_c^s.\end{aligned} \tag{1.8}$$

Theorems 1 and 2 will be proved in Section 3. We remark, as explained at the very end of Section 3, that an extension of arguments used by Aizenman and Newman (1984) for homogeneous percolation yields the inequality $\gamma^s \geq 1$ for a critical exponent describing cluster size divergence. We also note that the proof of Theorem 2 shows that for $s = 1$, its summability hypothesis suffices to yield $p_c^1 > p_c$; however, our current proof of the stronger conclusion that $p_c^1 = 1$ [i.e., (1.7)] requires the stronger summability hypothesis of Theorem 1. The following is our main application of Theorems 1 and 2.

COROLLARY 1. *If d is sufficiently large, then*

(a) *for $s = 1$, $\theta(p_1, p_c) = 0$ for any $p_1 \in [0, 1)$;*

(b) *for* $2 \le s \le d-3$, *there exists* p_c^s *in* $(p_c, p_c(\mathbb{Z}^s))$ *such that* (1.8) *is valid*.

PROOF. A major ingredient in the results of Hara and Slade (1990) on percolation critical exponents for high d is the highly nontrivial estimate (see their Theorem 1.1.) that when d is sufficiently large, than

$$\tau_{p_c}(x, y) \le \frac{C}{\|x-y\|^{d-2}}, \tag{1.9}$$

where $C = C_d < \infty$ and $\|x\| = \max\{|x_i|: i = 1, \dots, d\}$. Thus

$$\sum_{l \le 0,\ m \ge 1} \tau_{p_c}((0, \dots, 0, l), (0, \dots, 0, m)) \le \sum_{l \le 0,\ m \ge 1} \frac{C}{(m-l)^{d-2}} \tag{1.10}$$

A simple calculation shows that the right-hand side of (1.10) is finite for $d > 4$. Part (a) then follows from Theorem 1. For part (b), applying (1.9) again, we have that

$$\sum_{x \in \{0\}^{d-s} \times \mathbb{Z}^s} \tau_{p_c}(o, x) \le C' \sum_{n=1}^{\infty} n^{s-1} \frac{1}{n^{d-2}},$$

which is finite when $s < d-2$. Part (b) then follows from Theorem 2. □

For oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}$, which may be regarded as a discrete time contact process on $\mathbb{Z}^d$, analogues of Theorems 1 and 2 follow from analogous arguments. The analogue of Corollary 1 then follows by applying the Nguyen and Yang (1993) analogue of the infrared bound (1.9). The continuous time contact process will be the subject of Section 2.

The rest of this paper is organized as follows. In Section 2, we present the analogues of Theorems 1 and 2 for the (continuous time) contact process and discuss the status of the analogue of Corollary 1. The proofs of Theorems 1 and 2 and of the contact process analogues are given in Section 3.

We conclude this section with a brief discussion of some open problems. Part (a) of Corollary 1, conjectured for all $d \ge 2$ by Madras, Schinazi and Schonmann (1994), was proved by Zhang (1994) for $d = 2$ and is proved in this paper for large d; it is an open problem to prove it for intermediate values of d. We note in this regard that the summability condition of Theorem 1 is not expected to be valid for $d = 3$. We conjecture that Part (b) of Corollary 1 is valid for all $d \ge 3$ and $2 \le s \le d-1$. The upper bound $p_c^s < p_c(\mathbb{Z}^s)$, which follows from the results of Aizenman and Grimmett (1991), is valid for all these cases. It is an open problem to obtain the lower bound $p_c^s > p_c(\mathbb{Z}^d)$ both for large d and $s = d-2$ or $d-1$ and for intermediate d and any s with $1 \le s \le d-1$. We remark that in the former case, the summability condition of Theorem 2 is not valid. Another set of open problems is to extend Theorems 1 and 2 to the case of Ising models with low-dimensional inhomogeneity. In that context, we have obtained only partial results, which we do not present here.

2. Contact process. The contact process on $\mathbb{Z}^d$ can be defined by the usual graphical representation, as follows. Consider the space $\mathbb{Z}^d \times [0, \infty)$, in which $\mathbb{Z}^d$ represents the spatial component and $[0, \infty)$ represents time. Along each vertical time line $\{x\} \times [0, \infty)$ is positioned a Poisson process of points, called deaths, with density $\delta(x)$; between each ordered pair $\{x_1\} \times [0, \infty)$ and $\{x_2\} \times [0, \infty)$ of nearest neighbor lines, there is a Poisson process, with density λ, of bonds oriented in the direction x_1 to x_2. All these Poisson processes are taken to be independent of each other.

We are interested in the following two cases:

CASE 2a. $\delta(o) = \delta$ and $\delta(x) = 1$ for any $x \in \mathbb{Z}^d - \{o\}$, where o denotes the origin of $\mathbb{Z}^d$.

CASE 2b. $\delta(x) = \delta$ for any $x \in \{0\}^{d-s} \times \mathbb{Z}^s$ and $\delta(x) = 1$ for any $x \in \mathbb{Z}^d - (\{0\}^{d-s} \times \mathbb{Z}^s)$, where $1 \le s \le d - 1$.

We call this a (δ, λ)-model and write $P_{\delta, \lambda}$ for the resulting probability measure and $E_{\delta, \lambda}$ for the corresponding expectation. The case $\delta = 1$ corresponds to the homogeneous model, for which we will simply write P_λ rather than $P_{1, \lambda}$. A point (x, t_1) in $\mathbb{Z}^d \times [0, \infty)$ is said to be connected to another point (y, t_2) with $t_1 \le t_2$ if there is a path from (x, t_1) to (y, t_2) using vertical line segments touching no points of death, traversed in the upward direction, and oriented horizontal bonds. We denote this event by $(x, t_1) \to (y, t_2)$.

Let ξ_t be the set of sites x in $\mathbb{Z}^d$ such that the origin $(o, 0)$ of $\mathbb{Z}^d \times [0, \infty)$ is connected to (x, t). If we treat the contact process in the usual way as a model for the spread of infection, then ξ_t is the set of infected sites at time t when initially only the origin is infected. Let

$$C(x, t) = \{(y, s) : (x, t) \to (y, s)\}$$

denote the cluster of (x, t). We write C for $C(o, 0)$. Define the survival probability to be

$$\theta(\delta, \lambda) = P_{\delta, \lambda}(\xi_t \text{ is nonempty for all } t); \tag{2.1}$$

then the critical value (for the homogeneous model) is defined by

$$\lambda_c = \lambda_c(\mathbb{Z}^d) = \inf\{\lambda : \theta(1, \lambda) > 0\}. \tag{2.2}$$

A fundamental theorem of Bezuidenhout and Grimmett (1990) states that

$$\theta(1, \lambda_c) = 0. \tag{2.3}$$

Madras, Schinazi and Schonmann (1994) considered Case 2a and proved that if $\lambda < \lambda_c$, then

$$\theta(\delta, \lambda) = 0 \quad \text{for any } \delta \in (0, 1].$$

They conjectured that

$$\theta(\delta, \lambda_c) = 0 \quad \text{for any } \delta \in (0, 1]. \tag{2.4}$$

We remark that this conjecture, which would be a strengthening of the Bezuidenhout and Grimmett result, has not been proved even for the one-dimensional contact process, although for percolation, the analogous result for *two*-dimensional *nonoriented* percolation has been proved by Zhang (1994).

The next theorem addresses this conjecture, and the following one deals with Case 2b. For technical reasons, we extend the time coordinate from $[0, \infty)$ to $\mathbb{R} = (-\infty, \infty)$. It should be clear that the graphical representation can be extended to $\mathbb{Z}^d \times \mathbb{R}$.

THEOREM 3. *In Case* 2a, *if*

$$\int_{-\infty}^{0} \int_{1}^{\infty} P_{\lambda_c}((o, t) \to (o, u))\, du\, dt < \infty \tag{2.5}$$

then

$$\theta(\delta, \lambda_c) = 0 \quad \textit{for any } \delta \in (0, 1]. \tag{2.6}$$

THEOREM 4. *In Case* 2b, *if*

$$\sum_{x \in \{0\}^{d-s} \times \mathbb{Z}^s} \int_{0}^{\infty} P_{\lambda_c}((o, 0) \to (x, t))\, dt < \infty, \tag{2.7}$$

then there exists δ_c^s *in* $(\lambda_c(\mathbb{Z}^d)/\lambda_c(\mathbb{Z}^s), 1)$ *such that*

$$\begin{aligned} \theta(\delta, \lambda_c) &= 0 \quad \textit{if } \delta > \delta_c^s, \\ \theta(\delta, \lambda_c) &> 0 \quad \textit{if } \delta < \delta_c^s. \end{aligned} \tag{2.8}$$

The proofs of Theorems 3 and 4 are closely related to those of Theorems 1 and 2. We give the proof of Theorem 3 and sketch the proof of Theorem 4 in Section 3. At the end of Section 3, there is a remark about a critical exponent inequality.

As in Corollary 1, a sufficient condition for (2.5) and (2.7) (for large d) would be an infrared bound for the homogeneous contact process,

$$P_{\lambda_c}((x, t) \to (y, u)) \le \frac{C}{\|(x, t) - (y, u)\|^{(d+1)-2}}, \tag{2.9}$$

which is expected to be true when $d > 4$ [see Obukhov (1980) and Nguyen and Yang (1993)]. More precisely, if (2.9) holds for a given d, then we have:

1. For $d \ge 4$, (2.5) and hence (2.6) hold;
2. For $s \le d - 3$, (2.7) holds and hence (2.8) holds when $1 \le s \le d - 3$.

However (2.9) has not yet been rigorously proved.

3. Proofs of theorems.

PROOF OF THEOREM 1. We follow an approach taken by Schulman (1983) [extended by Aizenman and Newman (1986)] in the context of long range one-dimensional percolation. We call a site $(0,\dots,0,n)$ of $\{0\}^{d-1}\times\mathbb{Z}$ a breakpoint if the half line $\{(0,\dots,0,l)\in\{0\}^{d-1}\times\mathbb{Z}: l\le n\}$ is not connected to the half line $\{(0,\dots,0,m)\in\{0\}^{d-1}\times\mathbb{Z}: m\ge n+1\}$. Write B for the event that the (nearest neighbor) bond between $(0,0,\dots,0)$ and $(0,,\dots,0,1)$ is closed and write "$(0,\dots,0,l)\not\leftrightarrow(0,\dots,0,m)$ in $\mathbb{Z}^d-(\{0\}^{d-1}\times\mathbb{Z})$" for the complement of the event that there is a path of open bonds, not using any bonds in $\{0\}^{d-1}\times\mathbb{Z}$, connecting $(0,\dots,0,l)$ and $(0,\dots,0,m)$. Using the FKG inequalities, we have

$$\begin{aligned}
&P_{p_1,p_c}((0,0,\dots,0)\text{ is a breakpoint})\\
&\quad= P_{p_1,p_c}\Bigl(B\cap\bigcap_{l\le 0,\ m\ge 1}\{(0,\dots,0,l)\not\leftrightarrow(0,\dots,0,m)\}\\
&\qquad\qquad\qquad\text{in }\mathbb{Z}^d-\bigl(\{0\}^{d-1}\times\mathbb{Z}\bigr)\Bigr)\\
&\quad\ge P_{p_1,p_c}(B)\prod_{l\le 0,\ m\ge 1}P_{p_1,p_c}\Bigl((0,\dots,0,l)\not\leftrightarrow(0,\dots,0,m)\\
&\qquad\qquad\qquad\text{in }\mathbb{Z}^d-\bigl(\{0\}^{d-1}\times\mathbb{Z}\bigr)\Bigr)\\
&\quad=(1-p_1)\prod_{l\le 0,\ m\ge 1}P_{p_c,p_c}\Bigl((0,\dots,0,l)\not\leftrightarrow(0,\dots,0,m)\\
&\qquad\qquad\qquad\text{in }\mathbb{Z}^d-\bigl(\{0\}^{d-1}\times\mathbb{Z}\bigr)\Bigr)\\
&\quad\ge(1-p_1)\prod_{l\le 0,\ m\ge 1}\bigl(1-\tau_{p_c}((0,\dots,0,l),(0,\dots,0,m))\bigr).
\end{aligned}\tag{3.1}$$

The right-hand side of (3.1) is a strictly positive constant by the hypothesis of Theorem 1. So from the ergodicity and translation invariance in the last coordinate direction,

$$\begin{aligned}
&P_{p_1,p_c}((0,\dots,0,n)\text{ is a breakpoint for infinitely}\\
&\qquad\text{many positive and infinitely many negative }n)=1.
\end{aligned}\tag{3.2}$$

A consequence of (3.2) is that a.s. every cluster in $\mathbb{Z}^d$ can only intersect the line $\{0\}^{d-1}\times\mathbb{Z}$ at only finitely many places. Note that if there exists an infinite cluster in $\mathbb{Z}^d$ which intersects $\{0\}^{d-1}\times\mathbb{Z}$ at only finitely many places, then there exists an infinite cluster in $\mathbb{Z}^d-(\{0\}^{d-1}\times\mathbb{Z})$. Therefore, if $\theta(p_1,p_c)>0$, then there exists a.s. an infinite cluster in $\mathbb{Z}^d-(\{0\}^{d-1}\times\mathbb{Z})$; that is,

$$P_{p_1,p_c}\Bigl(\text{there exists an infinite cluster in }\mathbb{Z}^d-\bigl(\{0\}^{d-1}\times\mathbb{Z}\bigr)\Bigr)=1.\tag{3.3}$$

But (3.3) is equivalent to P_{p_c,p_c} (there exists an infinite cluster in $\mathbb{Z}^d-(\{0\}^{d-1}\times\mathbb{Z}))=1$, a contradiction to $\theta(p_c,p_c)=0$ (which itself follows from the hypothesis of the theorem). This proves Theorem 1. □

PROOF OF THEOREM 2. First, it is clear that if $p_1 > p_c(\mathbb{Z}^s)$ then $\theta(p_1, p_c) > 0$, since there is already percolation in $\{0\}^{d-s} \times \mathbb{Z}^s$, and hence there is percolation in $\mathbb{Z}^d$. Thus $p_c^s \le p_c(\mathbb{Z}^s)$. The strict inequality $p_c^s < p_c(\mathbb{Z}^s)$ follows from Theorem 1 of Aizenman and Grimmett (1991), because the bonds on $\mathbb{Z}^d - (\{0\}^{d-s} \times \mathbb{Z}^s)$ (each of which has probability p_c to be open) provide an "essential enhancement" of the percolation model on $\mathbb{Z}^s$. It remains to show that there exists $\varepsilon > 0$ such that

$$\theta(p_c + \varepsilon, p_c) = 0. \tag{3.4}$$

Write S for $\{0\}^{d-s} \times \mathbb{Z}^s$. Let

$$\overline{C} = C(o) \cap S = \{x \in S : o \leftrightarrow x \text{ in } \mathbb{Z}^d\} \tag{3.5}$$

be the intersection of S and the cluster of the origin. From the hypothesis of the theorem,

$$\chi \equiv E_{p_c, p_c} |\overline{C}| = \sum_{x \in s} \tau_{p_c}(o, x) < \infty. \tag{3.6}$$

The idea of proving (3.4) is to show that

$$E_{p_c + \varepsilon, p_c} |\overline{C}| < \infty. \tag{3.7}$$

To see how (3.4) follows from (3.7), decompose the event that the origin is in an infinite cluster as

$$\{|C(o)| = \infty\} = \{|C(o)| = \infty \text{ and } |\overline{C}| < \infty\} \cup \{|\overline{C}| = \infty\}. \tag{3.8}$$

From (3.7), we have

$$P_{p_c + \varepsilon, p_c}(|\overline{C}| = \infty) = 0. \tag{3.9}$$

Moreover,

$$\begin{aligned} &P_{p_c + \varepsilon, p_c}(|C(o)| = \infty \text{ and } |\overline{C}| < \infty) \\ &\quad \le P_{p_c + \varepsilon, p_c}(\text{there exists an infinite cluster in } \mathbb{Z}^d - S) \\ &\quad = P_{p_c, p_c}(\text{there exists an infinite cluster in } \mathbb{Z}^d - S) \\ &\quad \le \theta(p_c, p_c) = 0. \end{aligned} \tag{3.10}$$

The vanishing of $\theta(p_c, p_c)$ follows from the hypothesis of the theorem, as explained just before the statement of Theorem 1 in Section 1. Combining (3.8)–(3.10) proves (3.4). We now turn to the proof of (3.7).

The $(p_c + \varepsilon, p_c)$-model can be thought of as follows. Independently color (part of) each bond of $\mathbb{Z}^d$ blue with probability p_c, and then independently color (a different part of) each bond of S red with probability $\varepsilon/(1 - p_c)$; the blue-coloring and red-coloring processes are independent of each other, so bonds can be both colors (or one color or neither). Write P for the resulting probability measure and E for the corresponding expectation. Declare each bond of $\mathbb{Z}^d$ to be open if it is either blue or red (or both). Let

$$C^0 = \{y \in S : o \leftrightarrow y \text{ by a blue path in } \mathbb{Z}^d\} \tag{3.11}$$

and call it the level-0 cluster. Define

$$\partial C^0 = \{y \in S : y \notin C^0 \text{ but } y \text{ is a neighbor of some site of } C^0\} \tag{3.12}$$

to be the boundary of C^0 within S. Then $|\partial C^0| \le 2s|C^0|$, so

$$E|\partial C^0| \le 2sE|C^0| = 2sE_{p_c, p_c}|\overline{C}| = 2s\chi. \tag{3.13}$$

For each x in ∂C^0 which is connected by a *red* bond to C^0, define

$$C^1(x) = \{y \in S - C^0 : x \leftrightarrow y \text{ by a blue path in } \mathbb{Z}^d\} \tag{3.14}$$

and call it a level-1 cluster.

We proceed to stochastically bound $|C^1(x)|$ by a type of argument using "self-determined" sets that is quite standard in the percolation theory literature [see, e.g., Section 4.2 of Aizenman and Newman (1984)]. Denote by C_B^0 the blue cluster of o, that is, the set of sites in $\mathbb{Z}^d$ connected to o by a blue path in $\mathbb{Z}^d$. We claim that conditionally (on the blue cluster of o) each $|C^1(x)|$ is stochastically dominated by $|C^0|$; that is, for A any finite connected subset of $\mathbb{Z}^d$ containing o,

$$P\big(|C^1(x)| \ge k \mid C_B^0 = A\big) \le P(|C^0| \ge k) \tag{3.15}$$

for every x in ∂C^0 (with ∂C^0 determined here by A) and every k. This is because C^0 is formed in $\mathbb{Z}^d$ but $C^1(x)$ is formed in the smaller region, $\mathbb{Z}^d - C_B^0$; that is, conditioned on $C_B^0 = A$ and for any x in the resulting ∂C^0, the distribution of $C^1(x)$ is the same as that of

$$\{y \in S : x \leftrightarrow y \text{ by a blue path in } \mathbb{Z}^d - A\}.$$

That is so because, conditioned on $C_B^0 = A$, all bonds touching both A and its complement must be nonblue. Note that the union of the $C^1(x)$'s together with C^0 are those y's in S connected to o by a path of bonds which are either blue or red (or both) with at most *one* red (but not blue) bond.

Continue this procedure to define level-i clusters for any $i \ge 2$. The size of any level-i cluster, for $i \ge 1$, is again (conditionally) stochastically dominated by $|C^0|$. Here we condition on C_B^{i-1}, the set of sites in $\mathbb{Z}^d$ connected to o by a path of bonds that are either blue or red (or both) with at most $i-1$ red (but not blue) bonds. Define $\tilde{C}$ to be the union of all level-i clusters, for $i = 0, 1, 2, \dots$. The definitions are such that $\tilde{C}$ is the set of those y's in S connected to o by paths whose bonds are blue or red (or both) with no restriction on the number of purely red bonds. Thus $\tilde{C} = \overline{C}$ and

$$E_{p_c+\varepsilon, p_c}|\overline{C}| = E|\tilde{C}|. \tag{3.16}$$

We will bound the right-hand side of (3.16) as follows. Consider the Galton–Watson tree where each node v has a random number $2sW_v$ of bonds going forward, where W_v has the distribution of $|C^0|$ (recall that $|\partial C^0| \le 2s|C^0|$). Independently declare each bond to be open with probability $\varepsilon/(1-p_c)$. Then the expected number of nodes connected to the root by open paths is $1/(1-(\varepsilon/(1-p_c))2s\chi)$ when $(\varepsilon/(1-p_c))2s\chi < 1$, that is when $\varepsilon < (1-p_c)/(2s\chi)$. Let H denote the expected sum of the W_v's over these nodes connected to the root by open paths. Then it is easily seen that $\chi + (\varepsilon/(1-p_c))2s\chi H = H$ so $H = \chi/(1-(\varepsilon/(1-p_c))2s\chi)$. Finally, from the construction of the Galton–Watson tree and stochastic domination, it follows that

$$E\,|\tilde{C}| \le H = \frac{\chi}{1 - \big(\varepsilon/(1-p_c)\big)2s\chi}. \tag{3.17}$$

Therefore, (3.7) is valid when $0 < \varepsilon < (1-p_c)/2s\chi$. This completes the proof of Theorem 2. □

PROOF OF THEOREM 3. The proof is parallel to that of Theorem 1. We will write $(a, b]$ for $\{o\} \times (a, b]$, an interval on the vertical line $\{o\} \times \mathbb{R}$, located at o of $\mathbb{Z}^d$. We write $\{o\} \times t$ instead of (o, t) for a point on $\{o\} \times \mathbb{R}$ in order to distinguish a point from an interval on $\{o\} \times \mathbb{R}$ and we write $(a, b] \to (a', b']$ to denote the event that for *some* $a < t \le b$ and $a' < t' \le b'$, $\{o\} \times t \to \{o\} \times t'$. It is not hard to show that

$$\begin{aligned}
&\int_{-\infty}^{0}\int_{1}^{\infty} P_\lambda(\{o\} \times t \to \{o\} \times u)\,du\,dt \\
&\quad\le \sum_{l \le 0,\ m \ge 1} P_\lambda((l-1, l] \to (m, m+1]) \\
&\quad\le c\int_{-\infty}^{0}\int_{1}^{\infty} P_\lambda(\{o\} \times t \to \{o\} \times u)\,du\,dt,
\end{aligned} \tag{3.18}$$

where c is a finite constant. To see this, observe that for any $\{o\} \times t \in (l-1, l]$ and any $\{o\} \times u \in (m, m+1]$,

$$P_\lambda(\{o\} \times t \to \{o\} \times u) \le P_\lambda((l-1, l] \to (m, m+1]), \tag{3.19}$$

which implies the first inequality in (3.18). On the other hand, by the FKG inequalities [see Section 2 of Bezuidenhout and Grimmett (1991)]

$$\begin{aligned}
P_\lambda((l-1, l] \to (m, m+1]) \le cP_\lambda(&\{o\} \times (t-1) \\
&\to \{o\} \times (u+1))
\end{aligned} \tag{3.20}$$

with $c = 1/P_\lambda$ (no death in $(l-2, l] \cup (m, m+2]$). The second inequality in (3.18) then follows.

Analogously to the proof of Theorem 1, call a point $\{o\} \times n$ of $\{o\} \times \mathbb{R}$ (with n an integer) a breakpoint if $(-\infty, n-1] \not\to (n, \infty)$. Denote by "$(l-1, l] \not\to (m, m+1]$ in $(\mathbb{Z}^d - \{o\}) \times \mathbb{R}$" the complement of the event that there exist $\{o\} \times t \in (l-1, l]$ and $\{o\} \times u \in (m, m+1]$ such that $\{o\} \times t$ is connected to $\{o\} \times u$ by a path γ which intersects $\{o\} \times \mathbb{R}$ only at $\{o\} \times t$ and $\{o\} \times u$. Note that this event is independent of the Poisson process of deaths on $\{o\} \times \mathbb{R}$. Let B be the event that there is a death in the interval $(0, 1)$ and there is no bond leaving or entering $(0, 1]$. Then by the FKG inequalities, we have for any $0 < \delta < 1$ that

$$\begin{aligned}
&P_{\delta,\lambda_c}(\{o\} \times 1 \text{ is a breakpoint}) \\
&\quad\le P_{\delta,\lambda_c}\big(B \text{ occurs, and } (-\infty, 0] \not\to (1, \infty) \text{ in } (\mathbb{Z}^d - \{o\}) \times \mathbb{R}\big) \\
&\quad\ge P_{\delta,\lambda_c}(B) \prod_{l \le 0,\ m \ge 1} P_{\delta,\lambda_c}\big((l-1, l] \not\to (m, m+1] \\
&\qquad\qquad \text{in } (\mathbb{Z}^d - \{o\}) \times \mathbb{R}\big)
\end{aligned} \tag{3.21}$$

$$= P_{\delta, \lambda_c}(B) \prod_{l \le 0,\, m \ge 1} P_{1, \lambda_c}\big((l-1, l] \not\to (m, m+1] \text{ in } (\mathbb{Z}^d - \{o\}) \times \mathbb{R}\big)$$

$$\ge P_{\delta, \lambda_c}(B) \prod_{l \le 0,\, m \ge 1} \big(1 - P_{1, \lambda_c}((l-1, l] \to (m, m+1])\big).$$

The right-hand side of (3.21) is nonzero by the second inequality of (3.18) and the hypothesis of Theorem 3. Thus from the ergodicity and translation invariance along the time coordinate,

$$P_{\delta, \lambda_c}\big[\{o\} \times n \text{ is a breakpoint for infinitely many (both positive and negative) } n\big] = 1 \tag{3.22}$$

Arguing similarly to the last part of the proof of Theorem 1, it follows that $\theta(\delta, \lambda_c) = 0$ for any $\delta \in (0, 1]$. This completes the proof of Theorem 3. □

SKETCH OF PROOF OF THEOREM 4. First, it is not hard to see that if $\delta < \lambda_c(\mathbb{Z}^d)/\lambda_c(\mathbb{Z}^s)$ then $\theta(\delta, \lambda_c) > 0$, since if $\delta < \lambda_c/\lambda_c(\mathbb{Z}^s)$ then $\lambda_c/\delta > \lambda_c(\mathbb{Z}^s)$ and hence the infection already survives on $\{0\}^{d-s} \times \mathbb{Z}^s$. Thus $\delta_c \ge \lambda_c/\lambda_c(\mathbb{Z}^s)$. The strict inequality $\delta_c > \lambda_c/\lambda_c(\mathbb{Z}^s)$ again follows from the results of Aizenman and Grimmett, but now applied to the contact process [see pages 826 and 827 of Aizenman and Grimmett (1991)]. It remains to show that there exists $\varepsilon > 0$ such that

$$\theta(1 - \varepsilon, \lambda_c) = 0. \tag{3.23}$$

Again write S for $\{0\}^{d-s} \times \mathbb{Z}^s$. Let

$$\begin{aligned} C^* &= C(o, 0) \cap (S \times \mathbb{R}^+) \\ &= \{(x, t) : x \in S, (o, 0) \to (x, t) \text{ in } Z^d \times \mathbb{R}^+\} \end{aligned} \tag{3.24}$$

be the intersection of the cluster of the origin with $S \times \mathbb{R}^+$. For $A_x \subset \{x\} \times \mathbb{Z}^+$, we denote by $l(A_x)$ the Lebesgue measure of A_x and then for $A \subset \mathbb{Z}^d \times \mathbb{R}^+$, we define

$$l(A) = \sum_{x \in \mathbb{Z}^d} l(A \cap (\{x\} \times \mathbb{R}^+)). \tag{3.25}$$

From the hypothesis of the theorem,

$$\chi^* \equiv E_{1, \lambda_c} l(C^*) < \infty. \tag{3.26}$$

The idea of proving (3.23) is to show that

$$\chi^*(\varepsilon) \equiv E_{1-\varepsilon, \lambda_c} l(C^*) < \infty. \tag{3.27}$$

The reasoning for how (3.27) implies (3.23) is similar to that explained right after (3.7), except that one needs to use the simple fact that for each finite t, the set ξ_t of infected sites is almost surely finite.

A proof that $\chi^*(0) < \infty$ implies $\chi^*(\varepsilon) < \infty$ for small ε can be constructed which closely parallels the proof of Theorem 2. However, we find it more convenient to instead reformulate the earlier proof and then extend the reformulation to the contact process. The reformulation, itself a direct extension of arguments used by Aizenman and Newman (1984) (see especially

Lemma 3.1 there), is based on the differential inequality

$$(3.28)\quad (0 \le) \frac{d}{d\varepsilon}\chi(\varepsilon) \le 2s[\chi(\varepsilon)]^2 \quad \text{or} \quad (0 \ge) \frac{d}{d\varepsilon}\left[\chi(\varepsilon)^{-1}\right] \ge -2s,$$

where

$$(3.29)\qquad \chi(\varepsilon) \equiv E_{p_c+\varepsilon,\, p_c}|\overline{C}| = \sum_{x \in S} \tau_{p_c+\varepsilon,\, p_c}(0, x).$$

This implies that $\chi(\varepsilon)^{-1}$ cannot jump discontinuously from a strictly positive value to zero and hence that $\chi(\varepsilon)$ cannot jump from a finite value to infinity. More specifically $\chi(0)^{-1} = \chi^{-1} > 0$ implies the following slight improvement of (3.17):

$$(3.30)\qquad \chi(\varepsilon)^{-1} \ge \chi^{-1} - 2s\varepsilon \quad \text{or} \quad \chi(\varepsilon) \le \frac{\chi}{1 - \varepsilon 2s\chi}$$

for $0 < \varepsilon < (2s\chi)^{-1}$. Issues related to the differentiability of $\chi(\varepsilon)$ can be handled as in Aizenman and Newman (1984).

The above argument extends to the contact process because

$$(3.31)\qquad (0 \le) \frac{d}{d\varepsilon}\chi^*(\varepsilon) \le [\chi^*(\varepsilon)]^2.$$

This differential inequality can be derived analogously to (3.28), for example, by the arguments used in Barsky and Wu (1995) [see also Section 2 of Bezuidenhout and Grimmett (1991)] to obtain (see their Proposition 3.2)

$$(3.32)\qquad \frac{d}{d\lambda}[E_{1,\lambda} l(C(o,0))] \le 2d[E_{1,\lambda} l(C(o,0))]^2.$$

We note that there is no factor of $2d$ (or $2s$) in (3.31) because the derivative there is with respect to the death rate (in S) rather than the infection transmission rate λ. □

REMARK. As in Proposition 3.1 of Aizenman and Newman (1984), (3.28) implies that $\chi(\varepsilon)$ diverges at least as fast as (a constant times) $[\varepsilon_c^s - \varepsilon]^{-1}$ as $\varepsilon \uparrow \varepsilon_c^s$ so that the critical exponent γ^s for mean cluster size divergence satisfies $\gamma^s \ge 1$. Here ε_c^s is the minimum value of ε such that $\chi(\varepsilon) = \infty$. This should equal $p_c^s - p_c$, but we have not investigated the issue of ruling out the possibility (analogously to the situation for homogeneous percolation) that $\varepsilon_c^s < p_c^s - p_c$. Finally, we note that a similar critical exponent inequality follows for the inhomogeneous contact process from (3.31).

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