

## LIMIT THEOREMS FOR FUNCTIONALS OF MOVING AVERAGES

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Let  $X_n = \sum_{i=1}^{\infty} a_i \varepsilon_{n-i}$ , where the  $\varepsilon_i$  are i.i.d. with mean 0 and finite second moment and the  $a_i$  are either summable or regularly varying with index  $\in (-1, -1/2)$ . The sequence  $\{X_n\}$  has short memory in the former case and long memory in the latter. For a large class of functions  $K$ , a new approach is proposed to develop both central ( $\sqrt{N}$  rate) and noncentral (non- $\sqrt{N}$  rate) limit theorems for  $S_N \equiv \sum_{n=1}^N [K(X_n) - EK(X_n)]$ . Specifically, we show that in the short-memory case the central limit theorem holds for  $S_N$  and in the long-memory case,  $S_N$  can be decomposed into two asymptotically uncorrelated parts that follow a central limit and a noncentral limit theorem, respectively. Further we write the noncentral part as an expansion of uncorrelated components that follow noncentral limit theorems. Connections with the usual Hermite expansion in the Gaussian setting are also explored.

1. Introduction. This paper focuses on infinite moving averages defined by  $X_n = \sum_{i=1}^{\infty} a_i \varepsilon_{n-i}$ ,  $n \in \mathbf{Z}$ , where the innovations  $\varepsilon_i$  are mean-zero i.i.d. random variables having at least finite second moments, and the moving average coefficients  $a_i$  satisfy  $\sum_{i=1}^{\infty} a_i^2 < \infty$  and certain conditions to be described later. The goal is to investigate the asymptotic behavior of  $S_N \equiv \sum_{n=1}^N [K(X_n) - EK(X_n)]$ , as  $N \rightarrow \infty$ , for a general class of measurable functions  $K$  under those conditions. Recall that for  $K(x) = x$ , a complete characterization of the weak limit of  $S_N$  in terms of the  $a_i$  was given by Davydov (1970).

The first and primary case that we will consider is where the  $a_i$ 's are regularly varying with exponent  $-\beta$ , denoted by  $a_i \in RV_{-\beta}$ , for some  $\beta \in (1/2, 1)$ , that is,  $a_i = i^{-\beta} L(i)$  and  $L(i)$  is slowly varying at  $\infty$ . Notice that the covariance function  $\rho(m) \equiv EX_0 X_m$  of  $\{X_n\}$  in this case is regularly varying with exponent  $1 - 2\beta \in (-1, 0)$  and hence it is not summable. The study of stationary sequences with correlations decaying at hyperbolic rates presents interesting and challenging probabilistic as well as statistical problems. Progress has been steadily achieved for the last two decades or so. The expressions "long-memory sequence" and "sequence with long-range dependence" are nowadays commonly accepted as the descriptive terms for stationary time series whose correlation function is not summable. Detailed accounts on the development of long-memory sequences, both in theory and application, can be found in a large volume of review papers including Cox (1984), Rosenblatt (1984), Taqqu (1985), Sun and Ho (1985), Künsch (1986), Robinson (1994), Beran (1992) and

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Heyde (1995). A recent book, Beran (1994), also provides an up-to-date description of the statistical development in the area. The long-memory moving average processes considered in this paper cover the model known as the fractional ARIMA process [cf. Granger and Joyeux (1980), Hosking (1981)], which has motivated considerable interest in applied areas such as econometrics and hydrology [see, e.g., Lo (1991), Hipel and McLeod (1994)]. Suppose the  $X_n$  are normal and  $E K^2(X_1) < \infty$ . Then the asymptotics for  $S_N$  can be studied by exploiting the Hermite expansion of  $K$  along with the tool of multiple Wiener–Itô integrals, and are already well-known [see Taqqu (1979), Dobrushin and Major (1979), Breuer and Major (1983), Ho and Sun (1987), Giraitis and Surgailis (1985)]. Not much is known, however, about the case of nonnormal  $X_n$ . The difficulty is in part due to the lack of orthogonal polynomial expansion of  $K(X_n)$  in terms of  $X_n$  in a general setting. With the notion of so-called Appell polynomials, Surgailis (1982) and Giraitis (1985) proved non- $\sqrt{N}$  (noncentral) and  $\sqrt{N}$  (central) limit theorems, respectively, under very restrictive conditions that require  $K$  be analytic and  $\varepsilon_i$  have finite moments of all orders. Using the result in Surgailis (1982), Avram and Taqqu (1987) characterized the noncentral limits when  $K$  is any Appell polynomial of  $X_n$ . In this paper, we establish the asymptotic behavior of  $S_N$  using a new approach that bypasses the traditional polynomial expansion and hence requires considerably weaker regularity conditions for  $K$  than those of Surgailis (1982) and Giraitis (1985). It is the first time that a central limit theorem for  $S_N$  is proved in the long-memory setting without using the method of moments (cumulants).

The second case we shall consider is where the moving average coefficients  $a_i$  are summable and so  $\{X_n\}$  has short memory. A very straightforward question is whether one can prove a central limit theorem for the empirical process of  $X_1, \dots, X_n$  under no additional assumption on the  $a_i$ . This seems to be a natural problem and should have been thoroughly considered. Surprisingly, not much has been done towards this. Although there is a sizable literature on the theory of empirical processes for dependent random variables, with the exception of Chanda and Ruymgaart (1990) and Hesse (1990), the focus has been primarily on results based on various abstract conditions of short-range dependence, for example, strong-mixing conditions [see, e.g., Babu and Singh (1978), Basawa and Prakasa Rao (1980), Billingsley (1968), Deo (1973), Gastwirth and Rubin (1975), Mehra and Rao (1975), Sen (1971), Silverman (1983), Withers (1975)]. In the case of linear processes, the mixing conditions typically translate to very stringent conditions on the  $a_i$  and the  $\varepsilon_i$ . See Andrews (1984), Athreya and Pantula (1986), Bradley (1986), Chanda (1974), Gorodetskii (1977), Pham and Tran (1985). In our approach, we study  $S_N$  using a modification of the argument used for proving the central limit theorem in the long-memory setting and show that the assumption  $\sum_{i=1}^{\infty} |a_i| < \infty$  alone guarantees the central limit theorem for  $S_N$  for a very general class of  $K$ .

This paper is structured as follows. Section 2 introduces the notation and technical conditions. Sections 3 and 4, respectively, contain the results for the long and short-memory cases. The results are then applied to the Gaussian moving average process in Section 5 where certain known results are rediscov-

ered and connections with the theory based on Hermite expansion are given. For ease of reading, most of the proofs are collected in Section 6.

2. Notation and preliminaries. We first introduce some notation that will be useful for the whole paper. Throughout, the notation  $\sum_{1 \leq j_1 < \dots < j_r < \infty}$  denotes a sum over the set of all positive integers  $j_1, \dots, j_r$  such that  $j_1 < \dots < j_r$ . Define

$$Y_{N,0} = N,$$

and for  $r \geq 1$ ,

$$Y_{N,r} = \sum_{n=1}^N \sum_{1 \leq j_1 < \dots < j_r < \infty} \prod_{s=1}^r a_{j_s} \varepsilon_{n-j_s}.$$

If  $a_i \in RV_{-\beta}$  for some  $\beta \in (1/2, 1)$  and  $r$  is any positive integer such that  $r(2\beta - 1) < 1$ , then the process  $\{\sum_{1 \leq j_1 < \dots < j_r < \infty} \prod_{s=1}^r a_{j_s} \varepsilon_{n-j_s}; n \geq 1\}$  also has long memory and it is easily seen that

$$(2.1) \quad \sigma_{N,r}^2 \equiv \text{var}(Y_{N,r}) \sim N^{2-r(2\beta-1)} L^{2r}(N),$$

where  $L$  is the slowly varying component of  $a_i$ . Here and elsewhere in this paper, the notation  $b_N \sim c_N$  means  $b_N/c_N \rightarrow 1$  as  $N \rightarrow \infty$ .

Define  $X_{n,0} = 0$  and

$$X_{n,j} = \sum_{1 \leq i \leq j} a_i \varepsilon_{n-i}, \quad \tilde{X}_{n,j} = X_n - X_{n,j}, \quad j \geq 1$$

and

$$\tilde{X}_{n,j,\ell} = \tilde{X}_{n,j} - \tilde{X}_{n,\ell}, \quad 1 \leq j \leq \ell.$$

Let  $F_j, \tilde{F}_j, G_j$  and  $G$  be the distribution functions of  $X_{n,j}, \tilde{X}_{n,j}, a_j \varepsilon_1$ , and  $\varepsilon_1$ , respectively. For  $j \geq 0$ , define

$$(2.2) \quad K_j(x) = \int K(x+y) dF_j(y), \quad K_\infty(x) = \int K(x+y) dF(y)$$

and

$$K_\infty^{(r)}(x) = \frac{d^r}{dx^r} \int K(x+y) dF(y)$$

whenever they are well defined. For  $p \geq 0$ , define

$$S_{N,p} = \sum_{n=1}^N K(X_n) - \sum_{r=0}^p K_\infty^{(r)}(0) Y_{N,r},$$

and denote

$$S_N \equiv S_{N,0} = \sum_{n=1}^N [K(X_n) - EK(X_n)].$$

Now we define a technical condition that will be used throughout the paper. If the  $t$ th derivative  $K_j^{(t)}$  of  $K_j$  exists, define

$$K_{j,\lambda}^{(t)}(x) = \sup_{|y| \leq \lambda} |K_j^{(t)}(x + y)|, \quad \lambda \geq 0.$$

Let  $\tau$  and  $t$  be nonnegative integers, and let  $\lambda$  be a nonnegative real number. We say that the condition  $C(t, \tau, \lambda)$  holds, if:

1.  $K_\tau^{(t)}(x)$  exists for all  $x$  and  $K_\tau^{(t)}$  is continuous;
2. For all  $x \in \mathfrak{R}$ ,

$$\sup_{I \subset \{1, 2, \dots\}} E \left[ K_{\tau, \lambda}^{(t)} \left( x + \sum_{i \in I} a_i \varepsilon_i \right) \right]^4 < \infty,$$

where the sup is taken over all subsets  $I$  of  $\{1, 2, \dots\}$ .

**REMARK 1.** We will not require the full power of  $C(t, \tau, \lambda)$ . However,  $C(t, \tau, \lambda)$  conveniently implies a number of conditions required for the proof of Lemma 6.2. On the other hand, condition  $C(t, \tau, \lambda)$  is clearly satisfied if the  $t$ th derivative of  $K(\cdot)$  is bounded and continuous, in which case one can simply take  $\tau = 0$ . Similarly, any polynomial  $K$  also ensures  $C(t, \tau, \lambda)$ , provided that  $\varepsilon_i$  has finite moments of sufficiently high orders. The novelty here is that  $C(t, \tau, \lambda)$  can hold without  $K$  being smooth at all. One important example is  $K(x) = I(x \leq u)$  which is encountered in the case of the empirical process, where it is not difficult to see that if  $G$  has a continuous and integrable second derivative then, for  $1 \leq t \leq \tau$ ,

$$K_\tau^{(t)}(x) = \int \cdots \int K(x + a_1 y_1 + \cdots + a_\tau y_\tau) G^{(2)}(y_1) \cdots G^{(2)}(y_t) \\ \times G^{(1)}(y_{t+1}) \cdots G^{(1)}(y_\tau) dy_1 \cdots dy_\tau$$

is bounded and continuous.

The following basic result has a number of applications in this paper.

**LEMMA 2.1.** *Let  $q$  be a nonnegative integer. Suppose that there exist  $\tau$  and  $\lambda$  for which the condition  $C(t, \tau, \lambda)$  holds for all  $t = 0, 1, \dots, q$ . Then for each  $j \geq \tau + 1$ ,  $K_j$  is  $q$  times continuously differentiable and satisfies*

$$(2.3) \quad K_j^{(t)}(x) = EK_\tau^{(t)}(x + \tilde{X}_{1, \tau, j}) \\ = \int K_{j-1}^{(t)}(x + u) dG_j(u) \quad \text{for all } t = 0, 1, \dots, q.$$

**PROOF.** If  $t = 0$ , then (2.3) follows from the Fubini theorem and  $C(0, \tau, \lambda)$ . Next, suppose that we have shown (2.3) for  $t = 0, \dots, s$  for some  $s < q$ . Then,

$$\frac{K_j^{(s)}(x + \delta) - K_j^{(s)}(x)}{\delta} = E \left[ \frac{K_\tau^{(s)}(x + \delta + \tilde{X}_{1, \tau, j}) - K_\tau^{(s)}(x + \tilde{X}_{1, \tau, j})}{\delta} \right].$$

Hence, under  $C(s + 1, \tau, \lambda)$ , the first and the second equalities of (2.3) for  $t = s + 1$  follow from the Lebesgue dominated convergence theorem and the Fubini theorem, respectively.  $\square$

We also need the following definition.

**DEFINITION.** We say that  $K$  has power rank  $k$  for some positive integer  $k$ , if  $K_\infty^{(k)}(0)$  exists and is nonzero and  $K_\infty^{(r)}(0) = 0$  for  $1 \leq r < k$ .

**REMARK 2.** Suppose  $X_n$  is standard normal. Then for every function  $K$  with  $E K^2(X_n) < \infty$ , the following  $L_2$ -expansion holds:

$$K(X_n) = \sum_{j=0}^{\infty} \frac{h_j}{j!} H_j(X_n),$$

where  $H_j(x) = (-1)^j e^{x^2/2} (d^j e^{-x^2/2} / dx^j)$  denotes the  $j$ th Hermite polynomial and  $h_j = \int K(x) H_j(x) d\Phi(x)$ . In Taqqu (1979), the smallest  $j$  for which  $h_j$  is nonzero is called the Hermite rank of  $K$ . It is easy to check that  $h_j = K_\infty^{(j)}(0)$  for each  $j$  and hence the power rank is identical to the Hermite rank.

3. Limit theorems for long-memory moving averages. In this section we consider limit theorems for  $S_N$  in the long-memory case;  $a_i \in RV_{-\beta}$  and  $\beta \in (1/2, 1)$ . Theorem 3.1 below contains a stochastic Taylor expansion for  $S_N$  using the uncorrelated terms  $Y_{N,j}$ . Loosely speaking, the number of terms that can be included in the expansion is  $p$ , the integer part of  $(2\beta - 1)^{-1}$ . Theorem 3.2 shows that  $S_{N,p}$ , the remainder of the aforementioned expansion, follows a central limit theorem. The proof is based on a surprising argument that  $S_{N,p}$  can be approximated in distribution by the remainder of the same expansion for a finite moving average process. Theorems 3.1 and 3.2 are then utilized to determine the asymptotic distribution of  $S_N$  in Corollary 3.3.

**THEOREM 3.1.** *Let  $\beta \in (1/2, 1)$  and  $p$  be any positive integer satisfying  $p < (2\beta - 1)^{-1}$ . Assume that  $E \varepsilon_1^8 < \infty$ ,  $E K^2(X_1) < \infty$  and for some  $\tau \geq 0$  and  $\lambda > 0$ , condition  $C(t, \tau, \lambda)$  holds for  $t = 0, \dots, p + 2$ . Then for any  $\zeta > 0$ , there exists a constant  $C < \infty$  such that for all  $N \geq 1$ ,*

$$(3.1) \quad \text{var}(S_{N,p}) \leq C(N \vee N^{2-(p+1)(2\beta-1)+\zeta}).$$

Moreover, for any  $\lambda < ((2\beta - 1) \wedge (1 - p(2\beta - 1)))/2$ ,

$$(3.2) \quad \frac{N^\lambda}{\sigma_{N,p}} S_{N,p} \rightarrow 0 \quad \text{a.s. as } N \rightarrow \infty,$$

where  $\sigma_{N,p}$  is defined by (2.1).

It follows from (3.1) that if  $p + 1 > (2\beta - 1)^{-1}$  then  $\text{var}(S_{N,p}) \leq CN$ , from which one might conjecture that  $S_{N,p}$  follows a central limit theorem under proper conditions. That turns out to be true for a reason which is natural but far from obvious at first sight, as explained later. Fix  $\ell \geq 1$  and consider the moving average process  $\{X_{n,\ell}, n \geq 1\}$  with coefficients  $\{a_i, i \leq \ell\}$  and innovations  $\{\varepsilon_j\}$ . Clearly,  $\{X_{n,\ell}\}$  has short-range dependence (in fact,  $\ell$ -dependence). Now define

$$Y_{N,r,\ell} = \sum_{n=1}^N \sum_{1 \leq j_1 < \dots < j_r \leq \ell} \prod_{s=1}^r a_{j_s} \varepsilon_{n-j_s}, \quad 1 \leq r \leq \ell$$

and

$$S_{N,p,\ell} = \sum_{n=1}^N K(X_{n,\ell}) - \sum_{r=0}^p K_\ell^{(r)}(0) Y_{N,r,\ell}, \quad 0 \leq p \leq \ell.$$

The essence of the following result is that  $S_{N,p}$  behaves asymptotically like  $S_{N,p,\ell}$ , the partial sum of a short-memory process, and therefore follows a central limit theorem. It is important to remark that this approach is completely different from the traditional approach of the method of moments [cf. Arcones (1994), Breuer and Major (1983), Ho and Sun (1987)].

**THEOREM 3.2.** *Let  $\beta \in (1/2, 1)$  and  $p$  be any positive integer satisfying  $p + 1 > (2\beta - 1)^{-1}$ . Assume that  $E\varepsilon_1^8 < \infty$ ,  $EK^2(X_1) < \infty$  and*

$$(3.3) \quad E[K(X_1) - K(X_{1,\ell})]^2 \rightarrow 0 \quad \text{as } \ell \rightarrow \infty,$$

*In addition, assume that there exist some  $\tau \geq 0$  and  $\lambda > 0$  such that condition  $C(t, \tau, \lambda)$  holds for  $t = 0, \dots, p + 2$ . Then*

$$(3.4) \quad \lim_{\ell \rightarrow \infty} \limsup_{N \rightarrow \infty} N^{-1} \text{var}(S_{N,p} - S_{N,p,\ell}) = 0$$

and

$$(3.5) \quad N^{-1/2} S_{N,p} \rightarrow_d N(0, \sigma^2),$$

where

$$(3.6) \quad \sigma^2 = \lim_{N \rightarrow \infty} N^{-1} \text{var}(S_{N,p}) = \lim_{\ell \rightarrow \infty} \lim_{N \rightarrow \infty} N^{-1} \text{var}(S_{N,p,\ell}) \in [0, \infty).$$

The proofs of Theorems 3.1 and 3.2 are given in Section 6. Although we do not formally prove it here, we remark that under the conditions of Theorem 3.2, for each  $r < (2\beta - 1)^{-1}$ ,  $S_{N,p}$  and  $Y_{N,r}$  are asymptotically uncorrelated in the sense that

$$\lim_{N \rightarrow \infty} \text{cov}(N^{-1/2} S_{N,p}, \sigma_{N,r}^{-1} Y_{N,r}) = 0.$$

In the Gaussian case,  $S_{N,p}$  and  $Y_{N,r}$  are in fact uncorrelated for every  $N$ . See Section 5.

Combining the messages of Theorems 3.1 and 3.2 and assuming that  $(2\beta - 1)^{-1}$  is not an integer, we can write down the decomposition

$$\begin{aligned}
 S_N &= \text{noncentral part} + \text{central part} \\
 (3.7) \quad &= \sum_{r=1}^{[(2\beta-1)^{-1}]} K_\infty^{(r)}(0)Y_{N,r} + \left( \sum_{n=1}^N K(X_n) - \sum_{r=0}^{[(2\beta-1)^{-1}]} K_\infty^{(r)}(0)Y_{N,r} \right),
 \end{aligned}$$

where, as explained earlier, the noncentral and central parts are asymptotically uncorrelated. If the noncentral part is present then the asymptotic distribution of  $S_N$  is determined by the leading term there, and if the noncentral part is absent then the asymptotic distribution is determined by the central part. This is stated more formally in the following corollary.

**COROLLARY 3.3.** *Let  $\beta \in (1/2, 1)$  and  $k$  be the power rank of  $K(\cdot)$ . Assume, that for some  $\tau$  and  $\lambda$ , condition  $C(t, \tau, \lambda)$  holds for  $t = 0, \dots, k + 2$ , and  $E K^2(X_1) < \infty$ . If  $k < (2\beta - 1)^{-1}$  and  $E \varepsilon_1^{2k \vee 8} < \infty$  then*

$$(3.8) \quad \sigma_{N,k}^{-1} S_N \rightarrow_d K_\infty^{(k)}(0)Z_k \quad \text{as } N \rightarrow \infty,$$

where the random variable  $Z_k$  can be represented by the multiple Wiener-Itô integral

$$Z_k = \kappa(\beta, k) \int_{-\infty < u_1 < \dots < u_k < 1} \left\{ \int_0^1 \prod_{j=1}^k [(v - u_j)^+]^{-\beta} dv \right\} dB(u_1) \cdots dB(u_k),$$

with  $B$  denoting standard Brownian motion and

$$\kappa(\beta, k) = \left\{ k!(1 - k(\beta - 1/2))(1 - k(2\beta - 1)) \left[ \int_0^\infty (x + x^2)^{-\beta} dx \right]^{-k} \right\}^{1/2},$$

ensuring  $E Z_k^2 = 1$ . If  $k > (2\beta - 1)^{-1}$ ,  $E \varepsilon_1^8 < \infty$  and (3.3) holds, then

$$(3.9) \quad N^{-1/2} S_N \rightarrow_d N(0, \sigma^2) \quad \text{as } N \rightarrow \infty,$$

where  $\sigma^2 < \infty$  is as in (3.6).

**PROOF.** Since the power rank of  $K$  is  $k$ , from the definition of  $S_{N,p}$  we readily deduce

$$(3.10) \quad S_N = S_{N,k-1} = S_{N,k} + K_\infty^{(k)}(0)Y_{N,k}.$$

First, consider the case  $k < (2\beta - 1)^{-1}$ . Avram and Taqqu (1987) imposed  $E \varepsilon_1^{2k} < \infty$  and showed that  $\sigma_{N,k}^{-1} Y_{N,k}$  converges in distribution to  $Z_k$  [defined in (3.8)] as  $N \rightarrow \infty$ . Thus, (3.8) follows from Theorem 3.1 and the second equality of (3.10). Next, if  $k > (2\beta - 1)^{-1}$ , then (3.9) follows from Theorem 3.2 and the first equality of (3.10).  $\square$

In the preceding results if  $k = (2\beta - 1)^{-1}$  is an integer, then the asymptotic behavior of the various quantities in Corollary 3.3 can also be derived by slightly modifying the proofs, provided that the precise form of the slowly varying component of  $\alpha_i$  is available. In general, it is not easy to write limiting variance  $\sigma^2$  of (3.6) or (3.9) in a closed form, unless some very specific conditions are assumed about the functional  $K$  and the distribution of  $X_n$ . In Theorem 5.1 of Section 5, we derive a formula for  $\sigma^2$  when  $X_n$  is Gaussian.

4. A central limit theorem for short-memory moving averages. In this section we focus on short-memory linear processes  $\{X_n\}$ , where the coefficients  $\alpha_i$  satisfy  $\sum_{i=1}^{\infty} |\alpha_i| < \infty$ . In Theorem 4.1 we show that  $S_N$  is asymptotically normal. The proof, similar in spirit to that of Theorem 3.2, takes full advantage of the linear structure and, in particular, does not rely on whether certain mixing conditions hold. To the best of our knowledge, this is the first attempt to prove a central limit theorem for  $S_N$  in this general setting.

**THEOREM 4.1.** *Assume that  $E\varepsilon_1^4 < \infty$ , (3.3) holds and for some  $\tau$  and  $\lambda$ , condition  $C(t, \tau, \lambda)$  holds for  $t = 0, 1$ . Then*

$$(4.1) \quad \lim_{\ell \rightarrow \infty} \limsup_{N \rightarrow \infty} N^{-1} \text{var}(S_{N,0} - S_{N,0,\ell}) = 0$$

and

$$N^{-1/2} S_N \rightarrow_d N(0, \sigma^2),$$

where

$$\sigma^2 = \lim_{N \rightarrow \infty} N^{-1} \text{var}(S_N) = \lim_{\ell \rightarrow \infty} \lim_{N \rightarrow \infty} \text{var}(S_{N,0,\ell}) \in [0, \infty).$$

5. Gaussian processes. We revisit some results in the literature obtained by different arguments in the context of the Gaussian process.

Assume in this section that the  $\varepsilon_i$  are i.i.d. standard normal variables and the  $\alpha_i$  satisfy  $\sum_{i=1}^{\infty} \alpha_i^2 = 1$  so that  $\text{var}(X_n) = 1$ . We apply Corollary 3.3 to obtain the following result in which the noncentral limit theorem was originally obtained by Dobrushin and Major (1979) and Taqqu (1979), and the central limit theorem by Breuer and Major (1983).

**THEOREM 5.1.** *Suppose that  $\alpha_j \in RV_{-\beta}$  for some  $\beta \in (1/2, 1)$  and  $K$  has Hermite rank  $k$  (cf. Remark 2) and satisfies  $E K^2(X_n) < \infty$ . If  $k < (2\beta - 1)^{-1}$ , then*

$$(5.1) \quad \sigma_{N,k}^{-1} S_N \rightarrow_d h_k Z_k \quad \text{as } N \rightarrow \infty,$$

where  $Z_k$  is as described in Theorem 3.2. If  $k > (2\beta - 1)^{-1}$ , then

$$(5.2) \quad N^{-1/2} S_N \rightarrow_d N(0, \sigma^2),$$

where  $0 < \sigma^2 < \infty$  and is equal to

$$(5.3) \quad \sigma^2 = \sum_{j=k}^{\infty} \frac{h_j^2}{j!} \sum_{m=-\infty}^{\infty} \rho^j(m).$$

Note that Theorem 5.1 has no assumptions on  $K$  other than  $EK^2(X_n) < \infty$ , since normal distribution has bounded derivatives and moments of all order (cf. Remark 1). Our proof actually uses truncated Hermite expansions of  $K$  to approximate  $K$ . We note also that Theorem 4.1 together with the argument used for proving the central limit theorem above can be utilized to prove a central limit theorem for short-memory linear Gaussian processes. Taking a broader perspective, the following result explains the fundamental role of the  $Y_{N,i}$  for the Gaussian moving average process. Note that we do not distinguish between the long- and short-memory cases here.

**THEOREM 5.2.** *Assume the setting explained at the start of this section. Then for each  $N \geq 1$ , the infinite series  $\sum_{i=1}^{\infty} K_{\infty}^{(i)}(0)Y_{N,i}$  converges in  $L_2$ , and  $Y_{N,i}$ ,  $1 \leq i \leq \infty$ , are mutually uncorrelated, where  $Y_{N,\infty} \equiv S_N - \sum_{i=1}^{\infty} K_{\infty}^{(i)}(0)Y_{N,i}$ .*

The preceding result shows that we can write  $S_N$  as an infinite  $L_2$ -expansion using the mutually uncorrelated terms  $Y_{N,i}$ ,  $1 \leq i \leq \infty$ . This expansion, which collects all the ‘diagonal’ terms  $\alpha_j^k \varepsilon_{n-j}^k$  ( $k \geq 2$ ) in  $Y_{N,\infty}$ , is more natural than the Hermite expansion  $\sum_{i=1}^{\infty} (K_{\infty}^{(i)}(0)/i!) \sum_{n=1}^N H_i(X_n)$ , since it gives a clear picture of the fundamental role of each term in the expansion. For example, in the setting of Theorem 3.1 and under the terminology of (3.7) of Section 3, if the noncentral part is present in (3.7) then the expansion based on the  $Y_{N,i}$  reflects the fact that the diagonal terms make no contribution to the limiting distribution.

6. Proofs. We first give a representation of  $S_{N,p} - S_{N,p,\ell}$  which will be central in the proofs on a number of occasions. We will assume the existence of  $K_j^{(t)}$ ,  $j \geq \tau$ ,  $t \leq p$  and the  $L_2$  convergence of a number of telescoping series, where the justifications will be given when these representations are applied to specific cases. With that in mind, for a given nonnegative integer  $p$ , define for  $\ell = 0$  and  $\ell = \tau + 1, \tau + 2, \dots$ ,

$$\begin{aligned} T_{N,1,\ell}^{(t)} = & \left[ \sum_{n=1}^N \sum_{j=\tau+1}^{\infty} (K_{j-1}(\tilde{X}_{n,j-1}) - K_j(\tilde{X}_{n,j})) \right. \\ & - \sum_{r=1}^{t-1} K_{\infty}^{(r)}(0) \sum_{n=1}^N \sum_{\tau+1 \leq j_1 < \dots < j_r < \infty} \prod_{s=1}^r \alpha_{j_s} \varepsilon_{n-j_s} \\ & \left. - \sum_{n=1}^N \sum_{\tau+1 \leq j_1 < \dots < j_t < \infty} \left( \prod_{s=1}^t \alpha_{j_s} \varepsilon_{n-j_s} \right) K_{j_t}^{(t)}(\tilde{X}_{n,j_t}) \right] \end{aligned}$$

$$\begin{aligned}
 & - \left[ \sum_{n=1}^N \sum_{j=\tau+1}^{\infty} (K_{j-1}(\tilde{X}_{n, j-1, \ell}) - K_j(\tilde{X}_{n, j, \ell})) \right. \\
 & \quad - \sum_{r=1}^{t-1} K_{\ell}^{(r)}(0) \sum_{n=1}^N \sum_{\tau+1 \leq j_1 < \dots < j_r \leq \ell} \prod_{s=1}^r a_{j_s} \varepsilon_{n-j_s} \\
 & \quad \left. - \sum_{n=1}^N \sum_{\tau+1 \leq j_1 < \dots < j_t \leq \ell} \left( \prod_{s=1}^t a_{j_s} \varepsilon_{n-j_s} \right) K_{j_t}^{(t)}(\tilde{X}_{n, j_t, \ell}) \right], \quad 0 \leq t \leq p, \\
 T_{N, 2, \ell} & = \left[ \sum_{n=1}^N \sum_{\tau+1 \leq j_1 < \dots < j_p < \infty} \left( \prod_{s=1}^p a_{j_s} \varepsilon_{n-j_s} \right) (K_{j_p}^{(p)}(\tilde{X}_{n, j_p}) - K_{\infty}^{(p)}(0)) \right] \\
 & \quad - \left[ \sum_{n=1}^N \sum_{\tau+1 \leq j_1 < \dots < j_p \leq \ell} \left( \prod_{s=1}^p a_{j_s} \varepsilon_{n-j_s} \right) (K_{j_p}^{(p)}(\tilde{X}_{n, j_p, \ell}) - K_{\ell}^{(p)}(0)) \right], \\
 T_{N, 3, \ell} & = \left[ \sum_{n=1}^N \sum_{j=1}^{\tau} (K_{j-1}(\tilde{X}_{n, j-1}) - K_j(\tilde{X}_{n, j})) \right] \\
 & \quad - \left[ \sum_{n=1}^N \sum_{j=1}^{\tau} (K_{j-1}(\tilde{X}_{n, j-1, \ell}) - K_j(\tilde{X}_{n, j, \ell})) \right], \\
 T_{N, 4, \ell} & = - \sum_{r=1}^p K_{\infty}^{(r)}(0) \sum_{n=1}^N \sum_{\substack{j_1 < \dots < j_r < \infty, \\ 1 \leq j_1 \leq \tau}} \left( \prod_{s=1}^r a_{j_s} \varepsilon_{n-j_s} \right) \\
 & \quad + \sum_{r=1}^p K_{\ell}^{(r)}(0) \sum_{n=1}^N \sum_{\substack{j_1 < \dots < j_r \leq \ell, \\ 1 \leq j_1 \leq \tau}} \left( \prod_{s=1}^r a_{j_s} \varepsilon_{n-j_s} \right) \\
 & = - \sum_{r=1}^p K_{\infty}^{(r)}(0) \sum_{n=1}^N \sum_{\substack{j_1 < \dots < j_r, \\ 1 \leq j_1 \leq \tau, j_r \geq \ell+1}} \left( \prod_{s=1}^r a_{j_s} \varepsilon_{n-j_s} \right) \\
 & \quad + \sum_{r=1}^p (K_{\ell}^{(r)}(0) - K_{\infty}^{(r)}(0)) \sum_{n=1}^N \sum_{\substack{j_1 < \dots < j_r \leq \ell, \\ 1 \leq j_1 \leq \tau}} \left( \prod_{s=1}^r a_{j_s} \varepsilon_{n-j_s} \right).
 \end{aligned}$$

*Note.* We interpret  $\sum_{j=m}^n = 0$  when  $m > n$ ; for example, if  $\ell = 0$  then the second terms of the various quantities of  $T$  defined above are taken to be 0. Further, we write

$$\begin{aligned}
 T_{N, 1, \ell}^{(1)} & = \sum_{n=1}^N \sum_{j=\tau+1}^{\infty} U_{n, j, \ell}, \\
 T_{N, 1, \ell}^{(t+1)} - T_{N, 1, \ell}^{(t)} & = \sum_{n=1}^N \sum_{\tau+1 \leq j_1 < \dots < j_{t+1} < \infty} \left( \prod_{s=1}^t a_{j_s} \varepsilon_{n-j_s} \right) V_{n, j_{t+1}, \ell}^{(t)},
 \end{aligned}$$

$$T_{N,2,\ell} = \sum_{n=1}^N \sum_{\tau+1 \leq j_1 < \dots < j_p < \infty} \left( \prod_{s=1}^p a_{j_s} \varepsilon_{n-j_s} \right) W_{n,j_p,\ell},$$

$$T_{N,3,\ell} = \sum_{n=1}^N \sum_{j=1}^{\tau} P_{n,j,\ell},$$

where

$$U_{n,j,\ell} = [K_{j-1}(\tilde{X}_{n,j-1}) - K_j(\tilde{X}_{n,j}) - a_j \varepsilon_{n-j} K_{j-1}^{(1)}(\tilde{X}_{n,j})]$$

$$- [K_{j-1}(\tilde{X}_{n,j-1,\ell}) - K_j(\tilde{X}_{n,j,\ell}) - a_j \varepsilon_{n-j} K_{j-1}^{(1)}(\tilde{X}_{n,j,\ell})] I(j \leq \ell),$$

$$V_{n,j,\ell}^{(t)} = [K_{j-1}^{(t)}(\tilde{X}_{n,j-1}) - K_j^{(t)}(\tilde{X}_{n,j}) - a_j \varepsilon_{n-j} K_j^{(t+1)}(\tilde{X}_{n,j})]$$

$$- [K_{j-1}^{(t)}(\tilde{X}_{n,j-1,\ell}) - K_j^{(t)}(\tilde{X}_{n,j,\ell}) - a_j \varepsilon_{n-j} K_j^{(t+1)}(\tilde{X}_{n,j,\ell})] I(j \leq \ell),$$

$$W_{n,j,\ell} = [K_j^{(p)}(\tilde{X}_{n,j}) - K_{\infty}^{(p)}(0)] - [K_j^{(p)}(\tilde{X}_{n,j,\ell}) - K_{\ell}^{(p)}(0)] I(j \leq \ell)$$

and

$$(6.1) \quad P_{n,j,\ell} = [K_{j-1}(\tilde{X}_{n,j-1}) - K_j(\tilde{X}_{n,j})]$$

$$- [K_{j-1}(\tilde{X}_{n,j-1,\ell}) - K_j(\tilde{X}_{n,j,\ell})] I(j \leq \ell).$$

Now comes the most crucial observation in this paper. Fix  $n, n', j, j'$  and write

$$K_i(\tilde{X}_{m,i}) = E(K(X_m) | \mathcal{F}_{m-i-1})$$

where  $\mathcal{F}_s$  is the  $\sigma$ -field generated by  $\varepsilon_k, k \leq s$ . Suppose  $n - j \neq n' - j'$  and without loss of generality assume that  $n - j < n' - j'$ . Then

$$E[K_{j-1}(\tilde{X}_{n,j-1}) - K_j(\tilde{X}_{n,j})][K_{j'-1}(\tilde{X}_{n',j'-1}) - K_{j'}(\tilde{X}_{n',j'})]$$

$$= E\{[E(K(X_n) | \mathcal{F}_{n-j}) - E(K(X_n) | \mathcal{F}_{n-j-1})]$$

$$\times [E(K(X_{n'}) | \mathcal{F}_{n'-j'}) - E(K(X_{n'}) | \mathcal{F}_{n'-j'-1})]\}$$

$$= E\{[E(K(X_n) | \mathcal{F}_{n-j}) - E(K(X_n) | \mathcal{F}_{n-j-1})]$$

$$\times E[E(K(X_{n'}) | \mathcal{F}_{n'-j'}) - E(K(X_{n'}) | \mathcal{F}_{n'-j'-1}) | \mathcal{F}_{n-j}]\}$$

$$= E\{[E(K(X_n) | \mathcal{F}_{n-j}) - E(K(X_n) | \mathcal{F}_{n-j-1})]$$

$$\times [E(K(X_{n'}) | \mathcal{F}_{n-j}) - E(K(X_{n'}) | \mathcal{F}_{n-j})]\}$$

$$= 0.$$

By repeated applications of arguments of this nature, we obtain

$$(6.2) \quad \text{cov}(U_{n,j,\ell}, U_{n',j',\ell}) = 0 \quad \text{if } n - j \neq n' - j',$$

$$(6.3) \quad \text{cov}\left(\left(\prod_{s=1}^t a_{j_s} \varepsilon_{n-j_s}\right) V_{n,j_{t+1},\ell}^{(t)}, \left(\prod_{s=1}^t a_{j'_s} \varepsilon_{n'-j'_s}\right) V_{n',j'_{t+1},\ell}^{(t)}\right) = 0$$

if  $n - j_s \neq n' - j'_s$  for some  $1 \leq s \leq t$ ,

$$(6.4) \quad \text{cov}(P_{n,j,\ell}, P_{n',j',\ell}) = 0 \quad \text{if } n - j \neq n' - j',$$

$$(6.5) \quad \text{cov}\left(\left(\prod_{s=1}^p \alpha_{j_s} \varepsilon_{n-j_s}\right) W_{n,j_p,\ell}, \left(\prod_{s=1}^p \alpha_{j'_s} \varepsilon_{n'-j'_s}\right) W_{n',j'_p,\ell}\right) = 0$$

if  $n - j_s \neq n' - j'_s$  for some  $1 \leq s \leq p$ ,

$$(6.6) \quad \text{cov}\left(\prod_{s=1}^{r_1} \alpha_{j_s} \varepsilon_{n-j_s}, \prod_{s=1}^{r_2} \alpha_{j'_s} \varepsilon_{n'-j'_s}\right) = 0 \quad \text{if } r_1 \neq r_2 \text{ or if } r_1 = r_2 = r \text{ but}$$

$n - j_s \neq n' - j'_s$  for some  $1 \leq s \leq r$ .

In the following, let  $j' = (n' - n) + j$ . By (6.2)–(6.6), the Cauchy–Schwarz inequality, and the symmetry of the covariance function, we obtain

$$(6.7) \quad \text{var}(T_{N,1,\ell}^{(1)}) \leq 4 \sum_{n=1}^N \sum_{j=\tau+1}^{\ell} \sum_{n'=n}^{n+\ell-j} \text{cov}(U_{n,j,\ell}, U_{n',j',\ell})$$

$$+ 4 \sum_{n=1}^N \sum_{n'=n}^N \sum_{j=\ell+1}^{\infty} \text{cov}(U_{n,j,0}, U_{n',j',0}),$$

$$(6.8) \quad \text{var}(T_{N,1,\ell}^{(t+1)} - T_{N,1,\ell}^{(t)})$$

$$\leq 4 \sum_{n=1}^N \sum_{\tau+1 \leq j_1 < \dots < j_{t+1} \leq \ell} \sum_{n'=n}^{n+\ell-j_{t+1}} \prod_{s=1}^t \alpha_{j_s} \alpha_{j'_s} \text{cov}(V_{n,j_{t+1},\ell}^{(t)}, V_{n',j'_{t+1},\ell}^{(t)})$$

$$+ 4 \sum_{n=1}^N \sum_{n'=n}^N \sum_{\substack{\tau+1 \leq j_1 < \dots < j_{t+1}, \\ j_{t+1} \geq \ell+1}} \prod_{s=1}^t \alpha_{j_s} \alpha_{j'_s} \text{cov}(V_{n,j_{t+1},0}^{(t)}, V_{n',j'_{t+1},0}^{(t)}),$$

$$(6.9) \quad \text{var}(T_{N,2,\ell})$$

$$\leq 4 \sum_{n=1}^N \sum_{\tau+1 \leq j_1 < \dots < j_p \leq \ell} \sum_{n'=n}^{n+\ell-j_p} \left(\prod_{s=1}^p \alpha_{j_s} \alpha_{j'_s}\right) \text{cov}(W_{n,j_p,\ell}, W_{n',j'_p,\ell})$$

$$+ 4 \sum_{n=1}^N \sum_{n'=n}^N \sum_{\substack{\tau+1 \leq j_1 < \dots < j_p, \\ j_p \geq \ell+1}} \left(\prod_{s=1}^p \alpha_{j_s} \alpha_{j'_s}\right) \text{cov}(W_{n,j_p,0}, W_{n',j'_p,0}),$$

$$(6.10) \quad \text{var}(T_{N,3,\ell}) \leq 2 \sum_{n=1}^N \sum_{j=1}^{\tau} \sum_{n'=n}^{n+\tau-j} \text{cov}(P_{n,j,\ell}, P_{n',j',\ell}),$$

$$\begin{aligned}
 & \text{var}(T_{N,4,\ell}) \\
 & \leq 4E(\varepsilon_1^2) \sum_{r=1}^p \sum_{n=1}^N (K_\infty^{(r)}(0) - K_\ell^{(r)}(0))^2 \sum_{\substack{j_1 < \dots < j_r \leq \ell, \\ 1 \leq j_1 \leq \tau}} \sum_{n'=n}^{n+\tau-j_1} \left( \prod_{s=1}^r a_{j_s} a_{j'_s} \right) \\
 (6.11) \quad & + 4E(\varepsilon_1^2) \sum_{r=1}^p (K_\infty^{(r)}(0))^2 \sum_{n=1}^N \sum_{\substack{j_1 < \dots < j_r, \\ 1 \leq j_1 \leq \tau, j_r \geq \ell+1}} \sum_{n'=n}^{n+\tau-j_1} \left( \prod_{s=1}^r a_{j_s} a_{j'_s} \right).
 \end{aligned}$$

PROOF OF THEOREM 3.1. We prove (3.1) only, as (3.2) easily follows from (3.1) [cf. Lai and Stout (1980), Ho and Hsing (1996), Csörgő and Mielniczuk (1995)]. First, the derivatives  $K_j^{(t)}$ ,  $0 \leq t \leq p$ ,  $j \geq \tau$ , exist by conditions  $C(t, \tau, \lambda)$ ,  $0 \leq t \leq p + 2$ , and Lemma 2.1. Next, relations (6.12)–(6.15) below justify the existence of various infinite series in the following expression (in the  $L_2$  sense):

$$T_{N,1,0}^{(1)} + \sum_{t=1}^{p-1} (T_{N,1,0}^{(t+1)} - T_{N,1,0}^{(t)}) + T_{N,2,0} + T_{N,3,0} + T_{N,4,0}.$$

By the proof of (iii) of Lemma 6.2, it is easy to see that

$$K_j(\tilde{X}_{n,j}) \rightarrow_{L_2} K_\infty(0) \text{ as } j \rightarrow \infty,$$

which implies that

$$\begin{aligned}
 \sum_{n=1}^N [K(X_n) - EK(X_n)] &= \sum_{n=1}^N [K(X_n) - K_\infty(0)] \\
 &= \sum_{n=1}^N \sum_{j=1}^\infty [K_{j-1}(\tilde{X}_{n,j-1}) - K_j(\tilde{X}_{n,j})],
 \end{aligned}$$

where the infinite sum converges in  $L_2$ . Therefore we can write

$$\begin{aligned}
 S_{N,p} &= T_{N,1,0}^{(p)} + T_{N,2,0} + T_{N,3,0} + T_{N,4,0} \\
 &= T_{N,1,0}^{(1)} + \sum_{t=1}^{p-1} (T_{N,1,0}^{(t+1)} - T_{N,1,0}^{(t)}) + T_{N,2,0} + T_{N,3,0} + T_{N,4,0}.
 \end{aligned}$$

Our plan is to show that for some universal constant  $B < \infty$ ,

$$(6.12) \quad \text{var}(T_{N,1,0}^{(1)}) \leq BN,$$

$$(6.13) \quad \sum_{t=1}^{p-1} \text{var}(T_{N,1,0}^{(t+1)} - T_{N,1,0}^{(t)}) \leq BN,$$

$$(6.14) \quad \text{var}(T_{N,3,0}) + \text{var}(T_{N,4,0}) \leq BN,$$

and for any given small  $\zeta > 0$ , there exists  $C < \infty$ , independent of  $k$ , such that

$$(6.15) \quad \text{var}(T_{N,2,0}) \leq C(N \vee N^{2-(p+1)(2\beta-1)+\zeta}).$$

Clearly, by (i) of Lemma 6.2 and the Cauchy–Schwarz inequality, for each  $n, n', j, j' \geq 1$ , there exists a universal constant  $B$  such that

$$(6.16) \quad |\text{cov}(U_{n,j,0}, U_{n',j',0})| \leq Ba_j^2 a_{j'}^2,$$

$$(6.17) \quad \left| \text{cov}(V_{n,j_{t+1},0}^{(t)}, V_{n',j'_{t+1},0}^{(t)}) \right| \leq Ba_{j_{t+1}-1}^2 a_{j'_{t+1}-1}^2.$$

Since the  $a_j^2$  are summable, (6.12) clearly follows from (6.7) and (6.16). Now, by (6.8) and (6.17),

$$(6.18) \quad \begin{aligned} & \text{var}(T_{N,1,0}^{(t+1)} - T_{N,1,0}^{(t)}) \\ & \leq B \sum_{n=1}^N \sum_{n'=n}^N \sum_{\tau+1 \leq j_1 < \dots < j_t < \infty} \left( \prod_{s=1}^t |a_{j_s} a_{j'_s}| \right) \sum_{j_{t+1} > j_t} a_{j_{t+1}}^2 a_{j'_{t+1}}^2. \end{aligned}$$

For any  $\zeta > 0$  there exists  $C < \infty$  such that

$$|a_j| \leq C j^{-\beta+\zeta}.$$

Choose  $0 < \zeta < \beta - 1/2$ . Simple algebra shows that the right-hand side of (6.18) is bounded by

$$CN \sum_{n=0}^{N-1} \sum_{\tau+1 \leq j_1 < \dots < j_t < \infty} \left( \prod_{s=1}^t [j_s(n + j_s)]^{-\beta+\zeta} \right) \sum_{j_{t+1} > j_t} [j_{t+1}(n + j_{t+1})]^{-2\beta+\zeta}$$

for some  $C < \infty$ . By Lemma 6.3, for  $n \geq 1$ ,

$$\begin{aligned} & \sum_{\tau+1 \leq j_1 < j_2 < \dots < j_t < \infty} \left( \prod_{s=1}^t [j_s(n + j_s)]^{-\beta+\zeta} \right) \sum_{j_{t+1} > j_t} [j_{t+1}(n + j_{t+1})]^{-2\beta+\zeta} \\ & \leq C \sum_{j=\tau+1}^{\infty} [j(n + j)]^{-[(t+1)(\beta-\zeta-1/2)+\beta+1/2]} \\ & \leq C \sum_{j=\tau+1}^{\infty} [j(n + j)]^{-(\beta+1/2)} \leq Cn^{-(\beta+1/2)}, \end{aligned}$$

which is summable and ensures (6.13). Next we show (6.14). By the Jensen inequality,

$$E(K_j^2(\tilde{X}_{n,j})) = E(E^2(K(X_n)|\tilde{X}_{n,j})) \leq EK^2(X_n) < \infty \quad \text{for any } n, j \geq 1,$$

and hence by the Cauchy–Schwarz inequality,

$$|\text{cov}(P_{n,j,0}, P_{n',j',0})| \leq 4EK^2(X_n).$$

This, together with (6.10), takes care of the first term of (6.14). The second term there can be handled easily using (6.11). Finally, we show (6.15). It follows from

(iii) of Lemma 6.2 and the fact that  $\alpha_i \in RV_{-\beta}$  that for any  $0 < \zeta < \beta - 1/2$  there exists  $C$  such that

$$EW_{n,j,0}^2 \leq C \sum_{i=j}^{\infty} \alpha_i^2 < C j^{-2[(\beta-1/2)-\zeta]}.$$

This and (6.9) show that

$$\begin{aligned} & \text{var}(T_{N,2,0}) \\ & \leq C \sum_{n=1}^N \sum_{n'=n}^N \sum_{\tau+1 \leq j_1 < \dots < j_p < \infty} \left( \prod_{s=1}^p |\alpha_{j_s} \alpha_{j'_s}| \right) (j_p j'_p)^{-(\beta-1/2)+\zeta} \\ & \leq CN \sum_{n=0}^{N-1} \sum_{\tau+1 \leq j_1 < \dots < j_p < \infty} \left( \prod_{s=1}^p [j_s(n+j_s)]^{-\beta+\zeta} \right) [j_p(n+j_p)]^{-(\beta-1/2)+\zeta}. \end{aligned}$$

By Lemma 6.3, for  $n \geq 1$ ,

$$\begin{aligned} & \sum_{\tau+1 \leq j_1 < \dots < j_p < \infty} \left( \prod_{s=1}^p [j_s(n+j_s)]^{-\beta+\zeta} \right) [j_p(n+j_p)]^{-(\beta-1/2)+\zeta} \\ & \leq C \sum_{j=\tau+1}^{\infty} [j(n+j)]^{-[(p+1)(\beta-\zeta-1/2)+1/2]}. \end{aligned}$$

Note that there are two possibilities: for small  $\zeta > 0$ ,  $(p+1)(\beta-\zeta-1/2)+1/2$  is greater or less than one. By Lemma 6.3, in the former case, the right-hand side of the preceding inequality is summable in  $n$ , and in the latter case, it is bounded by  $Cn^{-2(p+1)(\beta-\zeta-1/2)}$ . Thus, for some  $C < \infty$ ,

$$\text{var}(T_{N,2,0}) \leq C(N \vee N^{2-2(p+1)(\beta-\zeta-1/2)}).$$

This shows (6.15) and concludes the proof of (3.1).  $\square$

**PROOF OF THEOREM 3.2.** Here is the strategy of our proof. Since  $S_{N,p,\ell}$  is the partial sum of an  $\ell$ -dependent sequence for any fixed  $\ell$ , it satisfies

$$(6.19) \quad N^{-1/2} S_{N,p,\ell} \xrightarrow{d} N(0, \sigma_\ell^2)$$

for some finite  $\sigma_\ell^2$ , where

$$\sigma_\ell^2 = \lim_{N \rightarrow \infty} N^{-1} \text{var}(S_{N,p,\ell}).$$

By the triangle inequality and the Cauchy-Schwarz inequalities,

$$\text{var}(S_{N,p,\ell_1} - S_{N,p,\ell_2}) \leq 2[\text{var}(S_{N,p} - S_{N,p,\ell_1}) + \text{var}(S_{N,p} - S_{N,p,\ell_2})].$$

Thus, if (3.4) holds, then

$$\lim_{\ell \rightarrow \infty} \sup_{\ell_1, \ell_2 \geq \ell} \limsup_{N \rightarrow \infty} N^{-1} \text{var}(S_{N,p,\ell_1} - S_{N,p,\ell_2}) = 0,$$

which implies that  $\{\sigma_\ell^2, \ell = 1, 2, \dots\}$  is a Cauchy sequence and therefore  $\sigma_\ell^2$  tends to some finite value  $\sigma^2$  as  $\ell \rightarrow \infty$ . Hence, assuming (3.4), both (3.5) and

(3.6) follow readily as a result of this fact and (6.19). Thus, the remaining part of the proof is dedicated to showing (3.4).

Using arguments similar to those used in the proof of Theorem 3.1, we obtain

$$S_{N,p} - S_{N,p,\ell} = T_{N,1,\ell}^{(1)} + \sum_{t=1}^{p-1} (T_{N,1,\ell}^{(t+1)} - T_{N,1,\ell}^{(t)}) + T_{N,2,\ell} + T_{N,3,\ell} + T_{N,4,\ell}.$$

Our goal is to prove

$$(6.20) \quad \lim_{\ell \rightarrow \infty} \limsup_{N \rightarrow \infty} N^{-1} \text{var}(T_{N,1,\ell}^{(1)}) = 0,$$

$$(6.21) \quad \lim_{\ell \rightarrow \infty} \limsup_{N \rightarrow \infty} N^{-1} \sum_{t=1}^{p-1} \text{var}(T_{N,1,\ell}^{(t+1)} - T_{N,1,\ell}^{(t)}) = 0$$

and

$$(6.22) \quad \lim_{\ell \rightarrow \infty} \limsup_{N \rightarrow \infty} N^{-1} \text{var}(T_{N,i,\ell}) = 0, \quad i = 2, 3, 4.$$

We first show (6.20) and (6.21) through bounding the terms on the right-hand side of (6.7) and (6.8). We first handle the first of the two terms in each of the bounds in (6.7) and (6.8). By (ii) of Lemma 6.2 and the Cauchy–Schwarz inequality,

$$|\text{cov}(U_{n,j,\ell}, U_{n',j',\ell})| \leq C \left( \sum_{i \geq \ell+1} \alpha_i^2 \right) \alpha_j^2 \alpha_{j'}^2$$

and

$$|\text{cov}(V_{n,j_{t+1},\ell}, V_{n',j'_{t+1},\ell})| \leq C \left( \sum_{i \geq \ell+1} \alpha_i^2 \right) \alpha_{j_{t+1}}^2 \alpha_{j'_{t+1}}^2.$$

Since the  $\alpha_i^2$  are summable, it is clear that

$$(6.23) \quad \lim_{\ell \rightarrow \infty} \limsup_{N \rightarrow \infty} N^{-1} \left| \sum_{n=1}^N \sum_{j=\tau+1}^{\ell} \sum_{n'=n}^{n+\ell-j} \text{cov}(U_{n,j,\ell}, U_{n',j',\ell}) \right| = 0$$

and

$$(6.24) \quad \lim_{\ell \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N} \left| \sum_{n=1}^N \sum_{\tau+1 \leq j_1 < \dots < j_{t+1} \leq \ell} \sum_{n'=n}^{n+\ell-j_{t+1}} \left( \prod_{s=1}^t \alpha_{j_s} \alpha_{j'_s} \right) \times \text{cov}(V_{n,j_{t+1},\ell}, V_{n',j'_{t+1},\ell}) \right| = 0.$$

Next, we consider the second of the two terms of each of the right-hand sides of (6.7) and (6.8). By (6.16), for any  $0 < \zeta < 2\beta - 1$ , there exists a finite constant

C such that

$$\begin{aligned} \left| \sum_{n=1}^N \sum_{n'=n}^N \sum_{j=\ell+1}^{\infty} \text{cov}(U_{n,j,0}, U_{n',j',0}) \right| &\leq C \sum_{n=1}^N \sum_{n'=n}^N \sum_{j=\ell+1}^{\infty} (jj')^{-2\beta+\zeta} \\ &\leq CN \sum_{n=0}^{N-1} \sum_{j=\ell+1}^{\infty} [j(n+j)]^{-2\beta+\zeta}, \end{aligned}$$

and it follows from Lemma 6.3 that the right hand side is bounded by

$$CN \sum_{n=0}^{N-1} (n \vee \ell)^{-2\beta+\zeta} \ell^{1-2\beta+\zeta} \leq CN \ell^{2(1-2\beta+\zeta)}.$$

Since  $1 - 2\beta + \zeta < 0$ , we have

$$(6.25) \quad \lim_{\ell \rightarrow \infty} \limsup_{N \rightarrow \infty} N^{-1} \left| \sum_{n=1}^N \sum_{n'=n}^N \sum_{j=\ell+1}^{\infty} \text{cov}(U_{n,j,0}, U_{n',j',0}) \right| = 0.$$

Similarly, by (6.17) and Lemma 6.3,

$$\begin{aligned} &\left| \sum_{n=1}^N \sum_{n'=n}^N \sum_{\substack{\tau+1 \leq j_1 < \dots < j_{t+1}, \\ j_{t+1} \geq \ell+1}} \prod_{s=1}^t a_{j_s} a_{j'_s} \text{cov}(V_{n,j_{t+1},0}^{(t)}, V_{n',j'_{t+1},0}^{(t)}) \right| \\ &\leq CN \sum_{n=0}^{N-1} \left( \sum_{1 \leq j_1 < \dots < j_t < \infty} \prod_{s=1}^t [j_s(n+j_s)]^{-\beta+\zeta} \right) \left( \sum_{j=\ell+1}^{\infty} [j(n+j)]^{-2\beta+\zeta} \right) \\ &\leq CN \ell^{1-2\beta+\zeta} \sum_{n=1}^{N-1} n^{-2\beta+\zeta} \\ &\leq CN \ell^{1-2\beta+\zeta}. \end{aligned}$$

Therefore, we have

$$(6.26) \quad \lim_{\ell \rightarrow \infty} \limsup_{N \rightarrow \infty} N^{-1} \left| \sum_{n=1}^N \sum_{n'=n}^N \sum_{\substack{1 \leq j_1 < \dots < j_{t+1}, \\ j_{t+1} \geq \ell+1}} \left( \prod_{s=1}^t a_{j_s} a_{j'_s} \right) \times \text{cov}(V_{n,j,0}^{(t)}, V_{n',j',0}^{(t)}) \right| = 0.$$

Thus, (6.20) follows from (6.7), (6.23) and (6.25) and (6.21) follows from (6.8), (6.24) and (6.26). Next we show (6.22) for  $i = 2$  through (6.9). The first term on the right of (6.9) is handled as follows. By (iv) of Lemma 6.2,

$$EW_{n,j,\ell}^2 \leq C \sum_{i \geq \ell} a_i^2 \quad \text{for all } n, j,$$

for some finite constant  $C$ . Then, there is a constant  $C$  independent of  $n, j, j'$  and  $\ell$  such that

$$|\text{cov}(W_{n,j,\ell}, W_{n',j',\ell})| \leq C \sum_{i \geq \ell} a_i^2 \leq C \ell^{-2(\beta-1/2-\zeta)},$$

where we choose  $\zeta > 0$  small enough so that  $2(p + 1)(\beta - 1/2 - \zeta) > 1$ . Consequently,

$$\begin{aligned} & \left| \sum_{n=1}^N \sum_{\tau+1 \leq j_1 < \dots < j_p \leq \ell} \sum_{n'=n}^{n+\ell-j_p} \left( \prod_{s=1}^p a_{j_s} a_{j'_s} \right) \text{cov}(W_{n, j_p, \ell}, W_{n', j'_p, \ell}) \right| \\ & \leq C \ell^{-2(\beta-1/2-\zeta)} \sum_{n=1}^N \sum_{n'=n}^{n+\ell-1} \sum_{\tau+1 \leq j_1 < \dots < j_p \leq \ell} \left( \prod_{s=1}^p |a_{j_s} a_{j'_s}| \right) \\ & \leq C \ell^{-2(\beta-1/2-\zeta)} \sum_{n=1}^N \sum_{n'=n}^{n+\ell-1} \left( \sum_{j=1}^{\ell} |a_j a_{j'}| \right)^p \\ & \leq C \ell^{-2(\beta-1/2-\zeta)} N \sum_{n=0}^{\ell-1} \left( \sum_{j=1}^{\ell} [j(n+j)]^{-\beta+\zeta} \right)^p. \end{aligned}$$

By Lemma 6.4 below, the last expression is bounded by

$$\begin{aligned} CN \ell^{-2(\beta-1/2-\zeta)} \sum_{n=1}^{\ell-1} n^{-2p(\beta-1/2-\zeta)} & \leq CN \ell^{-2(\beta-1/2-\zeta)} \ell^{-2p(\beta-1/2-\zeta)+1} \\ & = CN \ell^{-2(p+1)(\beta-1/2-\zeta)+1}. \end{aligned}$$

Hence, in view of the the restriction on  $\zeta$ , we conclude that

$$(6.27) \quad \lim_{\ell \rightarrow \infty} \limsup_{N \rightarrow \infty} N^{-1} \left| \sum_{n=1}^N \sum_{n'=n}^N \sum_{\tau+1 \leq j_1 < \dots < j_p \leq \ell} \left( \prod_{s=1}^p a_{j_s} a_{j'_s} \right) \times \text{cov}(W_{n, j_p, \ell}, W_{n', j'_p, \ell}) \right| = 0.$$

Now we consider the second term on the right of (6.9). By (iii) of Lemma 6.2, for any  $\zeta > 0$  there exists  $C < \infty$  such that

$$\begin{aligned} & \left| \sum_{n=1}^N \sum_{n'=n}^N \sum_{\substack{\tau+1 \leq j_1 < \dots < j_p, \\ j_p \geq \ell+1}} \left( \prod_{s=1}^p a_{j_s} a_{j'_s} \right) \text{cov}(W_{n, j_p, 0}, W_{n', j'_p, 0}) \right| \\ & \leq C \sum_{n=1}^N \sum_{n'=n}^N \sum_{\substack{1 \leq j_1 < \dots < j_p, \\ j_p \geq \ell+1}} \left( \prod_{s=1}^p |a_{j_s} a_{j'_s}| \right) (j_p j'_p)^{-(\beta-1/2)+\zeta} \\ & \leq CN \sum_{n=0}^{N-1} \sum_{\substack{1 \leq j_1 < \dots < j_p, \\ j_p \geq \ell+1}} \left( \prod_{s=1}^p [j_s(n+j_s)]^{-\beta+\zeta} \right) [j_p(n+j_p)]^{-\beta+1/2+\zeta} \\ & = CN \sum_{n=0}^{N-1} \sum_{\substack{1 \leq j_1 < \dots < j_p, \\ j_p \geq \ell+1}} \left( \prod_{s=1}^{p-1} [j_s(n+j_s)]^{-\beta+\zeta} \right) [j_p(n+j_p)]^{-2\beta+1/2+2\zeta}. \end{aligned}$$

By Lemma 6.6,  $\zeta$  here could be chosen small enough so that the preceding inequality gives

$$\lim_{\ell \rightarrow \infty} \limsup_{N \rightarrow \infty} N^{-1} \left| \sum_{n=1}^N \sum_{n'=n}^N \sum_{\substack{\tau+1 \leq j_1 < \dots < j_p, \\ j_p \geq \ell+1}} \left( \prod_{s=1}^p a_{j_s} a_{j'_s} \right) \text{cov}(W_{n, j_p, 0}, W_{n', j'_p, 0}) \right| = 0.$$

This and (6.27) imply (6.22) for  $i = 2$ .

The proof of (6.22) for  $i = 3$  follows from (6.10), using the fact that (3.3) implies that

$$\sup_{j \leq \ell} E[K_j(\tilde{X}_{1, j}) - K_j(\tilde{X}_{1, j, \ell})]^2 \leq E[K(X_1) - K(X_{1, \ell})]^2 \rightarrow 0 \quad \text{as } \ell \rightarrow \infty,$$

and hence

$$(6.28) \quad P_{n, j, \ell} \rightarrow_{L_2} 0 \quad \text{as } \ell \rightarrow \infty \text{ uniformly in } n \text{ and } j \leq \tau.$$

The proof of (6.22) for  $i = 4$  follows from (6.11) together with

$$K_\infty^{(r)}(0) - K_\ell^{(r)}(0) \rightarrow 0 \quad \text{as } \ell \rightarrow \infty \text{ for all } r = 1, \dots, p,$$

which is straightforward (see Lemmas 6.2 and 6.3).  $\square$

**PROOF OF THEOREM 4.1.** The proof is similar to that of Theorem 3.2 with  $p = 0$ ; it suffices to show (4.1). As before,

$$\text{var}(S_{N, 0} - S_{N, 0, \ell}) \leq R_{N, 1, \ell} + R_{N, 2, \ell} + R_{N, 3, \ell}$$

where, with the  $P_{n, j, \ell}$  defines by (6.1),

$$R_{N, 1, \ell} = 8 \sum_{n=1}^N \sum_{j=\tau+1}^{\ell} \sum_{n'=n}^{n+\ell-j} \text{cov}(P_{n, j, \ell}, P_{n', j', \ell}),$$

$$R_{N, 2, \ell} = 8 \sum_{n=1}^N \sum_{n'=n}^N \sum_{j=\ell+1}^{\infty} \text{cov}(P_{n, j, \ell}, P_{n', j', \ell}),$$

$$R_{N, 3, \ell} = 4 \sum_{n=1}^N \sum_{j=1}^{\tau} \sum_{n'=n}^{n+\tau-j} \text{cov}(P_{n, j, \ell}, P_{n', j', \ell}).$$

By (v) and (vi) of Lemma 6.2 and the fact that the  $|a_j|$  are summable,

$$\lim_{\ell \rightarrow \infty} \lim_{N \rightarrow \infty} N^{-1} \text{var}(R_{N, 1, \ell}) \leq C \lim_{\ell \rightarrow \infty} \left( \sum_{i=\ell+1}^{\infty} a_i^2 \right) \left( \sum_{j=1}^{\infty} |a_j| \right)^2 = 0$$

and

$$\lim_{\ell \rightarrow \infty} \lim_{N \rightarrow \infty} N^{-1} \text{var}(R_{N, 2, \ell}) \leq C \lim_{\ell \rightarrow \infty} \left( \sum_{j=\ell+1}^{\infty} |a_j| \right)^2 = 0.$$

Finally, it follows readily from (6.28) that

$$\lim_{\ell \rightarrow \infty} \lim_{N \rightarrow \infty} N^{-1} \text{var}(R_{N,3,\ell}) = 0.$$

This shows (4.1) and concludes the proof.  $\square$

To prove Theorems 5.1 and 5.2, we need a result due to Avram and Taqqu (1987). We first review some notation. For any fixed, preassigned positive integer  $r$ , let  $\pi = (p_1, \dots, p_l)$  be a vector of positive integers such that  $1 \leq p_1 < \dots < p_l$  and  $p_1 + \dots + p_l = r$ . We denote by  $(i)_t = (i_1, \dots, i_t)$  the  $t$ -tuples whose components  $i_s$ 's are all distinct positive integers. Any  $t$ -tuple  $(i)_t = (i_1, \dots, i_t)$  satisfying  $1 \leq i_s \leq \ell$ ,  $1 \leq s \leq t$ , is denoted by  $(i)_{t,\ell}$ .

PROOF OF THEOREM 5.1. Let  $\rho(m) = EX_n X_{n+m}$ . Define

$$X_{n,\ell} = \sum_{i=1}^{\ell} \alpha_i \varepsilon_{n-i} \quad \text{and} \quad X'_{n,\ell} = b_\ell^{-1} \sum_{i=1}^{\ell} \alpha_i \varepsilon_{n-i},$$

where  $b_\ell = \sum_{i=1}^{\ell} \alpha_i^2$  ensuring  $E(X'_{n,\ell})^2 = 1$ . We first note that

$$(6.29) \quad \rho(m) = \sum_{i=1}^{\infty} \alpha_i \alpha_{i+m}, \quad EX'_{n,\ell} X'_{n+m,\ell} = b_\ell^{-2} \sum_{i=1}^{\ell} \alpha_i \alpha_{i+m-n} I(m-n \leq \ell-i).$$

For positive integer  $M \geq k \vee (2\beta - 1)^{-1}$ , define

$$K^M(x) = \sum_{j=k}^M \frac{h_j}{j!} H_j(x).$$

Since  $K^M$  is a polynomial,

$$K_\infty^{M(r)}(0) \equiv (K_\infty^M)^{(r)}(0) = EK^M(X_n)H_r(X_n) = h_r, \quad r \geq 1,$$

and the technical conditions in Corollary 3.3 are easily verified with  $K^M$  in place of  $K$ . Thus, both the noncentral and central limit theorems there hold with  $K^M$  in place of  $K$ . Next, recall that

$$(6.30) \quad \begin{aligned} & E\left(\frac{1}{\sqrt{N}} \sum_{n=1}^N [K(X_n) - K^M(X_n)]\right)^2 \\ & \leq \sum_{j=M+1}^{\infty} \frac{h_j^2}{j!} \left(\frac{1}{N} \sum_{m,n=1}^N |\rho(m-n)|^j\right) \\ & \leq C \left(\sum_{m=-\infty}^{\infty} |\rho(m)|^{M+1}\right) \sum_{j=M+1}^{\infty} \frac{h_j^2}{j!} < \infty \end{aligned}$$

for some constant  $C$ . Since  $\sqrt{N} = o(S_{N,k})$  if  $k < (2\beta - 1)^{-1}$ ,  $\sigma_{N,k}^{-1} \sum_{n=1}^N K(X_n)$  has the same limiting distribution as  $\sigma_{N,k}^{-1} \sum_{n=1}^N K^M(X_n)$ , and (5.1) readily follows from the above discussion. Note also that the last sum on the

right of (6.30) converges to 0 as  $M \rightarrow \infty$ , so that if  $k > (2\beta - 1)^{-1}$  then  $N^{-1/2} \sum_{n=1}^N \mathbf{K}(X_n)$  satisfies the central limit theorem, and so does  $N^{-1/2} \sum_{n=1}^N \mathbf{K}^M(X_n)$ . Thus, it remains to verify that the variance  $\sigma^2$  of the limiting normal distribution is indeed as in (5.3). By (3.6) of Theorem 3.2, we need to show

$$(6.31) \quad \lim_{\ell \rightarrow \infty} \lim_{N \rightarrow \infty} N^{-1} E \left( \sum_{n=1}^N \mathbf{K}^M(X_{n,\ell}) - \sum_{r=0}^{k-1} \mathbf{K}_\ell^{M(r)}(0) Y_{N,r,\ell} \right)^2 = \sum_{j=k}^M \frac{h_j^2 c_j}{j!},$$

where

$$c_j = \frac{1}{j!} \lim_{N \rightarrow \infty} E \left( \frac{1}{\sqrt{N}} \sum_{n=1}^N H_j(X_n) \right)^2 = \sum_{m=-\infty}^{\infty} \rho^j(m).$$

Denote by  $P_j(x)$  the  $j$ th Appell polynomial associated with  $X_{n,\ell}$  [cf. Avram and Taqqu (1987)], defined recursively by

$$(6.32) \quad P'_j(x) = j! P_{j-1}(x), \quad P_0(x) = 1, \quad EP_j(X_{n,\ell}) = \delta_j.$$

It is known that the Appell polynomials for standard normal reduce to the orthogonal Hermite polynomials. Recall that  $\mathbf{K}_\ell^M(0) = E\mathbf{K}^M(X_{n,\ell})$ . Write

$$(6.33) \quad \mathbf{K}^M(X_{n,\ell}) - \mathbf{K}_\ell^M(0) = \sum_{j=1}^M \frac{h_{j,\ell}}{j!} P_j(X_{n,\ell}),$$

where  $h_{j,\ell} = \mathbf{K}_\ell^{M(j)}(0)$  by the recursive relations of (6.32). It is clear that

$$(6.34) \quad \lim_{\ell \rightarrow \infty} b_\ell = 1, \quad \lim_{\ell \rightarrow \infty} h_{j,\ell} = \begin{cases} 0, & \text{if } j < k, \\ h_j, & \text{if } j \geq k, \end{cases}$$

and

$$(6.35) \quad P_1(X_{n,\ell}) = b_\ell \sum_{i=1}^{\ell} (a_i/b_\ell) \varepsilon_{n-i} = b_\ell H_1(X'_{n,\ell}).$$

Using formula (3.9) of Avram and Taqqu (1987), we have for fixed  $j \geq 2$ ,

$$(6.36) \quad \begin{aligned} P_j(X_{n,\ell}) &= b_\ell^j \left[ j! \sum_{1 \leq i_1 < \dots < i_j \leq \ell} \prod_{s=1}^j (a_{i_s}/b_\ell) \varepsilon_{n-i_s} \right. \\ &\quad \left. + \sum_{\pi \neq (1, \dots, 1)} \binom{j}{p_1, \dots, p_t} \sum_{(i)_{t,\ell}} \prod_{s=1}^t (a_{i_s}/b_\ell)^{p_s} H_{p_s}(\varepsilon_{n-i_s}) \right] \\ &= b_\ell^j H_j(X'_{n,\ell}). \end{aligned}$$

The  $t$ -tuple  $\pi = (p_1, \dots, p_t)$  above satisfies  $\sum_{i=1}^t p_i = j$ . The summation  $\sum'_{(i)_{t,\ell}}$  requires that  $i_s < i_{s+1}$  if  $p_s = p_{s+1}$ . Combine (6.33), (6.35) and the last identity

of (6.36) to get

$$\begin{aligned} S_{N, k-1, \ell} &= \sum_{n=1}^N K^M(X_{n, \ell}) - \sum_{r=0}^{k-1} K_\ell^{M(r)}(0) Y_{N, r, \ell} \\ &= \sum_{j=1}^{k-1} h_{j, \ell} \left[ \frac{b_\ell^j}{j!} \sum_{n=1}^N H_j(X'_{n, \ell}) - Y_{N, j, \ell} \right] + \sum_{j=k}^M \frac{h_{j, \ell}}{j!} b_\ell^j \left[ \sum_{n=1}^N H_j(X'_{n, \ell}) \right] \\ &\equiv A_{N, \ell} + B_{N, \ell}. \end{aligned}$$

Since  $E[H_i(X)H_j(Y)] = \delta(i - j)i!E(XY)$ , for any bivariate normal  $(X, Y)$  with standard normal marginals, (6.36) implies  $EA_{N, \ell}B_{N, \ell} = 0$ ,

$$\begin{aligned} (6.37) \quad &E \left( \frac{b_\ell^i}{i!} \sum_{n=1}^N H_i(X'_{n, \ell}) - Y_{N, i, \ell} \right) \\ &\times \left( \frac{b_\ell^j}{j!} \sum_{n=1}^N H_j(X'_{n, \ell}) - Y_{N, j, \ell} \right) = 0, \quad i \neq j, \end{aligned}$$

and, for  $j \geq 2$ ,

$$\begin{aligned} (6.38) \quad &N^{-1} E \left[ \frac{b_\ell^j}{j!} \sum_{n=1}^N H_j(X'_{n, \ell}) - Y_{N, j, \ell} \right]^2 \\ &= \sum_{n=-N+1}^{N-1} \left( 1 - \frac{|n|}{N} \right) \sum_{\pi \neq (1, \dots, 1)} \frac{1}{p_1! \cdots p_t!} \\ &\quad \times \sum_{(i)_{t, \ell}} \prod_{s=1}^t (a_{i_s} a_{i_s+n})^{p_s} I(1 \leq i_s + n \leq \ell) \\ &\leq B \sum_{n=0}^{N-1} \sum_{\pi \neq (1, \dots, 1)} \frac{1}{p_1! \cdots p_t!} \sum_{1 \leq i_1 < \dots < i_t < \infty} \prod_{s=1}^t [i_s(i_s + |n|)]^{-\beta' p_s}, \end{aligned}$$

where  $\beta' \in (1/2, \beta)$  and the constant  $B$  is independent of  $N$  and  $\ell$ . Note that each  $t$ -tuple  $\pi = (p_1, \dots, p_t)$  satisfies  $t \leq j-1$  and  $p_1 + \dots + p_t = j$ . Therefore, the preceding bound in (6.38) can be further bounded by

$$(6.39) \quad C \sum_{n=0}^{N-1} \sum_{i=1}^{\infty} [i(i + |n|)]^{-j(\beta'-1/2)-1} \leq C_1 \sum_{n=1}^{\infty} |n|^{-j(\beta'-1/2)-1} < \infty,$$

using Lemma 6.3. Both constants  $C$  and  $C_1$  are independent of  $N$  and  $\ell$ . Relations (6.34), (6.35), (6.36) and (6.39) imply

$$\lim_{\ell \rightarrow \infty} \lim_{N \rightarrow \infty} N^{-1} EA_{N, \ell}^2 = 0.$$

Hence it is enough to consider  $B_{N,\ell}$ . From (6.29) and (6.34), we have

$$\begin{aligned} & \lim_{\ell \rightarrow \infty} \lim_{N \rightarrow \infty} N^{-1} EB_{N,\ell}^2 \\ &= \lim_{\ell \rightarrow \infty} \lim_{N \rightarrow \infty} \sum_{j=k}^M \frac{h_{j,\ell}^2}{j!} \left( \sum_{n=-\ell+1}^{\ell-1} \left(1 - \frac{|n|}{N}\right) \left[ \sum_{i=1}^{\ell} a_i a_{i+n} I(n+i \leq \ell) \right]^j \right) \\ &= \sum_{j=k}^M \frac{h_j^2}{j!} \left( \sum_{n=-\infty}^{\infty} \left[ \sum_{i=1}^{\infty} a_i a_{i+n} \right]^j \right) \\ &= \sum_{j=k}^M \frac{h_j^2 c_j}{j!}, \end{aligned}$$

which shows (6.31). It is clear that  $\sigma^2 \geq c_k > 0$ . The proof is complete.  $\square$

**PROOF OF THEOREM 5.2.** Fix  $r \geq 1$  and  $l$ -tuple  $\pi = (p_1, p_2, \dots, p_l)$  of positive integers such that  $\sum_{j=1}^l p_j = r$ . Define

$$T_{n,r,\pi} = \sum_{(i)_l} \prod_{s=1}^r a_{i_s}^{p_s} H_{p_i}(\varepsilon_{n-i_s}),$$

where  $\sum_{(i)_l}$  runs over all  $l$ -tuples  $(i)_l = (i_1, \dots, i_l)$  such that  $i_m \neq i_n$  for  $m \neq n$ , and  $i_m < i_{m+1}$  if  $p_m = p_{m+1}$ . By Theorem 1 of Avram and Taquq (1987),

$$H_r(X_n) = r! \sum_{1 \leq j_1 < \dots < j_r < \infty} \prod_{s=1}^r a_{j_s} \varepsilon_{n-j_s} + \sum_{\pi \neq (1, \dots, 1)} \binom{r}{p_1, \dots, p_l} T_{n,r,\pi}.$$

Thus,

$$\begin{aligned} \sum_{n=1}^N K(X_n) &= \sum_{r=1}^{\infty} \frac{K_{\infty}^{(r)}(0)}{r!} \sum_{n=1}^N H_r(X_n) \\ &= \sum_{r=1}^{\infty} K_{\infty}^{(r)}(0) Y_{N,r} + \sum_{n=1}^N \sum_{r=1}^{\infty} \sum_{\pi \neq (1, \dots, 1)} \binom{r}{p_1, \dots, p_l} T_{n,r,\pi}. \end{aligned}$$

Thus,

$$Y_{N,\infty} = \sum_{n=1}^N \sum_{r=1}^{\infty} \sum_{\pi \neq (1, \dots, 1)} \binom{r}{p_1, \dots, p_l} T_{n,r,\pi},$$

whose  $L_2$  convergence is guaranteed by that of the Hermite expansion. It suffices to show that for any positive integers  $n_1, n_2, r, l$ , any  $r$ -tuple  $(j_1, \dots, j_r)$  of positive integers such that  $j_1 < \dots < j_r$ , any  $l$ -tuple  $(p_1, \dots, p_l)$  of positive integers not equal to  $(1, \dots, 1)$  and any  $l$ -tuple  $(i_1, \dots, i_l)$  of positive integers,

$$(6.40) \quad E \left[ \left( \prod_{s=1}^r \varepsilon_{n_1-j_s} \right) \left( \prod_{t=1}^l H_{p_t}(\varepsilon_{n_2-i_t}) \right) \right] = 0.$$

Clearly, (6.40) holds if  $\{n_1 - j_s: 1 \leq s \leq r\} \neq \{n_2 - j_s: 1 \leq s \leq l\}$ . But if  $l = r$  and  $\{n_1 - j_s: 1 \leq s \leq r\} = \{n_2 - i_t: 1 \leq t \leq r\}$ , then

$$E \left[ \left( \prod_{s=1}^r \varepsilon_{n_1 - j_s} \right) \left( \prod_{t=1}^l H_{p_t}(\varepsilon_{n_2 - i_t}) \right) \right] = \prod_{i=1}^r E[H_1(\varepsilon_i)H_{p_i}(\varepsilon_i)],$$

which equals 0 since at least one of the  $p_i$  is not 1. This concludes the proof.  $\square$

The following lemmas handle some technical details required for the proofs above. In the sequel, given any bivariate function  $H(x_1, x_2)$ , we define

$$H_{d_1, d_2}(x_1, x_2) = \sup_{0 \leq |x| \leq d_1, 0 \leq |y| \leq d_2} |H(x_1 + x, x_2 + y)|.$$

For a smooth function  $H(x_1, x_2)$ , let

$$H^{(i, j)}(x_1, x_2) = \frac{\partial^{i, j} H(x_1, x_2)}{\partial x_1^i \partial x_2^j}$$

denote its  $(i, j)$ th partial derivatives, and let

$$H_{d_1, d_2}^{(i, j)}(x_1, x_2) = (H^{(i, j)})_{d_1, d_2}(x_1, x_2).$$

**LEMMA 6.1.** *Let  $H(x_1, x_2)$  be a function with continuous partial derivatives  $H^{(i, j)}(\cdot, \cdot)$ ,  $i = 0, 1, \dots, q + 1$  and  $j = 0, 1$ . Define*

$$(6.41) \quad D(x, y, u) = \left| H(x + y, u) - \sum_{i=0}^q (i!)^{(-1)} y^i H^{(i, 0)}(x, u) \right|^r, \quad r \geq 1.$$

Let  $\mu_i$ ,  $i = 1, 2, 3$  be probability measures on  $\mathfrak{R}$  and  $d\mu(x, y, u) = d\mu_1(x)d\mu_2(y)d\mu_3(u)$ . Assume that for all real  $x$ ,

$$H^{(i, 0)}(x, 0) = 0, \quad i = 0, 1, \dots, q + 1,$$

and there exists  $\lambda > 0$  such that the following integrals are finite for  $i = 1, \dots, q + 1$ :

$$(6.42) \quad \begin{aligned} I_1^{(i)} &= \int [H_{\lambda, \lambda}^{(i, 1)}(x, 0)]^{2r} d\mu_1(x), \\ I_2^{(i)} &= \int [H_{\lambda, 0}^{(i, 0)}(x, u)]^{2r} d\mu_1(x) d\mu_3(u), \\ I_3 &= \int [H_{0, \lambda}^{(0, 1)}(x + y, 0)]^{2r} d\mu_1(x) d\mu_2(y), \\ I_4 &= \int |H(x + y, u)|^{2r} d\mu. \end{aligned}$$

Then

$$(6.43) \quad \int D(x, y, u) d\mu(x, y, u) \leq A \left[ \int y^{2r(q+1)} d\mu_2(y) \int u^{2r} d\mu_3(u) \right]^{1/2},$$

where, for fixed  $q, r$  and  $\lambda$ ,  $A$  is a finite constant whose value depends only on  $\int_{i=1}^q \int y^{2ri} d\mu_2(y)$  and  $(\int_{i=1}^{q+1} I_1^{(i)}) \vee (\int_{i=1}^{q+1} I_2^{(i)}) \vee I_3 \vee I_4$  and can be taken to be increasing in both quantities.

PROOF. Write

$$\begin{aligned} & \int D(x, y, u) d\mu \\ &= \int_{\{|y| \leq \lambda, |u| \leq \lambda\}} + \int_{\{|y| \leq \lambda, |u| > \lambda\}} + \int_{\{|y| > \lambda, |u| \leq \lambda\}} + \int_{\{|y| > \lambda, |u| > \lambda\}} D(x, y, u) d\mu \\ &\equiv J_1 + J_2 + J_3 + J_4. \end{aligned}$$

We shall show that each  $J_i, i = 1, 2, 3, 4$  is bounded by the right-hand side of (6.43). There are two key steps in our argument. The first one is to use Taylor's theorem to obtain for each  $i = 0, 1, \dots, q + 1$ ,

$$D(x, y, u) \leq |y^{q+1} H_{\lambda,0}^{(q+1,0)}(x, u)|^r / q!, \quad |y| \leq \lambda, \quad x, u \in \mathfrak{R}$$

and

$$|H^{(i,0)}(x, u)|^r \leq |u|^r [H_{0,\lambda}^{(i,1)}(x, 0)]^r, \quad |u| \leq \lambda, \quad x \in \mathfrak{R}.$$

The second step is to observe that, by the Chebyshev inequality,

$$\int_{\{|y| > \lambda, |u| > \lambda\}} d\mu_2(y) d\mu_3(u) \leq \lambda^{-2r(q+1)-2r} \int y^{2r(q+1)} d\mu_2 \int u^{2r} d\mu_3.$$

Repeatedly using these two steps and Hölder's inequality, we obtain

$$\begin{aligned} J_1 &\leq \left[ \int (H_{\lambda,\lambda}^{(q+1,1)}(x, 0))^r d\mu_1(x) \right] \int |y|^{r(q+1)} d\mu_2(y) \int |u|^r d\mu_3(u), \\ J_2 &\leq \int |y|^{r(q+1)} I(|u| > \lambda) (H_{\lambda,0}^{(q+1,0)}(x, u))^r d\mu \\ &\leq \left[ \int (H_{\lambda,0}^{(q+1,0)}(x, u))^{2r} d\mu_1(x) d\mu_3(u) \right]^{1/2} \\ &\quad \times \int |y|^{r(q+1)} d\mu_2(y) \left[ \lambda^{-2r} \int |u|^{2r} d\mu_3(u) \right]^{1/2}, \\ J_3 &\leq \int I(|y| > \lambda, |u| \leq \lambda) |u|^r \\ &\quad \times \left[ H_{0,\lambda}^{(0,1)}(x + y, 0) + \sum_{i=0}^q (i!)^{-1} |y|^i H_{0,\lambda}^{(i,1)}(x, 0) \right]^r d\mu \\ &\leq \lambda^{-r(q+1)} \left\{ \int |y|^{2r(q+1)} d\mu_2(y) \int |u|^{2r} d\mu_3(u) \right\}^{1/2} \\ &\quad \times (q + 2)^r \left\{ \int [H_{0,\lambda}^{(0,1)}(x + y, 0)]^{2r} d\mu_1(x) d\mu_2(y) \right. \\ &\quad \left. + \sum_{i=0}^q (i!)^{-2r} \int |y|^{2ri} d\mu_2(y) \int [H_{0,\lambda}^{(i,1)}(x, 0)]^{2r} d\mu_1(x) \right\}^{1/2}, \end{aligned}$$

$$J_4 \leq \lambda^{-r(q+2)} \left\{ \int |y|^{2r(q+1)} d\mu_2(y) \int |u|^{2r} d\mu_3(u) \right\}^{1/2} \\ \times (q+2)^r \left\{ \int |H(x+y, u)|^{2r} d\mu + \sum_{i=0}^q (i!)^{-2r} \int |y|^{2ri} d\mu_2(y) \right. \\ \left. \times \int |H^{(i,0)}(x, u)|^{2r} d\mu_1(x) d\mu_3(u) \right\}^{1/2},$$

and (6.43) is then immediate.  $\square$

**LEMMA 6.2.** *If  $E\varepsilon_1^8 < \infty$  and for some  $\tau \geq 0$  and  $\lambda > 0$ , condition  $C(t, \tau, \lambda)$  holds for  $t = 0, \dots, p+2$ , then:*

(i) *For all  $t = 0, 1, \dots, p$  and  $k = i$  or  $i+1$  where  $i \geq \tau$ ,*

$$E[K_i^{(t)}(\tilde{X}_{n,k}) - K_{i+1}^{(t)}(\tilde{X}_{n,k+1}) - a_{k+1}\varepsilon_{n-k-1}K_k^{(t+1)}(\tilde{X}_{n,k+1})]^2 \leq C(a_k^4 + a_{k+1}^4).$$

(ii) *For all  $t = 0, 1, \dots, p$  and  $k = i$  or  $i+1$  with  $\tau \leq i < i+1 \leq \ell-1$  and  $\ell \geq \tau+1$ ,*

$$E\{[K_i^{(t)}(\tilde{X}_{n,k}) - K_{i+1}^{(t)}(\tilde{X}_{n,k+1}) - a_{k+1}\varepsilon_{n-k-1}K_k^{(t+1)}(\tilde{X}_{n,k+1})] \\ - [K_i^{(t)}(\tilde{X}_{n,k,\ell}) - K_{i+1}^{(t)}(\tilde{X}_{n,k+1,\ell}) - a_{k+1}\varepsilon_{n-k-1}K_k^{(t+1)}(\tilde{X}_{n,k+1,\ell})]\}^2 \\ \leq C\left(\sum_{m=\ell+1}^{\infty} a_m^2\right)(a_k^4 + a_{k+1}^4).$$

(iii) *For  $j \geq \tau+1$ ,*

$$E[K_j^{(p)}(\tilde{X}_{n,j}) - K_{\infty}^{(p)}(0)]^2 \leq C \sum_{m=j}^{\infty} a_m^2.$$

(iv) *For  $\tau+1 \leq j \leq \ell$ ,*

$$E\{[K_j^{(p)}(\tilde{X}_{n,j}) - K_{\infty}^{(p)}(0)] - [K_j^{(p)}(\tilde{X}_{n,j,\ell}) - K_{\ell}^{(p)}(0)]\}^2 \leq C \sum_{m=\ell+1}^{\infty} a_m^2,$$

where  $C$  is a universal constant.

*In addition, if  $E\varepsilon_1^4 < \infty$  and for some  $\tau \geq 0$  and  $\lambda > 0$ , condition  $C(t, \tau, \lambda)$  holds for  $t = 0, 1$ , then:*

(v) *For  $\tau+1 \leq j \leq \ell$ ,*

$$E\{[K_{j-1}(\tilde{X}_{n,j-1}) - K_j(\tilde{X}_{n,j})] - [K_{j-1}(\tilde{X}_{n,j-1,\ell}) - K_j(\tilde{X}_{n,j,\ell})]\}^2 \\ \leq C\left(\sum_{i=\ell+1}^{\infty} a_i^2\right)(a_{j-1}^2 + a_j^2).$$

(vi) *For  $j \geq \ell+1$ ,*

$$E[K_{j-1}(\tilde{X}_{n,j-1}) - K_j(\tilde{X}_{n,j})]^2 \leq C(a_{j-1}^2 + a_j^2),$$

where  $C$  is a universal constant.

PROOF. We only prove (i)–(iv) since the proofs of (v) and (vi) are similar.

(i) Taking into account that  $\int u dG(u) = 0$  and  $K_{i+1}^{(t)}(x) = \int K_i^{(t)}(x + a_{i+1}x') dG(x')$  (see Lemma 2.1), we write for  $k = i + 1$ ,

$$K_i^{(t)}(\tilde{X}_{n,i+1}) - K_{i+1}^{(t)}(\tilde{X}_{n,i+2}) - a_{i+2}\varepsilon_{n-i-2}K_{i+1}^{(t+1)}(\tilde{X}_{n,i+2}) = J_1 - J_2,$$

where

$$J_1 = \int [K_i^{(t)}(\tilde{X}_{n,i+2} + a_{i+2}\varepsilon_{n-i-2} + a_{i+1}x') - K_i^{(t)}(\tilde{X}_{n,i+2} + a_{i+1}x') - a_{i+2}\varepsilon_{n-i-2}K_i^{(t+1)}(\tilde{X}_{n,i+2} + a_{i+1}x')] dG(x')$$

and

$$J_2 = \int [K_i^{(t)}(\tilde{X}_{n,i+2} + a_{i+2}\varepsilon_{n-i-2} + a_{i+1}x') - K_i^{(t)}(\tilde{X}_{n,i+2} + a_{i+2}\varepsilon_{n-i-2}) - a_{i+1}x'K_i^{(t+1)}(\tilde{X}_{n,i+2} + a_{i+2}\varepsilon_{n-i-2})] dG(x').$$

We shall apply Lemma 6.1 to bound  $E|J_1|^2$  and  $E|J_2|^2$ . First, choose  $q = 1$ ,  $r = 2$ , and

$$H(x_1, x_2) = K_i^{(t)}(x_1)x_2, \quad t = 0, 1, \dots, p$$

in (6.41). By writing

$$J_1 = E[K_i^{(t)}(\tilde{X}_{n,i+2} + a_{i+2}\varepsilon_{n-i-2} + a_{i+1}\varepsilon_{n-i-1}) - K_i^{(t)}(\tilde{X}_{n,i+2} + a_{i+1}\varepsilon_{n-i-1}) - a_{i+2}\varepsilon_{n-i-2}K_i^{(t+1)}(\tilde{X}_{n,i+2} + a_{i+1}\varepsilon_{n-i-1}) | \tilde{X}_{n,i+2}, \varepsilon_{n-i-1}],$$

it is clear by the Cauchy–Schwarz inequality that

$$\begin{aligned} E|J_1|^2 &\leq E[K_i^{(t)}(\tilde{X}_{n,i+2} + a_{i+2}\varepsilon_{n-i-2} + a_{i+1}\varepsilon_{n-i-1}) - K_i^{(t)}(\tilde{X}_{n,i+2} + a_{i+1}\varepsilon_{n-i-1}) - a_{i+2}\varepsilon_{n-i-2}K_i^{(t+1)}(\tilde{X}_{n,i+2} + a_{i+1}\varepsilon_{n-i-1})]^2 \\ (6.44) \quad &= \int \int \int |H(x + y, u) - H(x, u) - yH^{(1,0)}(x, u)|^2 \\ &\quad \times d\mu_1(x)\mu_2(y)d\mu_3(u), \end{aligned}$$

where  $\mu_1$  is the convolution of the distributions  $\tilde{F}_{i+2}$  and  $G_{i+1}$ ,  $\mu_2$  is the distribution of  $G_{i+2}$ , and  $\mu_3$  is the probability measure concentrated at  $\{1\}$ . It is straightforward to check that the integral conditions (6.42) in Lemma 6.1 are satisfied, using conditions  $C(t, \tau, \lambda)$ . For example, with  $s = 0, 1, 2$ , by the Cauchy–Schwarz inequality,

$$\begin{aligned} \int [H_{\lambda,0}^{(s,0)}(x, u)]^4 d\mu_1(x)d\mu_3(u) &= \int [K_{i,\lambda}^{(t+s)}(x)]^4 d\mu_1(x) \\ &\leq E[K_{\tau,\lambda}^{(t+s)}(\tilde{X}_{n,\tau} - a_{i+2}\varepsilon_{n-(i+2)})]^4, \end{aligned}$$

which is uniformly bounded for all  $i$ , by (ii) of condition  $C(t, \tau, \lambda)$ . Hence, combining (6.43) and (6.44), we have

$$E|J_1|^2 \leq C \left[ \int y^8 d\mu_2 \right]^{1/2} \leq Ca_{i+2}^4$$

for some universal constant  $C$ . The proof for  $J_2$  is similar, except for switching the roles of  $a_{i+2}\varepsilon_{n-i-2}$  and  $a_{i+1}\varepsilon_{n-i-1}$ . Specifically, let  $\mu_1$  and  $\mu_2$  be the distributions of  $\tilde{F}_{i+1}$  (i.e., the convolution of  $\tilde{F}_{i+2}$  and  $G_{i+2}$ ) and  $G_{i+1}$ , respectively. Then, by the Cauchy-Schwarz inequality and Lemma 6.1 again,

$$E|J_2|^2 \leq Ca_{i+1}^4,$$

and (i) is immediate for  $k = i + 1$ . The case  $k = i$  is handled by the same argument and the details are omitted.

(ii) We focus on the case  $k = i + 1$  and write

$$\begin{aligned} & [K_i^{(t)}(\tilde{X}_{n,i+1}) - K_{i+1}^{(t)}(\tilde{X}_{n,i+2}) - a_{i+2}\varepsilon_{n-i-2}K_{i+1}^{(t+1)}(\tilde{X}_{n,i+2})] \\ & - [K_i^{(t)}(\tilde{X}_{n,i+1,\ell}) - K_{i+1}^{(t)}(\tilde{X}_{n,i+2,\ell}) - a_{i+2}\varepsilon_{n-i-2}K_{i+1}^{(t+1)}(\tilde{X}_{n,i+2,\ell})] \\ & = J_1 - J_2, \end{aligned}$$

where this time

$$\begin{aligned} J_1 &= [K_{i+1}^{(t)}(\tilde{X}_{n,i+2,\ell} + \tilde{X}_{n,\ell} + a_{i+2}\varepsilon_{n-i-2}) - K_{i+1}^{(t)}(\tilde{X}_{n,i+2,\ell} + a_{i+2}\varepsilon_{n-i-2})] \\ & - [K_{i+1}^{(t)}(\tilde{X}_{n,i+2,\ell} + \tilde{X}_{n,\ell}) - K_{i+1}^{(t)}(\tilde{X}_{n,i+2,\ell})] \\ & - a_{i+2}\varepsilon_{n-i-2}[K_{i+1}^{(t+1)}(\tilde{X}_{n,i+2,\ell} + \tilde{X}_{n,\ell}) - K_{i+1}^{(t+1)}(\tilde{X}_{n,i+2,\ell})], \\ J_2 &= \int \{ [K_i^{(t)}(\tilde{X}_{n,i+1,\ell} + \tilde{X}_{n,\ell} + a_{i+1}u') - K_i^{(t)}(\tilde{X}_{n,i+1,\ell} + a_{i+1}u')] \\ & - [K_i^{(t)}(\tilde{X}_{n,i+1,\ell} + \tilde{X}_{n,\ell}) - K_i^{(t)}(\tilde{X}_{n,i+1,\ell})] \\ & - a_{i+1}u'[K_i^{(t+1)}(\tilde{X}_{n,i+1,\ell} + \tilde{X}_{n,\ell}) - K_i^{(t+1)}(\tilde{X}_{n,i+1,\ell})] \} dG(u'). \end{aligned}$$

First of all we need to identify the function  $H(x_1, x_2)$  and the probability measures  $\mu_i$ ,  $i = 1, 2, 3$ . For  $J_1$ , set

$$H(x_1, x_2) = K_{i+1}^{(t)}(x_1 + x_2) - K_{i+1}^{(t)}(x_1),$$

and let  $\mu_1$  and  $\mu_2$  be the distributions of  $\tilde{X}_{n,i+2,\ell}$  and  $a_{i+2}\varepsilon_{n-i-2}$ , respectively. For  $J_2$ , set

$$H(x_1, x_2) = K_i^{(t)}(x_1 + x_2) - K_i^{(t)}(x_1),$$

and let  $\mu_1$  and  $\mu_2$  be the distributions of  $\tilde{X}_{n,i+2,\ell}$  and  $a_{i+1}\varepsilon_{n-i-1}$ , respectively. Let  $\mu_3$  be the distribution of  $\tilde{X}_{n,\ell}$  for both  $J_1$  and  $J_2$ . Applying the Cauchy-

Schwarz inequality and (6.43) with  $q = 1$ , we obtain

$$\begin{aligned} E|J_h|^2 &\leq C(E|\tilde{X}_{n,\ell}|^4 E|a_{i+h}\varepsilon_{n-i-2}|^8)^{1/2} \\ &\leq C\left(\sum_{m \geq \ell+1} a_m^2\right) a_{i+h}^4, \quad h = 1, 2. \end{aligned}$$

Again, as in part (i) above, the integral conditions (6.42) of Lemma 6.1 can be verified by using (ii) of condition  $C(t, \tau, \lambda)$ .

(iii) Write

$$K_j^{(p)}(\tilde{X}_{n,j}) - K_\infty^{(p)}(0) = \int (K_j^{(p)}(\tilde{X}_{n,j}) - K_j^{(p)}(y)) d\tilde{F}_j(y).$$

Hence

$$\begin{aligned} E[K_j^{(p)}(\tilde{X}_{n,j}) - K_\infty^{(p)}(0)]^2 &\leq 2\left[\int |K_j^{(p)}(y_1) - K_j^{(p)}(0)|^2 d\tilde{F}_j(y_1) \right. \\ &\quad \left. + \int |K_j^{(p)}(y_2) - K_j^{(p)}(0)|^2 d\tilde{F}_j(y_2)\right]. \end{aligned}$$

Set  $q = 0$  and

$$H(x_1, x_2) = K_j^{(p)}(x_1)x_2.$$

Choose the probability measures  $\mu_i$ ,  $i = 1, 2, 3$ , to be such that  $\mu_1(\{0\}) = 1$ ,  $\mu_3(\{1\}) = 1$  and  $\mu_2$  is  $\tilde{F}_j$  or  $\tilde{F}_{j-1}$ . Then, by the Cauchy-Schwarz inequality and (6.43) with  $r = 2$ ,

$$\begin{aligned} E[K_j^{(p)}(\tilde{X}_{n,j}) - K_\infty^{(p)}(0)]^2 &\leq B_1\left\{\left[\int |y|^4 d\tilde{F}_j(y)\right]^{1/2} + \left[\int |y|^4 d\tilde{F}_j(y)\right]^{1/2}\right\} \\ &\leq B_2 \sum_{m=j}^{\infty} a_m^2 \end{aligned}$$

for some constants  $B_1$  and  $B_2$ .

(iv) Write

$$[K_j^{(p)}(\tilde{X}_{n,j}) - K_\infty^{(p)}(0)] - [K_j^{(p)}(\tilde{X}_{n,j,\ell}) - K_\ell^{(p)}(0)] = J_1 - J_2,$$

where now

$$\begin{aligned} J_1 &= [K_j^{(p)}(\tilde{X}_{n,j,\ell} + \tilde{X}_{n,\ell}) - K_j^{(p)}(\tilde{X}_{n,j,\ell})], \\ J_2 &= \int [K_\ell^{(p)}(y) - K_\ell^{(p)}(0)] d\tilde{F}_\ell(y). \end{aligned}$$

Let

$$H(x_1, x_2) = K_j^{(p)}(x_1)x_2,$$

and let  $\mu_1$  and  $\mu_2$  be  $\tilde{F}_{j,\ell}$  and  $\tilde{F}_{\ell}$ , respectively. Let  $\mu_3$  satisfy  $\mu_3(\{1\}) = 1$ . Then by the Cauchy–Schwarz inequality and (6.43) with  $q = 0$  and  $r = 2$ ,

$$\begin{aligned} EJ_1^2 &\leq C \int |H(x + y, u) - H(x, u)|^2 d\mu_1(x) d\mu_2(y) \mu_3(u) \\ &\leq C \left( \int y^4 d\mu_2 \right)^{1/2} \leq C \sum_{m=\ell+1}^{\infty} a_m^2 \end{aligned}$$

for some finite constant  $C$ . The bound for  $EJ_2^2$  is the same, where the only changes are:  $H(x_1, x_2)$  is defined to be  $K_{\ell}^{(p)}(x_1)x_2$  instead of  $K_j^{(p)}(x_1)x_2$  and the probability measure  $\mu_1$  is now concentrated at  $\{0\}$ .  $\square$

**LEMMA 6.3.** *Given constants  $\gamma_1, \dots, \gamma_t > 1/2$ , and  $t \geq 1$ , there exists  $C < \infty$  such that for all  $n, \ell \geq 1$ ,*

$$\begin{aligned} &\sum_{\ell+1 \leq j_1 < j_2 < \dots < j_t < \infty} \prod_{s=1}^t [j_s(n + j_s)]^{-\gamma_s} \\ &\leq C \sum_{j=\ell+1}^{\infty} [j(n + j)]^{-\gamma} \\ &\leq \begin{cases} C(n \vee \ell)^{-2\gamma+1}, & \text{if } \gamma \in (1/2, 1), \\ Cn^{-1} \left[ I(n > \ell) \log\left(\frac{n}{\ell}\right) + I(n \leq \ell) \frac{n}{\ell} \right], & \text{if } \gamma = 1, \\ C(n \vee \ell)^{-\gamma} \ell^{1-\gamma}, & \text{if } \gamma \in (1, \infty), \end{cases} \end{aligned}$$

where  $\gamma = \sum_{s=1}^t \gamma_s - (t - 1)/2$ .

**PROOF.** The first inequality follows as in the proof of Lemma 6.4 of Ho and Hsing (1994). We illustrate the proof of the second inequality by considering the case  $\gamma \in (1, \infty)$ . By elementary arguments,

$$\begin{aligned} \sum_{j=\ell+1}^{\infty} [j(n + j)]^{-\gamma} &\leq n^{-2\gamma+1} \int_{\ell/n}^{\infty} (y^{-2\gamma} \wedge y^{-\gamma}) dy \\ &\leq \begin{cases} n^{-2\gamma+1} \left( \int_{\ell/n}^1 y^{-\gamma} dy + \int_1^{\infty} y^{-2\gamma} dy \right), & \text{if } n > \ell, \\ n^{-2\gamma+1} \int_{\ell/n}^{\infty} y^{-2\gamma} dy, & \text{if } n \leq \ell, \end{cases} \end{aligned}$$

so that

$$\sum_{j=\ell+1}^{\infty} [j(n + j)]^{-\gamma} \leq C(n \vee \ell)^{-\gamma} \ell^{1-\gamma}. \quad \square$$

**LEMMA 6.4.** *For  $\delta \in (1/2, 1)$ , there exists a constant  $C$  such that for all  $n, \ell \geq 1$ ,*

$$\sum_{j=1}^{\ell} [j(n + j)]^{-\delta} \leq C(n \wedge \ell)^{1-\delta} n^{-\delta}.$$

PROOF. As in the previous proof, we have

$$\sum_{j=1}^{\ell} [j(n+j)]^{-\delta} \leq \begin{cases} n^{1-2\delta} \int_0^{\ell/n} y^{-\delta} dy, & \text{if } n > \ell, \\ n^{1-2\delta} \left( \int_0^1 y^{-\delta} dy + \int_1^{\ell/n} y^{-2\delta} dy \right), & \text{if } n \leq \ell, \end{cases}$$

and hence the conclusion.  $\square$

LEMMA 6.5. *Given constants  $\gamma_1, \dots, \gamma_{t-1} \in (1/2, 1)$ , there exists  $C < \infty$  such that for all  $n, \ell \geq 1$ ,*

$$\sum_{\substack{1 \leq j_1 < j_2 < \dots < j_t, \\ j_i \geq \ell+1}} \prod_{s=1}^t [j_s(n+j_s)]^{-\gamma_s} \leq C \sum_{k=0}^{t-1} B(k, n, \ell),$$

where, with  $\gamma(k) = \sum_{s=k+1}^t \gamma_s - (t-k-1)/2$ ,

$$B(k, n, \ell) = (n \wedge \ell)^{\sum_{s=1}^k (1-\gamma_s)} n^{-\sum_{s=1}^k \gamma_s} \times \begin{cases} (n \vee \ell)^{-2\gamma(k)+1}, & \text{if } \gamma(k) \in (1/2, 1), \\ n^{-1} \left[ I(n > \ell) \log\left(\frac{n}{\ell}\right) + I(n \leq \ell) \frac{n}{\ell} \right], & \text{if } \gamma(k) = 1, \\ (n \vee \ell)^{-\gamma(k)} \ell^{1-\gamma(k)}, & \text{if } \gamma(k) \in (1, \infty). \end{cases}$$

PROOF. Observe that

$$\begin{aligned} & \sum_{\substack{1 \leq j_1 < j_2 < \dots < j_t, \\ j_i \geq \ell+1}} \prod_{s=1}^t [j_s(n+j_s)]^{-\gamma_s} \\ &= \sum_{j_1=\ell+1}^{\infty} \sum_{j_2=j_1+1}^{\infty} \dots \sum_{j_t=j_{t-1}+1}^{\infty} \prod_{s=1}^t [j_s(n+j_s)]^{-\gamma_s} \\ &+ \sum_{j_1=1}^{\ell} \sum_{j_2=\ell+1}^{\infty} \dots \sum_{j_t=j_{t-1}+1}^{\infty} \prod_{s=1}^t [j_s(n+j_s)]^{-\gamma_s} \\ &+ \dots + \sum_{j_1=1}^{\ell} \sum_{j_2=j_1+1}^{\ell} \dots \sum_{j_k=j_{k-1}+1}^{\ell} \sum_{j_{k+1}=\ell+1}^{\infty} \sum_{j_{k+2}=k_{k+1}+1}^{\infty} \dots \\ & \sum_{j_i=j_{i-1}+1}^{\infty} \prod_{s=1}^t [j_s(n+j_s)]^{-\gamma_s} \\ &+ \dots + \sum_{j_1=1}^{\ell} \sum_{j_2=j_1+1}^{\ell} \dots \sum_{j_{t-1}=j_{t-2}+1}^{\ell} \sum_{j_t=\ell+1}^{\infty} \prod_{s=1}^t [j_s(n+j_s)]^{-\gamma_s}, \end{aligned}$$

where the  $(k+1)$ th term on the right is bounded by

$$\prod_{s=1}^k \sum_{j=1}^{\ell} [j(n+j)]^{-\gamma_s} \sum_{j_{k+1}=\ell+1}^{\infty} \sum_{j_{k+2}=j_{k+1}+1}^{\infty} \dots \sum_{j_t=j_{t-1}+1}^{\infty} \prod_{s=k+1}^t [j_s(n+j_s)]^{-\gamma_s},$$

which, by Lemmas 6.3 and 6.4, is bounded by a constant times  $B(k, n, \ell)$ .  $\square$

LEMMA 6.6. *Let  $(p + 1)(2\beta - 1) > 1$ . Then there exists  $\zeta > 0$  such that*

$$\lim_{\ell \rightarrow \infty} \limsup_{N \rightarrow \infty} \sum_{n=0}^{N-1} \sum_{\substack{1 \leq j_1 < j_2 < \dots < j_p, \\ j_p \geq \ell+1}} \left( \prod_{s=1}^{p-1} [j_s(n + j_s)]^{-\beta+\zeta} \right) [j_p(n + j_p)]^{-2\beta+1/2+2\zeta} = 0.$$

PROOF. First consider the sum for  $n = 1$  to  $N - 1$ . Use the notation of Lemma 6.5 with  $t = p$  and  $\gamma_s = \beta - \zeta$  for  $1 \leq s \leq p - 1$  and  $2\beta - 1/2 - 2\zeta$  for  $s = p$ . Then

$$\begin{aligned} \gamma(k) &= \sum_{s=k+1}^{p-1} (\beta - \zeta) + 2\beta - 1/2 - 2\zeta - \frac{p - k - 1}{2} \\ &= (p - k + 1)(\beta - 1/2 - \zeta) + \frac{1}{2}, \quad 0 \leq k \leq p - 1. \end{aligned}$$

Clearly there exists a small  $\zeta > 0$  such that  $\gamma(0) > 1$  and

$$\gamma(k) > 1 \quad \text{or} \quad 1/2 < \gamma(k) < 1, \quad k = 1, \dots, p - 1.$$

If  $\gamma(k) > 1$  then

$$\begin{aligned} \sum_{n=1}^{N-1} B(n, k, \ell) &\leq \sum_{n=1}^{N-1} n^{\sum_{s=1}^k (1-2\gamma_s)} (n \vee \ell)^{-\gamma(k)} \ell^{1-\gamma(k)} \\ &\leq \sum_{n=1}^{N-1} (n \vee \ell)^{-\gamma(k)} \ell^{1-\gamma(k)} \\ &\leq C \ell^{2-2\gamma(k)}. \end{aligned}$$

Hence

$$(6.45) \quad \lim_{\ell \rightarrow \infty} \limsup_{N \rightarrow \infty} \sum_{n=1}^{N-1} B(n, k, \ell) = 0.$$

Now, for  $1/2 < \gamma(k) < 1$ ,  $k = 1, \dots, p - 1$ ,

$$\begin{aligned} \sum_{n=1}^{N-1} B(n, k, \ell) &= \sum_{n=1}^{N-1} (n \wedge \ell)^{k(1-\beta+\zeta)} n^{-k(\beta-\zeta)} (n \vee \ell)^{-2[(p-k+1)(\beta-1/2-\zeta)+1/2]+1} \\ &= \ell^{-2[(p-k+1)(\beta-1/2-\zeta)+1/2]+1} \sum_{n=1}^{\ell} n^{-2k(\beta-1/2-\zeta)} \\ &\quad + \ell^{k(1-\beta+\zeta)} \sum_{n=\ell+1}^{N-1} n^{-k(\beta-\zeta)-2[(p-k+1)(\beta-1/2-\zeta)+1/2]+1} \\ &\leq C \ell^{1-2(p+1)(\beta-1/2-\zeta)}, \end{aligned}$$

since

$$\begin{aligned} &-k(\beta - \zeta) - 2[(p - k + 1)(\beta - 1/2 - \zeta) + 1/2] + 1 \\ &< -k(1 - \beta) - 1 + (2p + 2 - k)\zeta < -1. \end{aligned}$$

Thus, (6.45) holds again. It remains to deal with the summand that corresponds to  $n = 0$  and show that

$$\lim_{\ell \rightarrow \infty} \limsup_{N \rightarrow \infty} \sum_{\substack{1 \leq j_1 < j_2 < \dots < j_p, \\ j_p \geq \ell + 1}} \left( \prod_{s=1}^{p-1} j_s^{-2(\beta-\zeta)} \right) j_p^{-2(2\beta-1/2-2\zeta)} = 0,$$

which is straightforward and the details are omitted. This concludes the proof.  $\square$

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